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DOUBLING CHAINS, SINGULAR ELEMENTS AND HYPER-$\mathscr{Z}$ l-GROUPS

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In a lattice-ordered group G a (*descending*) *doubling chain* is a sequence $a_1 > a_2 > \cdots > a_n > \cdots$ of positive elements of G such that $a_n \geqq 2a_{n+1}$. An element $0 < s \in G$ is *singular* if $0 \leqq g \leqq s$ implies that $g \wedge (s - g) = 0$. The main theorems are as follows: 1. The following two statements are equivalent: (a) every doubling chain in G is finite; (b) $G = \bigcup_{\tau<\alpha} G^\tau$($\tau$ ranging over all ordinals less than some α), where G^τ is an l-ideal of G, $\sigma < \tau$ implies that $G^\sigma \subseteq G^\tau$ and $G^{\tau+1}/G^\tau$ is generated by its singular elements, (i.e. a *Specker group*, à la Conrad). 2. If G is hyper-archimedean as well then either of the above conditions is equivalent to: (c) G is hyper-$\mathscr{Z}$, i.e. every totally ordered l-homomorphic image of G is cyclic.

The purpose of this investigation was to come up with an "elementwise" definition of the abelian lattice-ordered groups (henceforth abbreviated: l-groups) having the property that each l-homomorphic image which is totally ordered is cyclic. These l-groups are called *hyper-$\mathscr{Z}$*, and were first introduced by the author in [5]. Thus, G is hyper-$\mathscr{Z}$ if and only if G is abelian and G/P is cyclic, for each prime subgroup P of G. These l-groups are therefore hyper-archimedean, and they can in fact be characterized as those l-groups for which all the prime subgroups are maximal and have cyclic quotient; (see [3] and [5]). It should be stressed that in this characterization no assumptions need to be made with respect to commutativity. In [3] Conrad provided an example of an l-group which is hyper-archimedean and also a subdirect product of Z, the additive group of integers with the usual order, yet is not hyper-$\mathscr{Z}$.

An element $s > 0$ of the l-group G is *singular* provided $0 \leqq g \leqq s$ implies that $g \wedge (s - g) = 0$. An *S-group* (or *Specker group*) is one in which each positive element is a sum of singular elements. These S-groups are well explored in [3]; the main characterization is that each S-group can be embedded as an l-subring of bounded, integer-valued functions on a set, or alternatively, as an l-subgroup of bounded, integer-valued functions generated by characteristic functions. It was observed in [7] that the S-groups form a torsion class of l-groups; that is, they are closed under taking convex l-subgroups, l-homomorphic images, and if G is any l-group, and $\{C_\lambda \mid \lambda \in \Lambda\}$ a family of convex l-subgroups which are all S-groups then the convex l-subgroup they generate is an S-group. There is thus an associated

S-radical $\mathscr{S}(G)$ of an l-group G, and a "Loewy"-like ascending sequence $S(G) = \mathscr{S}^1(G) \subseteqq \cdots \subseteqq \mathscr{S}^\tau(G) \subseteqq \cdots$ for each ordinal τ, so that

(a) $\mathscr{S}(G)$ is the largest convex l-subgroup of G which is an S-group.

(b) For any convex l-subgroup A of G, $\mathscr{S}(A) = A \cap \mathscr{S}(G)$.

(c) If α is a limit ordinal $\mathscr{S}^\alpha(G) = \bigcup \{\mathscr{S}^\tau(G) \mid \tau < \alpha\}$,

(d) and otherwise $\mathscr{S}^\tau(G)$ is defined by the equation:

$$\mathscr{S}^\tau(G)/\mathscr{S}^{\tau-1}(G) = \mathscr{S}(G/\mathscr{S}^{\tau-1}(G)) .$$

Then we are able to define $\mathscr{S}^*(G) = \mathscr{S}^\tau(G)$, where τ is chosen so that $\mathscr{S}^\tau(G) = \mathscr{S}^{\tau+1}(G) = \cdots$; such a τ exists by a simple cardinality argument. G is said to be an *S^*-group* if $\mathscr{S}^*(G) = G$.

