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CONTINUITY OF LINEAR MAPS FROM C^*-ALGEBRAS

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The celebrated problem of automatic continuity in Banach algebras—whether or not an arbitrary homomorphism from the algebra $C(X)$ of all complex continuous functions on a compact Hausdorff space X is continuous—remains unsolved.

The lack of success on this point has generated quite a bit of effort to determine 'the extent' to which a homomorphism is continuous. In the basic work of W. G. Bade and P. C. Curtis around 1960 it was shown that a homomorphism is necessarily continuous on some dense subalgebra of the algebra $C(X)$.

Many of these results have later been shown to carry over to a much larger class of mappings, namely the separable maps (cf. Definition 1.1 below).

Recently, A. M. Sinclair has taken a new look at the homomorphism problems and succeeded in extending much of Bade's and Curtis's work to general C^*-algebras. In this paper we employ some of Sinclair's methods and obtain extensions of his main results, notably we prove (Theorem 3.7) that a separable linear map defined on a C^*-algebra A is necessarily continuous on a dense subalgebra of A.

The class of mappings considered here was introduced in [4]; we employ the generalized notion from [6].

DEFINITION 1.1. Let A, B, X, Y be normed linear spaces, $q: A \times B \rightarrow X$ a continuous bilinear form. Let $T: X \rightarrow Y$ be linear; T is said to be *separable* (with respect to q) if there are functions $f: A \rightarrow \boldsymbol{R}^+$ and $g: B \rightarrow \boldsymbol{R}^+$ (where $\boldsymbol{R}^+$ is the positive reals) such that $\| T(q(a, b)) \| \leqq f(a)g(b)$ for all $a \in A$, $b \in B$.

As mentioned in [4] this class of mappings contains all algebra-homomorphisms, derivations, multipliers (centralizers); in fact, as an easy norm-estimate shows, if T is a continuous linear mapping, then T is separable with respect to any continuous bilinear $q: A \times B \rightarrow X$.

Since discontinuous derivations are known (cf. e.g. [2]), not every separable linear map is continuous. Nevertheless, as the following will show, certain results concerning the degree to which separable linear maps are continuous can be obtained.

The techniques employed here owe much to the work of Sinclair [5]; it is only appropriate to add to the list of examples of separable maps the module homomorphisms considered in [5]: with the notation of Definition 1.1, let A be a Banach algebra, $B = X$ a Banach left A-module, Y a Banach space with continuous module operations (i.e. Y is an A-module and for each $a \in A$, $y \rightarrow ay$ is a continuous operator) and $T: X \rightarrow Y$ a module homomorphism. If $q(a, x) = ax$ is the module multiplication on X we have $\| T(q(a, x)) \| = \| T(ax) \| = \| aT(x) \| \leqq C_a \| T(x) \|$ where C_a is the norm-bound of the operator $y \rightarrow ay: Y \rightarrow Y$.

We list now the basic technical facts

LEMMA 1.2 ([6 Lemma 1.1]). *If A, B are Banach spaces, X, Y normed spaces, $q: A \times B \rightarrow X$ is a continuous bilinear form and $T: X \rightarrow Y$ is a linear map separable with respect to q, then, with $\{a_n\} \subset A$, $\{b_k\} \subset B$ such that $(q(a_n, b_k) = 0$ whenever $n \neq k$, we have that*

$$\sup_n \| T(q(a_n, b_n)) \| / \| a_n \| \| b_n \| < \infty .$$

Proof. Similar to the proof of [4, Lemma 1.1]

COROLLARY 1.3 (cf. [5, Lemma 2.1 (b)]). *Suppose A is a Banach algebra, X a Banach left A-module and Y a normed linear space. If $T: X \rightarrow Y$ is a linear map separable with respect to module multiplication, if $\{a_n\}$, $\{b_k\} \subset A$ are sequences for which $a_n b_k = 0$ whenever $n \neq k$ then $T \circ a_n b_n: X \rightarrow Y$ is continuous for all but finitely many n and the set $\{\| T \circ a_n b_n \| / \| a_n \| \| b_n \| \mid T \circ a_n b_n$ is continuous$\}$ is bounded.*

Proof. If the conclusion fails then we may assume all the maps $T \circ a_n b_n$ to be discontinuous and hence we can find $\{x_n\} \subset X$ such that $\| x_n \| \leqq (\| a_n \| \| b^n \|)^{-1}$ but $\| T(a_n b_n x_n) \| \geqq n$ for $n = 1, 2, \cdots$. Letting $b_n x_n$ play the rôle of b_n in Lemma 1.2 we have a contradiction. Then rest of the corollary is then immediate.

