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NORM ATTAINING OPERATORS ON $L^1[0, 1]$ AND THE RADON-NIKODÝM PROPERTY

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Let Y be a strictly convex Banach space. Then norm attaining operators mapping $L^1[0, 1]$ to Y are dense in the space of all linear operators from $L^1[0, 1]$ to Y if and only if Y has the Radon-Nikodým property.

Bishop and Phelps [1] have asked the general question—For which Banach spaces X and Y is the collection of norm attaining operators from X to Y dense in the space $B(X, Y)$ of all bounded (linear) operators from X to Y. Lindenstrauss in [8] investigated this question and related this question to existence of extreme points and exposed points in the closed unit ball of X. In the course of his paper Lindenstrauss showed that for some space Y the norm attaining operators in $B(L^1[0, 1], Y)$ are not dense in $B(L^1[0, 1], Y)$ due to the lack of extreme points in the closed unit ball of $L^1[0, 1]$. Left open is the following question: For which Banach spaces Y are the norm attaining operators dense in $B[L^1[0, 1], Y)$? Based on Lindenstrauss's work, one is led to believe that if the closed unit ball of Y has a rich extreme point or exposed point structure, then the norm attaining operators may be dense in $B(L^1[0, 1], Y)$. On the other hand the Radon-Nikodým property is intimately connected with extreme point structure (Rieffel [12], Maynard [10], Huff [6], Davis and Phelps [2], Phelps [11], Huff and Morris [7]). So there is some *prima facie* evidence to support the belief that the norm attaining operators are dense in $B(L^1[0, 1], Y)$ if and only if Y has the Radon-Nikodým property. The purpose of this paper is to verify this for strictly convex Banach spaces Y.

First a few well known results will be collected.

LEMMA A [4, 5]. *If (Ω, Σ, μ) is a finite measure space and $g: \Omega \to Y$ is μ-essentially bounded Bochner integrable function, then*

$$T(f) = Bochner - \int fgd\mu$$

defines a member T of $B(L^1(\mu), Y)$ with $\| T \| = \operatorname{ess\,sup} \| g \|_Y$.

LEMMA B [3]. *Any one of the following statements about Y implies all the others.*

(i) *Y has the Radon-Nikodým property.*

(ii) *If (Ω, Σ, μ) is a finite measure space and $G: \Sigma \to Y$ is a*

μ-continuous countably additive measure of bounded variation, then there exists a μ-Bochner integrable

$$g\colon \Omega \longrightarrow Y \textit{ with } G(E) = \int_E g d\mu \textit{ for all } E \in \Sigma \, .$$

(iii) *If μ is Lebesgue measure on* $[0, 1]$, *then for each* $T \in B(L^1[0, 1], Y)$ *there is a μ-essentially bounded $g\colon [0, 1] \rightarrow Y$ with*

$$T(f) = \int_{[0,1]} fgd\mu \textit{ for all } f \in L^1([0, 1], Y)$$

Moreover, if Y has the Radon-Nikodým property statement (iii) *is true for any finite measure space.*

The first theorem is a straight forward observation that is based on the definition of a measurable function.

THEOREM 1. *If Y has the Radon-Nikodým property and if (Ω, Σ, μ) is a finite measure space, then the norm attaining operators are dense in $B(L^1(\mu), Y)$.*

Proof. *Let $T \in B(L^1(\mu), Y)$* and $\varepsilon > 0$. Then there exists an essentially bounded Bochner integrable $g\colon \Omega \rightarrow Y$ such that $T(f) = \int_\Omega fgd\mu$ for all $f \in L^1(\mu)$ and there exists a countably valued function

$$h\colon \Omega \longrightarrow X \, , \quad h = \sum_{i=1}^{\infty} x_i \chi_{E_i} \, , \quad x_i \in X \, ,$$
$$E_i \in \Sigma \, , \quad \mu(E_i) > 0 \, , \quad E_i \cap E_j = \varnothing$$

for $i \neq j$, such that $\operatorname{ess\,sup} \| g - h \| < \varepsilon/2$. Define $T_1\colon L^1(\mu) \rightarrow Y$ by $T_1(f) = \int_\Omega fhd\mu$, $f \in L^1(\mu)$. Then $\| T - T_1 \| < (\varepsilon/2)$.