We should observe that if G is an S-group, it can be represented as an l-group of bounded, integer-valued functions, and it is therefore hyper-$\mathscr{Z}$; (see [3]). Examples of hyper-$\mathscr{Z}$ l-groups which are not S-groups are easy to construct.

In an l-group G, a *(descending) doubling chain* is a sequence $s_1 > s_2 > \cdots$ of positive elements of G so that $s_n \geqq 2s_{n+1}$, for each $n = 1, 2, \cdots$. Notice that the terms of a doubling chain may eventually be zero; in such a case it is a finite doubling chain.

We can now state our first result.

THEOREM 1. *G is an S^*-group if and only if every doubling chain for G is finite.*

Proof. Necessity. The proof proceeds by transfinite induction on the length of the Loewy sequence of $\mathscr{S}^\tau(G)$'s. The first thing to do is to show an S-group has this property. This is clear, because if G is an S-group, it can be represented as an l-group of bounded, integer-valued functions; and there are obviously no infinite doubling chains of such functions. Next, suppose $G = \mathscr{S}^\alpha(G)$ and $\mathscr{S}^\tau(G)$ has no infinite doubing chains, for each $\tau < \alpha$. Suppose by way of contradiction that $a_1 > a_2 > \cdots > a_n > \cdots$ is an infinite doubling chain for G; if α is a limit ordinal, then $a_1 \in \mathscr{S}^\beta(G)$ for some $\beta < \alpha$, and hence each $a_n \in \mathscr{S}^\beta(G)$, contradicting our assumption. If α has a predecessor, then no $a_n \in \mathscr{S}^{\alpha-1}(G)$, and consequently $a_1 + \mathscr{S}^{\alpha-1}(G) > a_2 + \mathscr{S}^{\alpha-1}(G) > \cdots$ is an infinite doubling chain in the S-group $G/\mathscr{S}^{\alpha-1}(G)$. This again is a contradiction, and we must conclude that $G = \mathscr{S}^\alpha(G)$ has no infinite doubling chains; this completes the proof of the necessity.

Sufficiency. Let us make a preliminary observation: for a given ordinal τ, an element $a > 0$ of an l-group G has the property that $a \geqq 2b \geqq 0$ implies that $b \in \mathscr{S}^\tau(G)$ if and only if $a \in \mathscr{S}^\tau(G)$ or else

$a + \mathscr{S}^{\tau}(G)$ is a singular element of $G/\mathscr{S}^{\tau}(G)$. If a has this property and $a \notin \mathscr{S}^{\tau}(G)$ we call a $(\tau+1)$-*singular*. (Note: for $\tau = 0$ we set $\mathscr{S}^{\tau}(G) = 0$; then 1-singular simply means: singular.)

Suppose now that every doubling chain of G is finite. If $0 < g \in G$ and τ is an ordinal, then if g is not $(\tau+1)$-singular we may find an element $0 < a_1 \in G$ such that $a_1 \notin \mathscr{S}^{\tau}(G)$ and $2a_1 \leqq g$. Inductively proceed to construct a doubling chain $g > a_1 > a_2 > \cdots > a_k > \cdots$, where a_k is the last entry outside $\mathscr{S}^{\tau}(G)$, and therefore $(\tau+1)$-singular. Thus, every positive element of G exceeds a $(\tau+1)$-singular element, for each ordinal τ.

If $G \neq \mathscr{S}^*(G)$, we pick $0 < g \in G \backslash \mathscr{S}^*(G)$, and an ordinal α such that $\mathscr{S}^*(G) = \mathscr{S}^{\alpha}(G)$. As we have indicated $g \geqq h$ for some $(\alpha+1)$-singular element h; that is, h is singular modulo $\mathscr{S}^{\alpha}(G)$, which is absurd. We must conclude that G is an S^*-group, and Theorem 1 is proved.

