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ON A FIXED POINT THEOREM OF KRASNOSELSKII FOR LOCALLY CONVEX SPACES

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Let $\mathcal{U}$ be a neighborhood basis of the origin consisting of absolutely convex open subsets of a separated locally convex topological vector space E and S a subset of E. Let a mapping $f: S \to E$ satisfy the condition: for each $U \in \mathcal{U}$ and $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, U) > 0$ such that if $x, y \in S$ and $x - y \in (\epsilon + \delta)U$, then $f(x) - f(y) \in \epsilon U$. In the present paper, sufficient conditions are given for the mapping f to have a fixed point in S. The result is extended to the sum of two mappings of Krasnoselskii type.

In a recent paper, Meir and Keeler **[8]** gave an interesting generalization of the Banach's contraction principle. Following **[8]**, a self mapping f of a metric space (X, d) is an (ϵ, δ) contraction iff for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in X$ with $\epsilon \leqq d(x, y) \leqq \epsilon + \delta$ implies $d(f(x), f(y)) < \epsilon$. The (ϵ, δ) contraction mappings clearly contain the class of strict contractions $(d(f(x), f(y)) \leqq \lambda d(x, y), 0 < \lambda < 1)$ and the nonlinear contractions investigated by Boyd and Wong **[4]**. In this paper, we consider mappings defined on a subset S of a locally convex vector space E with values in E (not necessarily S) and satisfy a certain condition similar to (ϵ, δ) contraction. The main result here generalizes a result of Cain and Nashed **[5]** and a recent result of Assad and Kirk **[2]** and provides a further generalization of a well-known result of Krasnoselskii **[7]**.

Throughout this paper, E is a separated locally convex topological vector space and $\mathcal{U}$ is a neighborhood basis of the origin consisting of absolutely convex open subsets of E. For each $U \in \mathcal{U}$, let p_U be the Minkowski's functional of U. Further, if $x, y \in E$ let

$$(x, y) = \{z \in E: z = \lambda x + (1 - \lambda)y,\ 0 < \lambda < 1\}$$

and $[x, y), = \{x\} \cup (x, y)$. For a set $A \subseteq E$, $\partial(A)$ denotes the boundary of A and $\mathrm{cl}(A)$ the closure of A in E. Also for $A, B \subseteq E$, $A - B = \{x - y: x \in A, y \in B\}$.

Let S be a nonempty subset of E. A mapping $f: S \to E$ is a U-contraction ($U \in \mathcal{U}$) iff for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon, U) > 0$ such that if $x, y \in S$ and if

$$(1) \qquad x - y \in (\epsilon + \delta)U, \quad \text{then} \quad f(x) - f(y) \in \epsilon U.$$

If $f: S \to E$ is a U-contraction for each $U \in \mathcal{U}$, then f is a $\mathcal{U}$-contraction. Note that if f is a $\mathcal{U}$-contraction, then f is continuous. (For a related definition of $\mathcal{U}$-contraction, see Taylor **[11]**.)

It may be remarked that if E is a normed space with $\mathcal{U} = \{x \in E: \|x\| < \epsilon, \epsilon > 0\}$ then (1) is equivalent to (ϵ, δ) contraction **[8]**.

The following lemma simplifies the proof of next theorem.

LEMMA 1. *Let $f: S \to E$ be a $\mathcal{U}$-contraction, then f is $\mathcal{U}$-contractive, that is for each $U \in \mathcal{U}$, $p_U(f(x) - f(y)) < p_U(x - y)$ if $p_U(x - y) \neq 0$ and 0 otherwise.*

Proof. Let $x, y \in S$ and suppose $p = p_U$, $p(x - y) = \epsilon > 0$. Then $x - y \in (\epsilon + \delta)U$ for each $\delta > 0$ and in particular $x - y \in (\epsilon + \delta_0)U$ where $\delta_0 = \delta(U, \epsilon)$. Therefore by (1) $(f(x) - f(y)) \in \epsilon U$. Since U is open, this implies that $p(f(x) - f(y)) < \epsilon = p(x - y)$. If $\epsilon = 0$, then $x - y \in \epsilon U$ for each $\epsilon > 0$ and hence by (1) $(f(x) - f(y)) \in \epsilon U$ which implies that $p(f(x) - f(y)) = 0$.

