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AN OPERATOR VERSION OF A THEOREM OF KOLMOGOROV[1]

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Let $\mathscr{G}$ be a (separable) Hausdorff space and let K be a continuous nonnegative-definite kernel (covariance) from $\mathscr{G} \times \mathscr{G}$ to C. The well known theorem of Kolmogorov states that in the case $\mathscr{G}$ is the set of integers there is a continuous mapping (stochastic process) $x(\cdot)$ from $\mathscr{G}$ into a (separable) Hilbert space $\mathscr{K}$ such that $K(s, t) = (x(s), x(t))$. The theorem is also known for any separable Hausdorff space. The purpose of this paper is to replace the complex numbers C by the algebra $B(\mathscr{H}, \mathscr{H})$ of bounded linear operators from a Hilbert space into itself. The factorization is then $K(t, s) = X(t)^*X(s)$ with X a continuous map from $\mathscr{G}$ to $B(\mathscr{H}, \mathscr{K})$ for a suitable Hilbert space $\mathscr{K}$. If $\mathscr{G}$ is separable we may take $\mathscr{K} = \mathscr{H}$.

Two proofs of this theorem are given. The first, for $\mathscr{G}$ separable and $\mathscr{H}$ of arbitrary dimension, uses an extension of the technique of [1] to obtain a triangular factorization for nonnegative-definite matrices with operator entries to construct the desired stochastic process $X(\cdot)$. The second, for $\mathscr{G}$ arbitrary and $\mathscr{H}$ of infinite dimension uses the techniques of reproducing kernel Hilbert spaces, and is a bit simpler.

Main results. Let $\mathscr{H}$ be a complex Hilbert space and let $B(\mathscr{H}, \mathscr{H})$ be the bounded linear operators on. Let $\mathscr{G}$ be a Hausdorff space and let $K: \mathscr{G} \times \mathscr{G} \to B(\mathscr{H}, \mathscr{H})$ be a (jointly) continuous function. We say that K is *nonnegative-definite* if for every $t_1, \cdots, t_n \in \mathscr{G}$ and $x_1, \cdots, x_n \in \mathscr{H}$ the sum

$$(1) \qquad \sum_{i,j=1}^{n} (K(t_i, t_j)x_j, x_i) \geqq 0 .$$

The generalization of the Kolmogorov theorem we wish to prove is contained in

THEOREM 1. *Let $\mathscr{G}$ be a separable Hausdorff space. If $K(\cdot, \cdot)$ is a continuous nonnegative-definite function from $\mathscr{G} \times \mathscr{G}$ into $B(\mathscr{H}, \mathscr{H})$ then there exists a separable Hilbert space $\mathscr{K}$ and a continuous function $X(t)$ from $\mathscr{G}$ into $B(\mathscr{H}, \mathscr{K})$ such that*

$$X^*(t)X(s) = K(t, s) .$$

[1] This generalization was suggested to the authors by Professor P. Masani in January 1975.

In order to prove this theorem we require a number of facts about operator-valued matrices and about the solution of operator equations. The first result, is due to Douglas [2]. (See also Fillmore-Williams [4].) We will denote the *range* of the operator A by $\mathscr{R}(A)$, and the kernel of A by $\mathscr{N}(A)$.

LEMMA 1. *Let A and B be bounded operators on $\mathscr{H}$. Then the following conditions are equivalent*:

(i) $\mathscr{R}(A) \subset \mathscr{R}(B)$,

(ii) $A = BC$, *for some bounded operator C on $\mathscr{H}$*,

(iii) $AA^* \leqq \lambda^2 BB^*$, *for some $\lambda > 0$*.

Moreover, the operator C can be chosen so that $\mathscr{N}(C^*) \supset \mathscr{N}(B)$ and $\mathscr{R}(C) \subset \overline{\mathscr{R}(B)}$.

COROLLARY. *If B is bounded and nonnegative then $\mathscr{R}(\sqrt{B}) \supset \mathscr{R}(B)$*.