2. Commutative regular algebras. Throughout this section A is assumed to be a regular semi-simple commutative unital Banach algebra, X is a Banach left A-module, Y a normed linear space and $T: X \rightarrow Y$ a linear map, separable with respect to the module multiplication.

We now introduce the first version of the continuity ideal of T and study its size in A. The following results are analogues and generalizations of those of [6, § 3]; the methods of proof that Stein employs are due to Bade and Curtis [1] and could be used here, too. Instead we have chosen the not dissimilar approach of [5, § 2].

DEFINITION 2.1. With the assumptions of the first paragraph of this section let the continuity ideal of T, I_T, be

$$I_T = \{a \in A \mid x \longrightarrow T(ax): X \longrightarrow Y \text{ continuous}\} .$$

It is clear that I_T is indeed an ideal.

As usual, if I is an ideal in a commutative Banach algebra B with maximal ideal space M_B, the *hull* of I is the set $\{\varphi \in M_B \mid \varphi(I) = \{0\}\}$. It F is a closed subset of M_B we let $J(F)$ denote the set

$$J(F) = \{x \in B \mid \varphi(x) = 0 \text{ for every } \varphi \text{ in a neighborhood of } F\} .$$

We assume known the fact that if B is regular and semisimple then $J(F)$ is minimal among all ideals in B with hull F.

The following generalizes [5, Theorem 2.2]

THEOREM 2.2. *Let A be a regular semisimple commutative unital Banach algebra, let X be a left Banach A-module and Y be a normed linear space. Suppose $T: X \rightarrow Y$ is separable with respect to module multiplication. Then $I_T = \{a \in A | x \rightarrow T(ax)$ is continuous$\}$ is an ideal in A with finite hull F; moreover $\exists$ constant C such that*

$$\| T \circ ab \| \leqq C \| a \| \| b \|$$

for all $a, b \in J(F)$.

Proof. Let F be the hull of I_T and suppose F is infinite; then we can extract from F a sequence $\{\varphi_n\}$ which does not converge to any of the points φ_n. We can therefore choose open sets $U_1, U_2, \cdots$ in M_A such that $\varphi_n \in U_n$, $n = 1, 2, \cdots$ and such that $U_j \cap U_n = \emptyset$ for all $j < n$ and $n = 1, 2, \cdots$. Now employ the regularity of A to find functions $\{a_n\}$ such that $\hat{a}_n$ vanishes off U_n and $\hat{a}_n(\varphi_n) = 1$. By choice of a_n we have $a_n^2 \notin I_T$, $n = 1, 2, \cdots$; also $a_n a_m = 0$ if $n \neq m$. Since $a_n^2 \notin I_T$, $T \circ a_n^2$ is discontinuous for each n. This contradicts Corollary 1.3.

To prove the boundedness of $f(a, b) = \| T \circ ab \| / \| a \| \| b \|$ on $J(F) \times J(F)$ we proceed as in [5, proof of Theorem 2.2] and suppose f unbounded on $J(F) \times J(F)$.

First we show that if $U \subset M_A$ is an open neighborhood of F then

$$K_U = \{f(a, b) \mid a, b \in J(F) \cap J(M_A \backslash U)\}$$

is unbounded. Suppose K_U is bounded by C. Since A is normal on M_A we can find $h \in A$ for which h is zero in a neighborhood of F and 1 in a neighborhood of $M_A \backslash U$. Since h is in $J(F)$ and thus in I_T, $T \circ h$ is continuous. Let $a, b \in J(F)$; then

$$\begin{aligned}\| T \circ ab \| &= \| T \circ (a - ha)(b - hb) + T \circ h(2 - h)ab \| \\ &\leqq \| T \circ (a - ha)(b - hb) \| + \| T \circ h(2 - h)ab \| ;\end{aligned}$$

since $a - hb$, $b - ha \in J(F) \cap J(M_A \backslash U)$ we have $\| T \circ (a - ha)(b - hb) \| \leqq C \| a - ha \| \| b - hb \|$; also

$$\| T \circ h(2 - h)ab \| \leqq \| ab \| \| T \circ h(2 - h) \| ,$$

since $h(2 - h) \in I_T$. Collecting these estimates we get

$$\| T \circ ab \| \leqq C \| 1 - h \|^2 \| a \| \| b \| + \| T \circ h(2 - h) \| \| a \| \| b \|$$

which contradicts the assumption that f be unbounded.