THEOREM 1. *Let S be a sequentially complete subset of E and $f: S \to E$ be a $\mathcal{U}$-contraction. If f satisfies the condition:*

(2) *for each $x \in S$ with $f(x) \notin S$, there is a $z \in (x, f(x)) \cap S$ such that $f(z) \in S$*

then f has a unique fixed point in S.

Proof. Let $x_0 \in S$ and choose a sequence $\{x_n\} \subseteq S$ defined inductively as follows: for each $n \in I$ (positive integers) if $f(x_n) \in S$, set $x_{n+1} = f(x_n)$ and if $f(x_n) \notin S$, let x_{n+1} be any element of $(x_n, f(x_n)) \cap S$ such that $f(x_{n+1}) \in S$ (such x_{n+1} exists by (2)). It then follows that for each $n \in I$, there is a $\lambda_n \in [0, 1)$ satisfying

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) f(x_n). \tag{3}$$

We show that the sequence $\{x_n\}$ so constructed satisfies

$$\text{(a)} \quad x_{n+1} - x_n \to 0 \qquad \text{(b)} \quad x_n - f(x_n) \to 0 \tag{4}$$

To establish (4), note that by (3)

$$x_{n+1} - x_n = (1 - \lambda_n)(f(x_n) - x_n), \quad \text{and} \tag{5}$$

$$f(x_n) - x_{n+1} = \lambda_n (f(x_n) - x_n). \tag{6}$$

Therefore, for a $U \in \mathcal{U}$ with $p = p_U$, it follows by the above lemma that

$$\begin{aligned} p(f(x_{n+1})-x_{n+1}) &\leqq p(f(x_{n+1})-f(x_n))+p(f(x_n)-x_{n+1}) \\ &\leqq p(x_{n+1}-x_n)+\lambda_n(f(x_n)-x_n). \end{aligned}$$

Thus by (5) $p(f(x_{n+1})-x_{n+1}) \leqq p(f(x_n)-x_n)$ for each $n \in I$, that is $\{p(f(x_n)-x_n)\}$ is a nonincreasing sequence of nonnegative reals and hence for each $p = p_U$, $U \in \mathcal{U}$, there is a $r(U) \geqq 0$ with

$$r(U) \leqq p(f(x_n)-x_n) \rightarrow r(U) \geqq 0. \tag{7}$$

We claim that $r(U) \equiv 0$. Suppose $r(U) > 0$. Choose a $\delta = \delta(r(U), U) > 0$ satisfying (1). Then by (7) there is a $n_0 \in I$ such that $p(f(x_n)-x_n) < r(U)+\delta$ for all $n \geqq n_0$. Now choose an $m \in I$, $m \geqq n_0$ such that $x_{m+1} = f(x_m)$, (let $m = n_0$ if $f(x_{n_0}) \in S$, otherwise let $m = n_0+1$, then $x_{m+1} = f(x_m) \in S$). Thus for this m,

$$p(x_m - x_{m+1}) = p(x_m - f(x_m)) < r(U)+\delta.$$

and hence by (1)

$$p(x_{m+1}-f(x_{m+1})) = p(f(x_m)-f(x_{m+1})) < r(U),$$

which contradicts (7). Thus $r(U) = 0$ for each $U \in \mathcal{U}$ and this implies that the sequence $x_n - f(x_n) \rightarrow 0$. This establishes 4(b) and 4(a) now, follows by (5).