If we restrict K, of Theorem 1, to a finite subset of $\mathscr{G}$ the kernel K becomes a $n \times n$ matrix whose (i, j) entry is $K_{ij} = K(t_i, t_j)$, $1 \leqq i, j \leqq n$. This matrix is nonnegative-definite in the sense that for every $x_1, x_2, \cdots, x_n \in \mathscr{H}$,

$$\sum_{i,j=1}^{n} (K_{ij}x_j, x_i) \geqq 0 . \tag{2}$$

Denote by $\mathscr{H}_n$ the space which is a direct sum of n copies of $\mathscr{H}$, $\mathscr{H}_n = \mathscr{H} \oplus \cdots \oplus \mathscr{H}$, with the natural inner product. Suppose that K is an operator on $\mathscr{H}_n$; that is, K is an $n \times n$ operator-valued matrix. Then (2) means that $(Kx, x) \geqq 0$ for every $x = (x_1, \cdots, x_n) \in \mathscr{H}_n$, that is K is a nonnegative operator on $\mathscr{H}_n$. Note that if K is nonnegative-definite, $K_{ij} = K_{ji}^*$, for all $1 \leqq i, j \leqq n$. If K is an $n \times n$ operator-valued matrix and $m \leqq n$, we write K_m for the upper left $m \times m$ submatrix of K.

LEMMA 2. *Let K be an $n \times n$ nonnegative definite, bounded operator-valued mat1 Then there is a positive constant λ so that*

$$K_{ii} \geqq \lambda K_{ij}K_{ij}^*, 1 \leqq i < j \leqq n . \tag{3}$$

Proof. Let $V_i: \mathscr{H} \to \mathscr{H}_n$ where $V_ih = (0, \cdots, h, 0 \cdots 0)$, h being in the ith position. If $h \in \mathscr{H}$, then

$$\begin{aligned} |K_{ij}^*h|^2 &= |K_{ji}h|^2 = |V_j^*KV_ih|^2 \leqq |KV_ih|^2 \\ &= (V_i^*K^2V_ih, h) \leqq |K|(V_i^*KV_ih, h) = |K|(K_{ii}h, h) . \end{aligned} \tag{4}$$

Thus $K_{ij}K_{ij}^* \leqq |K|K_{ii}$.

We must show that

$$T_{n-1,n-1}T_{n-1,n} = K_{n-1,n} - \sum_{i=1}^{n-2} T_{i,n-1}^{*}T_{i,n} \tag{10}$$

has a bounded solution for $T_{n-1,n}$. By the Remark we have for any $z_{n-1} \in \mathscr{H}$ a vector $z_{n-2} \in \mathscr{H}$ such that

$$T_{n-2,n-2}z_{n-2} + T_{n-2,n-1}z_{n-1} = 0 .$$

Thus, proceeding sequentially we can solve the equations

$$\sum_{j=i}^{n-1} T_{ij}z_j = 0 , \qquad i = n-2, n-3, \cdots, 1 \tag{11}$$

for $z_{n-2}, z_{n-3}, z_{n-4}, \cdots, z_1$, given z_{n-1}. Now, if $z = (z_1, \cdots, z_n)$, an application of (11) gives

$$\begin{aligned}(Kz, z) = &\sum_{j=1}^{n-2}(K_{nj}z_j, z_n) + \sum_{j=1}^{n-2}(K_{jn}z_n, z_j) + (K_{n-1,n}z_n, z_{n-1}) \\ &+ (K_{n,n-1}z_{n-1}, z_n) + (K_{nn}z_n, z_n) + (T_{n-1,n-1}^2 z_{n-1}, z_{n-1}) .\end{aligned} \tag{12}$$

By (9),

$$\begin{aligned}\sum_{j=1}^{n-2}(K_{jn}z_n, z_j) &= \sum_{j=1}^{n-2}((T_{jj}T_{jn} + \sum_{i=1}^{j-1} T_{ij}^{*}T_{in})z_n, z_j) \\ &= \sum_{j=1}^{n-2}(T_{jn}z_n, T_{jj}z_j) + \sum_{j=1}^{n-2}\sum_{i=1}^{j-1}(T_{ij}^{*}T_{in}z_n, z_j) .\end{aligned} \tag{13}$$

