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NONCOMPLETENESS OF THE WEIL-PETERSSON METRIC FOR TEICHMÜLLER SPACE

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Let T_g be the Teichmüller space of a compact Riemann surface R of genus g with $g \geqq 2$. In the present paper it is shown thatthe Weil-Petersson length of a large class of rays is finite, deduced that the metric is not complete and indicated how the proof can be extended to the Teichmüller space of an arbitrary finitely generated Fuchsian group of the first kind. The proof is carried out by estimating the Weil-Petersson length of Teichmüller geodesic rays in directions corresponding to a certain class of quadratic differentials.

Metrics dealing with various properties of T_g have been defined. Among them are the Teichmüller, Kobayashi, Carathéodory, Bergman, and Weil-Petersson metrics. Royden has shown that the Teichmüller and Kobayashi metrics coincide, [8]. The Teichmüller-Kobayashi and Carathéodory metrics are known to be complete. Furthermore the Weil-Petersson metric is Hermitian, Kähler and has negative holomorphic sectional curvature, [3].

A trajectory of a quadratic differential φdz^2 of R is a curve along which $\varphi dz^2 > 0$. Zeros (or poles) of φdz^2 are referred to as critical points and a trajectory meeting such points as a critical trajectory. By a Jenkins-Strebel differential we mean a quadratic differential φdz^2 such that the complement on R of the critical trajectories of φdz^2 is a finite union of conformal annuli A_j, $1 < |z_j| < \exp(|c_j|^{-1/2} \log r_j)$ with $\varphi dz^2 = c_j(dz_j \backslash z_j)^2$ on A_j where $c_j < 0, j = 1, \cdots, n$. The existence of such differentials on finite Riemann surfaces is a consequence of the solution by J. Jenkins of a class of free homotopy a conmodule problems, [7]. A. Douady and J. Hubbard recently confirmed jecture of K. Strebel that such differentials represent a dense subset of the space of all analytic quadratic differentials, [6].

It has been communicated to the author that Mr. T. C. Chu of Columbia University has found a similar result. The author would like to take this opportunity to thank professors Clifford Earle and Halsey Royden for their patience and assistance with this investigation.

1. Description of the curve and its tangent vectors. A path leading to the boundary of the Teichmüller space is given by surfaces R_t that are determined by the Beltrami differential $((t-1)/(t+1))(\overline{\varphi dz^2}/|\varphi dz^2|)$, where $t \geqq 1$ and $-\varphi dz^2$ is a Jenkins-Strebel differential.

In a neighborhood of a point z_0 which is not a zero of φdz^2 the local coordinates on R_t are given by

$$z_t = \left(\int_{z0}^{z} \sqrt{\varphi dz^2} + \frac{t-1}{t+1}\int_{z0}^{z} \sqrt{\overline{\varphi dz^2}}\right)\left(1 - \frac{t-1}{t+1}\right)^{-1} .$$

If S_j is a vertical strip in the ζ_j plane s.t. $z_j = \exp(|c_j|^{-1/2}\zeta_j)$ is a covering of the annulus A_j then the formula becomes

$$\zeta_{j,t} = \left(\zeta_j + \frac{t-1}{t+1}\bar{\zeta}_j\right)\left(1 - \frac{t-1}{t+1}\right)^{-1}, \qquad 0 < \operatorname{Re}\zeta_j < \log r_j \tag{1}$$

or equivalently $\zeta_{j,t} = t\xi_j + i\eta_j$ where $\zeta_j = \xi_j + i\eta_j$ and $(d\zeta_{j,t})^2$ is the quadratic differential on $S_{j,t}$ associated with this mapping in the sense of Teichmüller's theorem. By (1) we view the strip $S_{j,t}$, $0 < \operatorname{Re}\zeta_{j,t} < t\log r_j$ in the $\zeta_{j,t}$ plane as the quasiconformal image of the strip $0 < \operatorname{Re}\zeta_j < \log r_j$ in the ζ_j plane. The $S_{j,t}$ cover annuli $A_{j,t}$ by the maps $z_{j,t} = \exp(|c_j|^{-1/2}\zeta_{j,t})$ and due to the nature of the definition of the new coordinates it is clear that the annuli $A_{j,t}$ are identified to form R_t in the same manner in which the A_j are identified to form R.