Thus K_U is an unbounded set for any open $U \supsetneqq F$. We use this to construct sequences that will contradict Corollary 1.3.

First choose a_1, b_1 in $J(F)$ such that $f(a_1, b_1) \geqq 1$ and let U_1 be an open proper neighborhood of $\{\varphi \in M_A \mid \varphi(a_1) = \varphi(b_1) = 0\}$. Assume next that $a_1, \cdots, a_n, b_1, \cdots, b_n$ have been found in $J(F)$ such that $f(a_j, b_j) \geqq j$ and $a_j, b_j \in J(M_A \backslash U_{j-1})$ where U_{j-1} is a proper open neighborhood of

$$\{\varphi \in M_A \mid \varphi(a_k) = \varphi(b_k) = 0,\ k = 1, \cdots, j - 1\}$$

for $j = 2, \cdots, n$. Since K_{U_n} is unbounded we can find $a_{n+1}, b_{n+1} \in J(F) \cap J(M_A \backslash U_n)$ for which $f(a_{n+1}, b_{n+1}) \geqq n + 1$. Since a_n vanishes where b_m is supported (if $n \neq m$) the semi-simplicity of A implies that $a_n b_m = 0$ whenever $n \neq m$. We have obtained a contradiction of Corollary 1.3.

The following is a bilinear variant of Theorem 2.2 which will be the basis of the work in the next section.

THEOREM 2.3. *Let A be a commutative regular semi-simple unital Banach algebra, X a Banach algebra and $\Pi: A \rightarrow X$ a homomorphism. Let Y be a normed linear space, $T: X \rightarrow Y$ a linear map, separable with respect to the algebra multiplication in X. Let*

$$I_T = \{a \in A \mid (b, c) \longrightarrow T(b\Pi(a)c): X \times X \longrightarrow Y \text{ is continuous}\}$$

be the continuity ideal. Then I_T has finite hull F and $\exists$ constant C such that the bilinear map

$$\theta_{a,b}: X \times X \longrightarrow Y: (x_1, x_2) \longrightarrow T(x_1 \Pi(ab) x_2)$$

satisfies $\| \theta_{a,b} \| \leqq C \| \Pi(a) \| \| \Pi(b) \|$ for all $a, b \in J(F)$.

Proof. Except for the changes necessitated by the linearity being replaced by bilinearity the proof proceeds exactly as that of the preceding Theorem, so we omit it.

THEOREM 2.4. *In addition to the assumptions of Theoren 2.2 suppose $J(F)$ has a bounded approximate identity. Then T is continuous on the subspace $J(F)X$ of X.*

Proof. Since $J(F)X = \text{span}\,\{ax \mid a \in J(F),\, x \in X\}$ we see that if $z \in J(F)X$ then there is a $b \in J(F)$ such that $z = bz$. Suppose $\{e_\alpha\}$ is an approximate identity for $J(F)$ with bound M and let C be the bound of Theorem 2.2. Since $\{e_\alpha^2\}$ is also an approximate identity we may choose $e \in \{e_\alpha\}$ such that $\|b - e^2 b\| \leqq 1/\|b\|$. Then

$$\begin{aligned}\|Tz\| = \|Tb^2 z\| &\leqq \|T(b - e^2 b)bz\| + \|Te^2 z\| \\ &\leqq C\|b - e^2 b\|\,\|b\|\,\|z\| + C\|e\|^2\|z\| \leqq C(1 + M^2)\|z\| .\end{aligned}$$

REMARK. Recall that a commutative unital Banach algebra A is said to satisfy Ditkin's condition if $J(\{\varphi\})$ contains a bounded approximate identity for each $\varphi \in M_A$. Then any ideal $J(F)$ for which F is finite will contain a bounded approximate identity. Thus, the above result is a generalization of [6, Proposition 3.2] and is also an analogue of part of [5, Theorem 2.3].

The technique of the proof of Theorem 2.4 yields an analogous bilinear result, based on Theorem 2.3.

THEOREM 2.5. *With the assumptions of Theorem 2.3 and the additional assumptions that $\Pi: A \to X$ be continuous and $J(F) \subset A$ have a bounded approximate identity the bilinear map*

$$X \times J(F)X \longrightarrow Y: (x, t) \longrightarrow T(xt)$$

is continuous.