We interpret all sums over not well defined limits to be zero (e.g. $\sum_{i=1}^{0}(\cdot) = 0$). From (11) we have

$$T_{jj}z_j = -\sum_{i=j+1}^{n-1} T_{ji}z_i .$$

Substitution into (13) gives

$$\begin{aligned}\sum_{j=1}^{n-2}(K_{jn}z_n, z_j) &= -\sum_{j=1}^{n-2}\sum_{i=j+1}^{n-1}(T_{jn}z_n, T_{ji}z_i) + \sum_{j=1}^{n-2}\sum_{i=1}^{j-1}(T_{in}z_n, T_{ij}z_j) \\ &= -\sum_{j=1}^{n-3}\sum_{i=j+1}^{n-2}(T_{jn}z_n, T_{ji}z_i) + \sum_{j=1}^{n-2}\sum_{i=1}^{j-1}(T_{in}z_n, T_{ij}z_j) \\ &\quad - \sum_{j=1}^{n-2}(T_{jn}z_n, T_{j,n-1}z_{n-1}) .\end{aligned} \tag{14}$$

The last term of (14) is

$$-\sum_{j=1}^{n-2}(T_{j,n-1}^{*}T_{jn}z_n, z_{n-1}) .$$

Interchanging limits in the second term on the right hand side of (14), the equation (14) becomes we will employ Lemma 1 (iii). (T_{13} is obtained in the same way as T_{12}.) According to the Remark above we take $T_{11}z_1 = -T_{12}z_2$, for some $z_2 \in \mathscr{H}$. If $z = (z_1, z_2, z_3)$, then

$$\begin{aligned}0 \leqq (K_3 z, z) &= (T_{11}^2 z_1, z_1) + (T_{11}T_{12}z_2, z_1) + (T_{12}^* T_{11} z_1, z_2)\\ &\quad + ((T_{22}^2 + T_{12}^* T_{12})z_2, z_2) + (T_{11}T_{13}z_3, z_1)\\ &\quad + (K_{23}z_3, z_2) + (K_{32}z_2, z_3) + (K_{33}, z_3, z_3)\ ,\end{aligned}$$

which, since $T_{11}z_1 = -T_{12}z_2$ equals

$$(T_{22}^2 z_2, z_2) + ((K_{23} - T_{12}^* T_{13})z_3, z_2) + ((K_{32} - T_{13}^* T_{12})z_2, z_3) + (K_{33}z_3, z_3)\ .$$

In matrix form this means, for every $z_2, z_3 \in \mathscr{H}$,

$$\left(\begin{pmatrix} T_{22}^2 & K_{23} - T_{12}^* T_{13} \\ (K_{23} - T_{12}^* T_{13})^* & K_{33}\end{pmatrix}\begin{pmatrix} z_2 \\ z_3\end{pmatrix},\ \begin{pmatrix} z_2 \\ z_3\end{pmatrix}\right) \geqq 0\ .$$

By Lemma 2, then, there is a positive λ such that

$$T_{22}^2 \geqq \lambda(K_{23} - T_{12}^* T_{13})(K_{23} - T_{12}^* T_{13})^*\ ,$$

and hence by the Corollary and Lemma 2 (ii) T_{23} exists and is a bounded operator. Moreover, by Lemma 1 $\mathscr{R}(T_{23}) \subset \overline{\mathscr{R}(T_{22})}$. (This last fact together with the Remark is interpreted to mean that for any $y \in \mathscr{H}$ there is an $x \in \mathscr{H}$ so that $T_{22}x + T_{23}y = 0$.)

To show that T_{33} exists is now routine. Let $z = (z_1, z_2, z_3)$. Then

$$\begin{aligned}((K_{33} - T_{13}^* T_{13} - T_{23}^* T_{23})z_3, z_3) &= (Kz, z) - |T_{22}z_2 + T_{23}z_3|^2\\ &\quad - |T_{11}z_1 + T_{12}z_2 + T_{13}z_3|^2\\ &\geqq -|T_{22}z_2 + T_{23}z_3|^2 - |T_{11}z_1 + T_{12}z_2 + T_{13}z_3|^2\ .\end{aligned}$$

This inequality, combined with the Remark above gives the nonnegativity of $K_3 - T_{13}^* T_{13} - T_{23}^* T_{23}$ and hence the existence of and boundedness of T_{33}.