We assert that $\{x_n\}$ is a Cauchy sequence in E. Suppose not. Let for each $k \in I$, $A_k = \{x_n : n \geqq k\}$. Then by assumption there is $U \in \mathcal{U}$ such that $A_k - A_k \not\subseteq U$ for any $k \in I$. Choose an ϵ with $0 < \epsilon < 1$ and a δ with $0 < \delta < \delta(\epsilon, U)$ satisfying $\epsilon + \delta < 1$. It follows that $A_k - A_k \not\subseteq (\epsilon + \delta/2)U$ for any $k \in I$. Thus for each $k \in I$, there exist integers $n(k)$ and $m(k)$ with $k \leqq n(k) < m(k)$ such that

$$x_{n(k)} - x_{m(k)} \not\in (\epsilon + \delta/2)U. \tag{8}$$

Let $m(k)$ be the least integer exceeding $n(k)$ satisfying (8). Then by (8)

$$\begin{aligned} x_{n(k)} - x_{m(k)} &= (x_{n(k)} - x_{m(k)-1}) + (x_{m(k)-1} - x_{m(k)}) \\ &\in (x_{m(k)-1} - x_{m(k)}) + (\epsilon + \delta/2)U. \end{aligned} \tag{9}$$

Now by (4) there is a $k_0 \in I$ such that $x_k - f(x_k) \in (\delta/4)U$ and $x_{k-1} - x_k \in (\delta/4)U$ whenever $k \geqq k_0$, and hence by (9)

$$x_{n(k)} - x_{m(k)} \subseteq (\epsilon + \delta)U, \qquad k \geqq k_0.$$

It follows, that for all $k \geqq k_0$

$$f(x_{n(k)}) - f(x_{m(k)}) \in \epsilon U.$$

However, for $k \geqq k_0$,

$$x_{n(k)} - x_{m(k)} = (x_{n(k)} - f(x_{n(k)})) + (f(x_{n(k)}) - f(x_{m(k)})) + (f(x_{m(k)}) - x_{m(k)})$$

and therefore,

$$x_{n(k)} - x_{m(k)} \in \left(\frac{\delta}{4} U + \epsilon U + \frac{\delta}{4} U\right) \subseteq \left(\epsilon + \frac{\delta}{2}\right) U, \qquad k \geqq k_0,$$

which contradicts (8). Thus $\{x_n\}$ is a Cauchy sequence in S and the sequential completeness implies that there is a $u \in S$ such that $x_n \to u$. Since f is continuous, it follows by (4b) that $u = f(u)$. This proves the existence of the fixed point of f. Since E is separated, the uniquencess is an immediate consequence of the Lemma 1.

The following result was proven in **[10]** and its proof here is given for completeness.

LEMMA 2. *Let S be a closed or sequentially complete subset of E. If $x \in S$ and $y \notin S$ then there is a $\lambda \in [0, 1]$ such that $z = (1 - \lambda)x + \lambda y \in \partial(S)$. Further, if $x \notin \partial(S)$ then $0 < \lambda < 1$.*

Proof. Let $A = \{\mu \geqq 0: (1 - \alpha)x + \alpha y \in S$ for all α with $0 \leqq \alpha \leqq \mu\}$. Since $x \in S$, $A \neq \emptyset$. The hypothesis $y \notin S$ implies that $\lambda = \sup\{\mu : \mu \in A\} \leqq 1$. Now if S is closed or sequentially complete, it follows that $z = (1 - \lambda)x + \lambda y \in S$ and hence $\lambda < 1$. To show that $z \in \partial(S)$, it suffices to show that for each $U \in \mathcal{U}$, $(z + U) \cap c(S) \neq \emptyset$, where $c(S)$ is the complement of S in E. Choose a $\beta_0 > \lambda$ with $(\beta_0 - \lambda)p(x - y) < 1$ where $p = p_U$. By definition of λ, there is a β with $\lambda < \beta \leqq \beta_0$ such that $z_1 = (1 - \beta)x + \beta y \notin S$. Since $p(z - z_1) = (\beta - \lambda)p(x - y) < 1$, it follows that $z_1 \in (z + U)$ and hence $z \in \partial(S)$. If $x \notin \partial(S)$ but $x \in S$, then clearly $0 < \lambda < 1$.

The following is now an immediate consequence of Theorem 1.

THEOREM 2. *Let S be sequentially complete subset of E and $f: S \to E$ be a $\mathcal{U}$-contraction. If $f(S \cap \partial(S)) \subseteq S$, then f has a unique fixed point.*

It may be noted that if S is closed then $S \cap \partial(S) = \partial(S)$.