Proof. As in the proof of Theorem 2.4, if $t \in J(F)X$, $t = \Sigma\Pi(a_i)x_i$, then $\exists b \in J(F)$ such that $t = \Pi(b)t$; also we can find $e \in J(F)$ with $\|\Pi(e)\| \leqq \|\Pi\|M$ such that $\|\Pi(b - c^2 b)\| \leqq 1/\|\Pi(b)\|$. If $x \in X$ then $\|T(xt)\| \leqq \|T(x\Pi(b - e^2 b)\Pi(b)t)\| + \|T(x\Pi(e)^2 t)\| \leqq C\|\Pi(b - e^2 b)\| \times \|\Pi(b)\|\,\|x\|\,\|t\| + C\|\Pi\|^2 M^2\|x\|\,\|t\| \leqq C(1 + \|\Pi\|^2 M^2)\|x\|\,\|t\|$.

3. C^*-algebras. In this section we restrict attention to C^*-algebras; we let A be a unital C^*-algebra, B a normed linear space and $T: A \to B$ a linear map, separable with respect to the algebra multiplication on A. The main result is Theorem 3.7 which states that T is continuous on a dense subalgebra of A thus establishing the general version of [4, Theorem 5.6]. As in the previous sections

the techniques are mainly those of [5].

PROPOSITION 3.1. *Let C be a unital commutative C^*-subalgebra of A. Then there is a closed cofinite ideal* $\ker F$ *in C such that*

$$\theta_1 : A \ker F \times I_T \longrightarrow B$$

and

$$\theta_2 : I_T \times \ker FA \longrightarrow B$$

defined by $\theta_1(a, b) = T(ab)$, $\theta_2(b, a) = T(ba)$, are continuous bilinear maps. Here

$$I_T = \{b \in A \mid (a, c) \longrightarrow T(abc) : A \times A \longrightarrow B \text{ is continuous}\} .$$

Proof. Let F_1 be the singularity set of T as in Theorem 2.3, let $a \in A \ \ker F_1$ and let $b \in I_T$. Let e denote the unit of A and choose $\varepsilon > 0$. By the Cohen factorization theorem applied to $A \ker F_1$ there is $c_1 \in A \ \ker F_1$ and $c_2 \in \ker F_1$ such that $a = c_1c_2$ with $\|c_1\| \leqq 2\|a\|$, $\|c_2\| = 1$. Choose c_3, c_4 and c_5 in $J(F)$ such that $c_3 = c_4c_5$, such that $\|c_2 - c_3\| < \varepsilon_1$ (ε_1 to be specified) and such that $\|c_4\|\|c_5\| \leqq 2\|c_3\|$. This is yet another application of Cohen's factorization theorem. By the analogous version of Theorem 2.5 the bilinear map $(x, z) \rightarrow T(xz) : AJ(F_1) \times A \rightarrow B$ is continuous and has a norm bound which does not exceed $2C_1$, where C_1 is the constant of Theorem 2.3. Let C_b denote the norm bound of the continuous bilinear operator $(x_1, x_2) \rightarrow T(x_1bx_2)$; and note that $\|c_2 - c_3\| < \varepsilon_1$, so $\|c_3\| < 1 + \varepsilon_1$. We then have

$$\begin{aligned}
\|T(ab)\| &= \|T(c_1c_2b)\| \\
&\leqq \|T(c_1(c_2 - c_3)be)\| + \|T(c_1c_3b)\| \\
&\leqq \|c_1\|\|c_2 - c_3\| C_b \|e\| + 2C_1\|c_1\|\|c_3\|\|b\| \\
&\leqq \varepsilon_1\|c_1\| C_b + 2C_1 \cdot 2\|a\|(1 + \varepsilon_1)\|b\| \\
&= \varepsilon_1(2C_b + 4C_1\|b\|)\|a\| + 4C_1\|a\|\|b\| .
\end{aligned}$$

If

$$\varepsilon_1 < \frac{\varepsilon}{(2C_b + 4C_1\|b\|)\|a\|}$$

then

$$\|T(ab)\| \leqq 4C_1\|a\|\|b\| + \varepsilon$$

which proves that θ_1 is continuous on $A \ \ker F_1 \times I_T$. Similarly we get a finite set F_2 such that θ_2 is continuous on $I_T \times \ker F_2A$. If we take $F = F_1 \cup F_2$ the proposition follows.

COROLLARY 3.2. *The trilinear maps*

$$\Psi_1: A \times \ker F \times I_T \longrightarrow B$$
$$\Psi_2: I_T \times \ker F \times A \longrightarrow B$$

defined by $\Psi_1(a, b, c) = T(abc)$, $\Psi_2(c, b, a) = T(cba)$ *are continuous.*

On the basis of the above corollary we make the following definition.