We pass to the induction. Assume that $T_k^* T_k = K_k$, $k = 1, 2, \cdots, n-1$. Solve for T_{1n} in the same way as for T_{12}. Proceeding, once again, by induction we assume that the T_{kn} exist and are bounded for $k = 2, 3, \cdots, n-2$, and also that $\mathscr{R}(T_{kn}) \subset \overline{\mathscr{R}(T_{kk})}$, which makes the Remark applicable. The formula for the T_{km} are given by

$$T_{kk}T_{km} = K_{km} - \sum_{i=1}^{k-1} T_{ik}^* T_{im},\ k \leqq m\ ,$$

or

$$K_{km} = \sum_{i=1}^{k} T_{ik}^* T_{im}\ . \tag{9}$$

We must show that

$$T_{n-1,n-1}T_{n-1,n} = K_{n-1,n} - \sum_{i=1}^{n-2} T^*_{i,n-1}T_{i,n} \tag{10}$$

has a bounded solution for $T_{n-1,n}$. By the Remark we have for any $z_{n-1} \in \mathscr{H}$ a vector $z_{n-2} \in \mathscr{H}$ such that

$$T_{n-2,n-2}z_{n-2} + T_{n-2,n-1}z_{n-1} = 0 \, .$$

Thus, proceeding sequentially we can solve the equations

$$\sum_{j=i}^{n-1} T_{ij}z_j = 0 \, , \qquad i = n-2, n-3, \cdots, 1 \tag{11}$$

for $z_{n-2}, z_{n-3}, z_{n-4}, \cdots, z_1$, given z_{n-1}. Now, if $z = (z_1, \cdots, z_n)$, an application of (11) gives

$$\begin{aligned}(Kz, z) = \sum_{j=1}^{n-2} (K_{nj}z_j, z_n) + \sum_{j=1}^{n-2} (K_{jn}z_n, z_j) + (K_{n-1,n}z_n, z_{n-1}) \\ + (K_{n,n-1}z_{n-1}, z_n) + (K_{nn}z_n, z_n) + (T^2_{n-1,n-1}z_{n-1}, z_{n-1}) \, .\end{aligned} \tag{12}$$

By (9),

$$\begin{aligned}\sum_{j=1}^{n-2} (K_{jn}z_n, z_j) &= \sum_{j=1}^{n-2} ((T_{jj}T_{jn} + \sum_{i=1}^{j-1} T^*_{ij}T_{in})z_n, z_j) \\ &= \sum_{j=1}^{n-2} (T_{jn}z_n, T_{jj}z_j) + \sum_{j=1}^{n-2}\sum_{i=1}^{j-1} (T^*_{ij}T_{in}z_n, z_j) \, .\end{aligned} \tag{13}$$

We interpret all sums over not well defined limits to be zero (e.g. $\sum_{i=1}^{0} (\cdot) = 0$). From (11) we have

$$T_{jj}z_j = - \sum_{i=j+1}^{n-1} T_{ji}z_i \, .$$

Substitution into (13) gives

$$\begin{aligned}\sum_{j=1}^{n-2} (K_{jn}z_n, z_j) &= - \sum_{j=1}^{n-2}\sum_{i=j+1}^{n-1} (T_{jn}z_n, T_{ji}z_i) + \sum_{j=1}^{n-2}\sum_{i=1}^{j-1} (T_{in}z_n, T_{ij}z_j) \\ &= - \sum_{j=1}^{n-3}\sum_{i=j+1}^{n-2} (T_{jn}z_n, T_{ji}z_i) + \sum_{j=1}^{n-2}\sum_{i=1}^{j-1} (T_{in}z_n, T_{ij}z_j) \\ &\quad - \sum_{j=1}^{n-2} (T_{jn}z_n, T_{j,n-1}z_{n-1}) \, .\end{aligned} \tag{14}$$

The last term of (14) is

$$- \sum_{j=1}^{n-2} (T^*_{j,n-1}T_{jn}z_n, z_{n-1}) \, .$$

Interchanging limits in the second term on the right hand side of (14), the equation (14) becomes

$$- \sum_{j=1}^{n-3} \sum_{i=j+1}^{n-2} (T_{jn}z_n, T_{ji}z_i) + \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} (T_{in}z_n, T_{ij}z_j)$$
$$- \sum_{j=1}^{n-2} (T^*_{j,n-1}T_{jn}z_n, z_{n-1}) .$$

Upon interchanging i and j we obtain

$$\sum_{j=1}^{n-2} (K_{jn}z_n, z_j) = - \sum_{j=1}^{n-2} (T^*_{j,n-1}T_{jn}z_n, z_{n-1}) . \tag{15}$$

Similarly

$$\sum_{j=1}^{n-2} (K_{nj}z_j, z_n) = -\left(\sum_{j=1}^{n-2} T^*_{jn}T_{j,n-1}z_{n-1}, z_n \right) . \tag{16}$$