In the following, let $\mathcal{P} = \{p = p_U$ for some $U \in \mathcal{U}\}$, R^+ the nonnegative reals and Ψ a family of mappings defined as $\Psi = \{\phi: R^+ \to R^+: \phi$ is continuous and $\phi(t) < t$ if $t > 0\}$. A mapping $f: S \to E$ is a nonlinear $\mathcal{P}$ contraction (see also Boyd and Wong **[4]**) iff for each $p \in \mathcal{P}$, there is a $\phi_p \in \Psi$ such that $p(f(x) - f(y)) \leqq \phi_p(p(x - y))$ for all $x, y \in S$. If this inequality holds with $\phi_p(t)=\alpha_p t$, $0<\alpha_p<1$, then f is called $\mathcal{P}$-contraction (see [5]). Since a nonlinear $\mathcal{P}$ contraction is a $\mathcal{U}$-contraction, the following result immediately follows by Theorem 1 and provides an extension of a result in [5], (see also Assad [1]).

THEOREM 3. *Let S be a sequentially complete subset of E and $f: S \to E$ be a nonlinear $\mathcal{P}$ contraction. If f satisfies* (2) *then f has a unique fixed point in S.*

As an application of Theorem 3, we give here a generalization of a well-known result of Krasnoselskii [7] which has been extended recently to locally convex spaces in [5]. The following extension of Tychonoff's theorem [12] is due to Singball [3] (see also Himmelberg [6]) and is used in the proof of Theorem 5.

THEOREM 4. *Let S be a closed and convex subset of E and $f: S \to S$ be a continuous mapping such that the range $f(S)$ is contained in a compact set. Then f has fixed point.*

In the rest of this paper, a mapping $f: S \to E$ is completely continuous if it is continuous and $f(S)$ is contained in a compact subset of E. Further, if $A: S \to E$ is a nonlinear $\mathcal{P}$ contraction and $B: S \to E$ is completely continuous, then for each fixed $x \in S$, the mapping $f_x: S \to E$ is defined by $f_x(y) = A(y)+B(x)$. Note that since E is separated, the mapping $(I-A): S \to E$ is one-to-one, where I is the identity map of S.

The following lemma follows immediately from Theorem 3.

LEMMA 3. *Let S be a sequentially complete subset of E and $A: S \to E$ be a nonlinear $\mathcal{P}$ contraction. Suppose for a $x \in E$, the mapping $f: S \to E$ defined by $f(y)=A(y)+x$ satisfies* (2), *then there exists a unique $u(x) \in S$ with $f(u(x))=u(x)$, that is $(I-A)^{-1}x = u(x) \in S$.*

THEOREM 5. *Let S be a convex and complete subset of E. Let $A: S \to E$ be a nonlinear $\mathcal{P}$ contraction and $B: S \to E$ be completely continuous. If for each $x \in S$, the mapping $f_x: S \to E$ satisfies* (2) *and $(I-A)^{-1}B(S)$ is a bounded subset of S, then there is a $u \in S$ satisfying $A(u)+B(u)=u$.*

Proof. For each fixed $x \in S$, the mapping f_x satisfies the conditions of Lemma 3 and hence there is a unique $u_x \in S$ with $f_x(u_x)=u_x$. Define a mapping $L: S \to S$ by

$$L(x)=u_x=A(L(x))+B(x), \qquad x \in S. \tag{10}$$

Then, for each $x \in S$, $L(x) = (I - A)^{-1}B(x)$. If follows by hypothesis that $L(S)$ is a bounded subset of E. We show that L in (10) is continuous. Let $\{x_\alpha : \alpha \in \Gamma\} \subseteq S$ be a net such that $x_\alpha \to x \in S$ and suppose $L(x_\alpha)$ does not converge to $L(x)$. Then there is a $p \in \mathcal{P}$ and an $\epsilon > 0$ and a subnet $\{p(L(x_\alpha) - L(x)) \colon \alpha \in \Gamma_1\}$ of the net $\{p(L(x_\alpha) - L(x)) \colon \alpha \in \Gamma\}$ such that

$$p(L(x_\alpha) - L(x)) > \epsilon \quad \text{for each} \quad \alpha \in \Gamma_1. \tag{11}$$