DEFINITION 3.3.

$$K_T = \{a \in A \mid (c, d) \longrightarrow T(cad): A \times I_T \longrightarrow B \text{ and } (d, c) \longrightarrow T(dac): I_T \times A \longrightarrow B \text{ are continuous}\}.$$

About this set we have the following

PROPOSITION 3.4. *K_T is a closed cofinite two-sided ideal that contains I_T.*

Proof. It is immediate that K_T is a two-sided ideal that contains I_T. By [5, Lemma 3.2], to show that K_T is closed and cofinite, it is sufficient to show that $K_T \cap C$ is closed and cofinite for any commutative unital C^*-subalgebra of C. Corollary 3.2 shows that the hull of $K_T \cap C$ is finite. If $\ker F \subsetneqq K_T \cap C$ let $c_1, c_2, \cdots, c_n$ be chosen in $K_T \cap C$ such that $c_i c_j = 0$ if $i \neq j$ and such that $c_i = 1$ in a neighborhood of φ_i where

$$\{\varphi_1, \cdots, \varphi_n\} = F \backslash \text{hull } (K_T \cap C)$$

and such that $c_i(\varphi_j) = 0$ if $j \neq i$.

If $a \in K_T \cap C$ then $a - \Sigma a(\varphi_i)\, c_i \in \ker F$ and since $\ker F$ is closed it follows from the definition of K_T that $K_T \cap C$ is closed: suppose $\{a_m\} \subset K_T \cap C$ converges to a; then $a_m - \Sigma a_m(\varphi_i)c_i \rightarrow a - \Sigma a(\varphi_i)c_i$, so if $c \in X$ and $d \in I_T$ then

$$T(c(a - \Sigma a(\varphi_i)c_i)d) = T(cad) - \Sigma a(\varphi_i)T(cc_i d)$$

which shows that (c, d) $\rightarrow T(cad)$ is continuous, i.e. that $a \in K_T \cap C$.

COROLLARY 3.5. *The bilinear maps*

$$(a, b) \longrightarrow T(ab): K_T \times I_T \longrightarrow B$$

and

$$(b, a) \longrightarrow T(ba): I_T \times K_T \longrightarrow B$$

are continuous.

The next result corresponds to [5, Theorem 3.8] and is a generalization of [4, Theorem 5.1].

THEOREM 3.6. *In addition to the above assumptions suppose B to be a Banach space. Then T is continuous on the two sided ideal $K_T I_T$ which is dense in I_T.*

Proof. Let Ψ be the extension by continuity to $K_T \times I_T^-$ of the mapping $(a, b) \rightarrow T(ab)$ mentioned in Corollary 3.5. We may also consider Ψ a continuous linear operator from the projective tensor product $K_T \otimes_\gamma I_T^-$ into B. Let $\theta: K_T \otimes_\gamma I_T^- \rightarrow I_T^-$ be defined by $\theta(\Sigma x_i \otimes y_i) = \Sigma x_i y_i$. By Cohen's factorization theorem θ is surjective so by the open mapping theorem $\theta^{-1}: I_T^- \rightarrow K_T \otimes_\gamma I_T^- / \ker(\theta)$ is continuous.

As we shall see below, $\ker \theta \subseteqq \ker \Psi$ so the composition $\Phi = \Psi \circ \theta^{-1}: I_T^- \rightarrow B$ is well defined and continuous. On $K_T I_T \subseteqq I_T^-$ we have $\Phi(ab) = \Psi(a, b) = T(ab)$ and consequently T is continuous on $K_T I_T$.

It remains to see that $\ker \theta \subseteqq \ker \Psi$.

To that end we can refer to Sinclair's proof in [5, Theorem 3.8] in which a similar claim is established.

The density of $K_T I_T$ in I_T is clear from the fact that $K_T \supseteqq I_T$. That completes the proof.

We are now in a position to state and prove the main result of this section (cf. the introduction to this section).

THEOREM 3.7. *Let A be a C^*-algebra, B a Banach space, T a linear map of A into B, separable with respect to algebra multiplication. Then T is continuous on a dense subalgebra of A.*

Proof. By Theorem 3.6 T is continuous on the two-sided ideal $K_T I_T$ which has cofinite closure [4, p. 501]. By [3, Theorem 6] there is a finite dimensional subspace F of A such that $K_T I_T + F$ is a dense subalgebra on which T is clearly continuous. That proves the theorem.

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