Substituting (15) and (16) into (12) and writing the result in matrix form we have

$$0 \leqq (K_n x, x)$$
$$= \left(\begin{pmatrix} T^2_{n-1,n-1} & K_{n-1,n} - \sum_{j=1}^{n-2} T^*_{j,n-1}T_{jn} \\ \left(K_{n-1,n} - \sum_{j=1}^{n-2} T^*_{j,n-1}T_{jn}\right)^* & K_{nn} \end{pmatrix} \begin{pmatrix} z_{n-1} \\ z_n \end{pmatrix}, \begin{pmatrix} z_{n-1} \\ z_n \end{pmatrix} \right)$$

An application of Lemmas 1 and 2 and the Corollary (ii), (iii) gives that there is a bounded operator $T_{n-1,n}$ satisfying (10) and moreover that $\mathscr{R}(T_{n-1,n}) \subseteqq \overline{\mathscr{R}(T_{n-1,n-1})}$.

To show that T_{nn} exists a similar argument is used. This completes the induction and the Lemma is proved.

Lemma 3 works in any Hilbert space $\mathscr{H}$, finite or infinite dimensional. The following result, a considerable improvement of Lemma 3, applies only to infinite dimensional Hilbert spaces.

LEMMA 3′. (a) *Suppose* $\dim \mathscr{H} = \infty$ *of* K *is a nonegative* $n \times n$ *matrix with entries* $K_{ij} \in B(\mathscr{H}, \mathscr{H})$ *then there exist* $X_1, X_2, \cdots, X_n$ *in* $B(\mathscr{H}, \mathscr{H})$ *such that* $K_{ij} = X_i^* X_j (1 \leqq i, j \leqq n)$. *Hence* $K = X^*X$ *where* X *is the* $n \times n$ *matrix whose first row is* $(X_1 X_2 \cdots X_n)$ *and whose other entries are all* 0.

(b) *If* A *is an* $n \times n$ *matrix with entries* $A_{ij} \in B(\mathscr{H}, \mathscr{H})$ *then there exists a partial isometry* $U = (U_{ij})$ *in* $B(\mathscr{H}_n, \mathscr{H}_n)$ *and a matrix* X *as in* (a) *such that* $A = UX$, $X = U^*A$.

(c) *If* $A \geqq 0$ *then* U *may be chosen to be an isometry in* (b).

Proof. (a) Let V_i be the isometry from $\mathscr{H}$ into $\mathscr{H}_n$ given by $V_i h = (0, 0, \cdots, 0, h, 0, \cdots)$ where the vector h appears as the ith coordinate. If h, k belong to $\mathscr{H}$ then $(K_{ij}h, k) = (KV_j h, V_i k) = (\sqrt{K}V_j h, \sqrt{K}V_i k)$. Hence $K_{ij} = (\sqrt{K}V_i)^*(\sqrt{K}V_j)$. Let Φ be an isometry from $\mathscr{H}$ onto $\mathscr{H}_n$. Then $X_i = \Phi^*\sqrt{K}V_i \in B(\mathscr{H}, \mathscr{H})$ and $X_i^* X_j = K_{ij}$.

(b), (c) Choose X as in (a) so that $A^*A = X^*X$. Then

$$V\sqrt{A^*A}f = Xf$$

defines an isometry V from $\mathscr{R}(\sqrt{A^*A})^-$ onto $\mathscr{R}(X)^-$. Since $\mathscr{R}(X)^\perp = \Phi^*(\mathscr{N}(\sqrt{A^*A})) \oplus \mathscr{H} \oplus \mathscr{H} \oplus \cdots \oplus \mathscr{H}$ it is clear V can be extended to an isometry on $\mathscr{H}_n$. This proves (c) and to complete the proof of (b) use the polar factorization $A = W\sqrt{A^*A}$ and put $U = WV^*$.

REMARK. Lemma 3′ is also valid for infinite matrices K(or A) that define bounded operators on the direct sum of countably many copies of $\mathscr{H}$.