It will be necessary to know the tangents to the curve expressed as tangent vectors based at the points R_t of Teichmüller space, [2], let f_t be the map from the ζ plane to the ζ_t plane defined by $\zeta_t = t\xi + i\eta$. We are interested in the map f^ρ from the ζ_t plane to the ζ_τ plane satisfying $f_\tau = f^\rho \circ f_t$. Clearly $f^\rho(\zeta_t)$ is defined by $f^\rho(\zeta_t) = (\tau/t)\xi_t + i\eta_t$ where $\zeta_t = \xi_t + i\eta_t$ or equivalently

$$f^\rho(\zeta_t) = \left(\zeta_t + \frac{\tau - t}{\tau + t}\bar{\zeta}_t\right)\left(1 - \frac{\tau - t}{\tau + t}\right)^{-1} .$$

It is clear that this is the Teichmüller map associated with Beltrami differential $((\tau - t)/(\tau + t))\overline{(d\zeta_t)^2}/|(d\zeta_t)^2|$. Taking the τ derivative of $(\tau - t)\backslash(\tau + t)$ and setting $\tau = t$ yields the quantity $1/2t$. Hence the Beltrami differential $(1/2t)\overline{(d\zeta_t)^2}/|(d\zeta_t)^2|$ is the tangent to the curve based at the point R_t.

2. The finite length estimate. For a compact Riemann surface S of genus g, $g \geqq 2$, one can identify the cotangent space at the point S of Teichmüller space with the regular quadratic differentials of S, and the tangent space at S with the Beltrami differentials modulo those which are infinitesimally trivial, [2]. The Weil-Petersson cometric is induced by the norm $\|\varphi\| = \left(\int_S |\varphi|^2\lambda_s^{-2}\right)^{1/2}$ on the space of regular quadratic differentials φ of S, where λ_s is the Poincaré metric for S. The Weil-Petersson metric is induced by the norm $\|\mu\| = \sup_\varphi |[\mu, \varphi]|/\|\varphi\|$, where μ is a Beltrami differential, φ

ranges over the regular quadratic differentials of S and $[\mu, \varphi] = \int_S \mu\varphi$, [10]. For all t, $t \geqq 1$, it is clear that the union of the regions $0 < \operatorname{Re}\zeta_t < t\log r_j$, $0 < \operatorname{Im}\zeta_t < |c_j|^{1/2}2\pi$ can be considered as a domain in R_t and that its complement on the surface is a set of smooth curves. Hence area integrals for R_t can be computed on this union of regions. Since the annulus obtained by taking the quotient of the strip $0 < \operatorname{Re}\zeta_t < t\log r_j$ by the group $\{\zeta_t \to \zeta_t + n\,|c_j|^{1/2}2\pi i,\ n \in \boldsymbol{Z}\}$ is contained in R_t, the Poincaré metric for the annulus bounds the Poincaré metric of R_t restricted to the annulus. If $\|\varphi\|_t$ is the Weil-Petersson norm of a quadratic differential of R_t, we have

$$\|\varphi\|_t^2 \geqq \sum_j \int_{\substack{0<\operatorname{Im}\zeta_{j,t}<|c_j|^{1/2}2\pi \\ 0<\operatorname{Re}\zeta_{j,t}<t\log r_j}} |\varphi|^2\lambda_j^{-2}\,,$$

where λ_j is the Poincaré metric for the $S_{j,t}$, and consequently

$$\left\|\frac{1}{2t}\frac{\overline{(d\zeta_t)^2}}{|(d\zeta_t)^2|}\right\|_t = \sup_\varphi \frac{\left|\left[\frac{1}{2t}\frac{\overline{(d\zeta_t)^2}}{|(d\zeta_t)^2|}, \varphi\right]\right|}{\|\varphi\|_t} \leqq \sup_\varphi \frac{\left|\left[\frac{1}{2t}\frac{\overline{(d\zeta_t)^2}}{|(d\zeta_t)^2|}, \varphi\right]\right|}{\left(\sum_j \int |\varphi|^2\lambda_j^{-2}\right)^{1/2}}\,.$$