A hyper-archimedean l-group is characterized by the condition that each prime subgroup be maximal [3]. Therefore, every totally ordered l-homomorphic image of a hyper-archimedean l-group is a subgroup of the additive reals, by Hölder's theorem. Now let us prove:

THEOREM 2. *Suppose G is hyper-archimedean; then it is hyper-$\mathscr{Z}$ if and only if every doubling chain for G is finite.*

Proof. Suppose G is hyper-$\mathscr{Z}$, yet $a_1 > a_2 > \cdots > a_n > \cdots$ is an infinite doubling chain. The a_i are contained in an ultrafilter of the positive cone of G, and thus a minimal prime subgroup P exists so that $a_n \notin P$ for each $n = 1, 2, \cdots$. (Recall that an *ultrafilter* is a subset U of strictly positive elements of an l-group H, maximal with respect to the property: $a, b \in U$ imply that $a \wedge b \in U$. For an account of the correspondence between ultrafilters and minimal prime subgroups we refer the reader to [1] or [2].)

Continuing then, $a_1 + P > a_2 + P > \cdots$ is an infinite descending chain for the archimedean o-group G/P; G/P can therefore not be cyclic, and we have a contradiction.

Conversely, suppose every doubling chain of G is finite; then G is an S^*-group by Theorem 1, and it is easy to verify from this that each totally ordered quotient of G is cyclic, since the class of S^*-groups is closed under l-homomorphic images; (see [7]).

This is enough to establish Theorem 2.

COROLLARY. *If G is hyper-archimedean, and A is an l-ideal of G so that A and G/A are both hyper-$\mathscr{Z}$, then G is hyper-$\mathscr{Z}$.*

The following example illustrates the use of hyper-archimedeaneity in Theorem 2 and the above corollary. Let G be the l-group of sequences of integers by the eventually constant sequences and $a = (1, 2, 3, \cdots)$. This example was discussed in [6], and it was shown there that G is not hyper-archimedean. Yet G is an extension of an S-group by Z, and all its doubling chains are finite.

Finally, we state a corollary which says something about the underlying group of a hyper-$\mathscr{Z}$ l-group.

COROLLARY. *If G is a hyper-$\mathscr{Z}$ l-group, then G is free, qua abelian group.*

Proof. As an S-group is a subgroup of bounded, integer-valued functions it is free abelian; this result goes back to Nöbeling [8], and it is further discussed by Hill in [4] and Conrad in [3]. If G is a hyper-$\mathscr{Z}$ l-group then it is an S^*group, say $G = \mathscr{S}^{\alpha}(G)$; we assume that $\mathscr{S}^{\tau}(G)$ is free abelian for each ordinal $\tau < \alpha$, and that a free basis X_τ for $\mathscr{S}^{\tau}(G)$ can be picked so that $X_\sigma = X_\tau \cap \mathscr{S}^{\sigma}(G)$, if $\sigma < \tau < \alpha$. If α is a limit ordinal, we let $X = \bigcup \{X_\tau \mid \tau < \alpha\}$; it is easy to verify that X is a free basis for $\mathscr{S}^{\alpha}(G)$. Otherwise, we have that $\mathscr{S}^{\alpha-1}(G)$ is free, and so is the S-group $\mathscr{S}^{\alpha}(G)/\mathscr{S}^{\alpha-1}(G)$; therefore $\mathscr{S}^{\alpha}(G)$ is the direct sum of $\mathscr{S}^{\alpha-1}(G)$ and $\mathscr{S}^{\alpha}(G)/\mathscr{S}^{\alpha-1}(G)$. Clearly then $\mathscr{S}^{\alpha}(G)$ is free and there is a free basis for it extending $X_{\alpha-1}$.

This proves the corollary; it should be noted that it is valid for any abelian S^*-group.

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