Proof of Theorem 1. Define the Hilbert space $\mathscr{K} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \cdots$ where $\mathscr{H}_i = \mathscr{H}$, $i = 1, 2, \cdots$, with the natural inner product. Let $V_i (i \geqq 1)$ be the isometry from $\mathscr{H}$ into $\mathscr{K}$ given by $V_i h = (h_1, h_2, \cdots)$ where $h_i = h$ and $h_j = 0$ for $j \neq i$. Let $\mathscr{R} = \{t_i : i = 1, 2, \cdots\}$ be a dense set of points in $\mathscr{G}$. Define the non negative-definite, bounded operator-valued matrices

$$K^{(n)} = K(t_i, t_j)\ , \qquad i, j = 1, \cdots, n\ .$$

By Lemma 3 there is an upper triangular operator-valued matrix $T^{(n)}$ for which $T^{(n)*}T^{(n)} = K^{(n)}$ and moreover from the construction, if $m \leqq n$ then $K^{(m)} = K_m^{(n)} = (T_m^{(n)})^*(T_m^{(n)})$. Let T be the formal infinite upper triangular matrix whose n^{th} column is the n^{th} column of $T^{(n)}$, $n = 1, 2, \cdots$. For each $t_l \in \mathscr{R}$ define

$$\tilde{X}(t_l) = \sum_{i=1}^{l} V_i T_{il}\ .$$

Then, if $m = \min(k, l)$,

$$\begin{aligned}\tilde{X}(t_k)^*\tilde{X}(t_l) &= \Big(\sum_{j=1}^{k} V_j T_{jk}\Big)^*\Big(\sum_{i=1}^{l} V_i T_{il}\Big) \\ &= \sum_{j=1}^{k}\sum_{i=1}^{l} T_{jk}^* V_j^* V_i T_{il} \\ &= \sum_{i=1}^{m} T_{ik}^* T_{il} = K(t_k, t_l)\ .\end{aligned}$$

From this it follows that

$$|\tilde{X}(t) - \tilde{X}(s)| \leqq |K(t, t) - K(s, t)| + |K(s, s) - K(t, s)|\ ,$$

for any t, s in $\mathscr{R}$. Using the completeness of $B(\mathscr{H}, \mathscr{K})$ and the continuity of K we can therefore extend $\tilde{X}$ to a function X from $\mathscr{G}$ into $B(\mathscr{H}, \mathscr{K})$ that satisfies the same inequalities for all t, s in

$\mathscr{G}$. The function X is then continuous and $X(t)^*X(s) = K(t, s)$.

In the following theorem the condition of separability is removed from $\mathscr{G}$. However, $\mathscr{K}$ will be a nonseparable Hilbert space. The construction below seems to have originated with Naimark [5].

THEOREM 2. *Let $\mathscr{G}$ be a Hausdorff space, and let $K(\cdot, \cdot)$ be as in Theorem 1. Then there is a Hilbert space $\mathscr{K}$ and a continuous function $X(t)$ from $\mathscr{G}$ into $B(\mathscr{H}, \mathscr{K})$ such that $X^*(t)X(s) = K(t, s)$.*

Proof. Let $\mathscr{L}$ be the vector space of functions $\xi: \mathscr{G} \rightarrow \mathscr{H}$ that vanish at all but a finite number of points of $\mathscr{G}$, and for ξ, η in $\mathscr{L}$ put

$$(\xi, \eta) = \sum_{s,t} (K(s, t)\xi(t), \eta(s)) .$$

Let $\mathscr{N} = \{\xi \in \mathscr{L}: (\xi, \xi) = 0\}$. Then $\mathscr{N}$ is a subspace of $\mathscr{L}$ and

$$(\xi + \mathscr{N}, \xi + \mathscr{N}) = (\xi, \eta)$$

defines an inner product on $\mathscr{K}_0 = \mathscr{L}/\mathscr{N}$. Let $\mathscr{K}$ be the completion of $\mathscr{K}_0$. For $s \in \mathscr{G}$ and $h \in \mathscr{H}$ define

$$\xi_s h(t) = \begin{cases} h & \text{if} \quad t = s \\ 0 & \text{if} \quad t \neq s . \end{cases}$$

Then $X(s)h = \xi_s h + \mathscr{N}$ defines a bounded operator $X(s)$ from $\mathscr{H}$ into $\mathscr{K}$. A simple computation shows that $X(t)^*X(s) = K(t, s)$. This implies $|X(t) - X(s)|^2 \leqq |K(t, t) - K(t, s)| + |K(s, s) - K(s, t)|$, so the continuity of the map $s \rightarrow X(s)$ follows from that of K.

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