Noting that

$$\lambda_j = \frac{\pi}{t\log r_j}\csc\frac{\pi}{t\log r_j}\operatorname{Re}\zeta_{j,t}\,|d\zeta_{j,t}|$$

and that the metric does not depend on $\operatorname{Im}\zeta_{j,t}$, we see that the extremal value of the last quotient is estimated by $\varphi = (d\zeta_t)^2$. Since the λ_j are known, the quantity

$$\sum_j \int |(d\zeta_{j,t})^2|^2\lambda_j^{-2}$$

can be computed directly, or the dependence on t can be determined by considering the invariance properties of the λ_j in either case

$$\sum_j \int |(d\zeta_{j,t})^2|^2\lambda_j^{-2} = Ct^3\,, \qquad C > 0\,. \tag{2}$$

Furthermore we have

$$\begin{aligned}\left[\frac{1}{2t}\frac{\overline{(d\zeta_t)^2}}{|(d\zeta_t)^2|}, (d\zeta_t)^2\right] &= \int_{R_t}\frac{1}{2t}\frac{\overline{(d\zeta_t)^2}}{|(d\zeta_t)^2|}(d\zeta_t)^2 \\ &= \frac{1}{t}\sum_j\int\frac{i}{2}d\zeta_{j,t}\overline{d\zeta_{j,t}} = 2\pi\sum_j|c_j|^{1/2}\log r_j\,.\end{aligned} \tag{3}$$

It follows from (2) and (3) that the length of the tangent vector $(1/2t)\overline{(d\zeta_t)^2}/|(d\zeta_t)^2|$ is bounded above by

$$\frac{C_1}{t^{3/2}} \quad \text{for} \quad t \geqq 1\,, \quad C_1 > 0\,.$$

Since $\int_1^\infty dt/t^{3/2}$ converges, we see that this curve has finite length in the Weil-Petersson metric.

The Teichmüller distance between R and R_t is $\log t$ and since this metric is complete we see that for any sequence of t's tending to infinity the surfaces R_t cannot converge. The Weil-Petersson and Teichmüller metrics induce the same topology and thus such a sequence cannot converge in the Weil-Petersson metric even though it is Cauchy. We conclude that the Weil-Petersson metric is not complete.

3. **Further remarks.** If $\dot{R}$ is obtained from R by the removal of finitely many points and the upper half plane U is taken as a finitely ramified cover of $\dot{R}$ then we again choose φdz^2 as a Jenkins-Strebel differential where in this case the possible poles of φdz^2, the punctures of $\dot{R}$ as well as those points above which the cover is ramified are all considered as critical points. With this convention the branch points lie on the boundaries of the resulting annuli and the Poincaré metrics of the annuli again provide a bound for the restriction to the annuli of the Poincaré metric of the surface. In this way the proof is extended to the general case of the Teichmüller space of a finitely generated Fuchsian group of the first kind.

The curve under consideration is a geodesic in the Teichmüller metric and is readily seen to be a "pinching" of a given Riemann surface as in the work of L. Bers, [5]. An immediate consequence of the present investigation is that the Weil-Petersson metric is not uniformly equivalent to the Teichmüller-Kobayashi or Carathéodory metrics. We also observe that the Teichmüller space of a 4 times punctured sphere is conformally equivalent to the unit disc. In this case the Weil-Petersson metric is not uniformly equivalent to the Bergman metric since the latter is the Poincaré metric which is complete.

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