Since $\{p(L(x_\alpha) - L(x)) \colon \alpha \in \Gamma_1\}$ is a bounded subset of the reals, it has a subnet $\{p(L(x_\alpha) - L(x)) \colon \alpha \in \Gamma_2 \subseteq \Gamma_1\} \to r \geqq 0$. However, by (10) for any $\alpha \in \Gamma_2$

$$p(L(x_\alpha) - L(x)) \leqq p(B(x_\alpha) - B(x)) + \phi_p(p(L(x_\alpha) - L(x))),$$

which implies that $r = 0$. This contradicts (11) and consequently L is continuous. We now show that $L(S)$ is relatively compact in S. If $\{L(x_\alpha) \colon \alpha \in \Gamma\}$ is a net in $L(S)$, then there is a net $\{B(x_\alpha) \colon \alpha \in \Gamma_1\}$ which is convergent. We assert that $\{L(x_\alpha) \colon \alpha \in \Gamma_1\}$ is a Cauchy subnet. Suppose not. Then there is a $p \in \mathcal{P}$ and an $\epsilon > 0$ such that for each $\alpha \in \Gamma_1$ there are elements $n(\alpha)$ and $m(\alpha)$ in Γ_1 with $n(\alpha) \geqq \alpha$, $m(\alpha) \geqq \alpha$, satisfying

$$r_\alpha = p(L(x_{n(\alpha)}) - L(x_{m(\alpha)})) > \epsilon, \qquad \alpha \in \Gamma_1. \tag{12}$$

Since $\{B(x_\alpha) \colon \alpha \in \Gamma_1\}$ is a Cauchy net, there is an $\alpha_0 \in \Gamma_1$ such that $p(B(x_\alpha) - B(x_\beta)) < \epsilon$ for all $\alpha, \beta \geqq \alpha_0$, $\alpha, \beta \in \Gamma_1$. However, $\{r_\alpha \colon \alpha \in \Gamma_1\}$ being a bounded subset of reals has a convergent subnet $\{r_\alpha \colon \alpha \in \Gamma_2\} \to r \geqq 0$. The same argument as above implies that $r = 0$ and this contradicts (12). This proves the assertion. It now follows by Theorem 4, that $L(u) = u$ for some $u \in S$ and hence by (10) $A(u) + B(u) = u$.

The following consequence of Theorem 5 appears new and generalizes a result of Nashed and Wong (Theorem 1 **[9]**). Note that in a normed linear space E a mapping $f \colon S \to E$ is a nonlinear contraction (see **[4]**) if there exists a $\phi \in \Psi$ such that $\|f(x) - f(y)\| \leqq \phi(\|x - y\|)$ for all $x, y \in S$.

COROLLARY 1. *Let S be a closed, bounded and convex subset of a Banach space E. If $A \colon S \to E$ is a nonlinear contraction and $B \colon S \to E$ is completely continuous such that for each $x \in \partial(S)$, $f_x(\partial(S)) \subseteq S$, then $A(u) + B(u) = u$ for some $u \in S$.*

As another consequence, we have the following extension of a result of Cain and Nashed **[5]**.

COROLLARY 2. *Let S be a convex and complete subset of E. Let $A: S \to E$ be a $\mathcal{P}$ contraction and $B: S \to E$ be a completely continuous mapping. If for each $x \in S$, f_x satisfies* (2) *then $A(u) + B(u) = u$ for some $u \in S$.*

Proof. It suffices to show that for each $p \in \mathcal{P}$, $p((I - A)^{-1}B(S))$ is a bounded subset of reals. Now it follows by (10) that for all $x, y \in S$

$$p(L(x) - L(y)) \leqq p(B(x) - B(y)) + \alpha_p p(L(x) - L(y)),$$

which implies that $p(L(x) - L(y)) \leqq (1 - \alpha_p)^{-1} p(B(x) - B(y))$ and hence $L(S) = (I - A)^{-1}B(S)$ is bounded.

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