Now T_1 will be approximated within $\varepsilon/2$ by an operator which attains its norm. If $T_1 = 0$, there is nothing to prove. Otherwise $\beta = \sup \| y_i \| > 0$. Choose i_0 such that $\beta - \| y_{i_0} \| < \varepsilon/2$ and $\alpha > 1$ such that $\varepsilon/4 < (\alpha - 1) \| y_{i_0} \| < \varepsilon/2$ and define

$$T_2(f) = \int_{\substack{\cup E_i \\ i \neq i_0}} fhd\mu + \alpha y_{i_0} \int_{E_{i_0}} fd\mu \, .$$

It is easy to verify that $\| T_1 - T_2 \| < \varepsilon/2$ and that $\| T_2 \| = \alpha \| y_{i_0} \| = \| T_2(x_{E_{i_0}}/\mu(E_{i_0})) \|$. Hence T_2 attains its norm and $\| T - T_2 \| < \varepsilon$, as required.

The operator T_2 constructed in the proof of Theorem 1 has two important properties. First it attains its norm and second there exists $E \in \Sigma$, $\mu(E) > 0$ and $y_0 \in Y$ with $\|y_0\| = \|T\|$ and $T_2(f\chi_E) = \int_E f d\mu y_0$ for all $f \in L^1(\mu)$. If Y is strictly convex and real, this property is shared by all norm attaining operators in $B(L^1(\mu), Y)$.

LEMMA 2. *Let (Ω, Σ, μ) be a finite measure space and Y be a strictly convex Banach space. If $T \in B(L^1(\mu), Y)$ attains its norm then there exists a set $E_0 \in \Sigma$ with $\mu(E_0) > 0$, $g \in L^\infty(\mu)$ with $|g| = 1$ on E_0, and $y_0 \in Y$ with $\|y_0\| = \|T\|$ such that*

$$T(f\chi_{E_0}) = \int_{E_0} fgd\mu\, y_0$$

for all $f \in L^1(\mu)$.

If Y is a real Banach space, g may be taken as the constant function 1.

Proof. If $\|T\| = 0$, there is nothing to prove.

Otherwise, choose $f_0 \in L^1(\mu)$ with $\|T(f_0)\| = \|T\|$ and $\|f_0\| = 1$. With the help of the Hahn-Banach theorem, choose $y_0^* \in Y^*$ with $\|y_0^*\| = 1$ and

$$y_0^* T(f_0) = \|T(f_0)\| = \|T\| .$$

Next choose $h \in L^\infty(\mu)$ with $\|h\|_\infty = \|T\|$ such that

$$y_0^* T(f) = \int_\Omega fhd\mu$$

for all $f \in L^1(\mu)$. A simple computation reveals that $h = \overline{\operatorname{sgn}} f_0 \|T\|$ on the support of f_0. (Here $\operatorname{sgn} f_0 = f_0/|f_0|$.) Let E_0 be the support of f_0. Thus if $f \in L^1(\mu)$,

$$y_0^* T(f\chi_{E_0}) = \int_{E_0} f\, \overline{\operatorname{sgn}} f_0 \|T\| d\mu .$$

Next suppose $E \subset E_0$, $E \in \Sigma$ and $\mu(E)$, $\mu(E_0 - E) > 0$. (The rest of the proof is trivial if E_0 is an atom of μ.) Then

$$y_0^* T\Big(\frac{\chi_E}{\mu(E)} \operatorname{sgn} f_0\Big) = \int_{E_0} \frac{\chi_E}{\mu(E)} \|T\| d\mu = \|T\| ,$$

$$y_0^* T\Big(\frac{\chi_{E_0 - E}}{\chi(E)} \operatorname{sgn} f_0\Big) = \int_{E_0} \frac{\chi_{E_0 - E}}{\mu(E_0 - E)} \|T\| d\mu = \|T\| ,$$

and

$$y_0^* T\Big(\frac{\chi_{E_0}}{\mu(E_0)} \operatorname{sgn} f_0\Big) = \int_{E_0} \frac{\chi_{E_0}}{\mu(E_0)} \|T\| d\mu = \|T\| .$$

From these equalities, one obtains

$$\begin{aligned}\|T\|\,\mu(E_0) &= \|T(\chi_{E_0}\operatorname{sgn} f_0)\| = \|T(\chi_E \operatorname{sgn} f_0) + T(\chi_{E_0-E}\operatorname{sgn} f_0)\| \\ &\leqq \|T(\chi_{E_1}\operatorname{sgn} f_0)\| + \|T(\chi_{E_0-E}\operatorname{sgn} f_0)\| \\ &= \|T\|\,\mu(E) + \|T\|\,\mu(E_0 - E) = \|T\|\,\mu(E_0)\,.\end{aligned}$$

This combined with the fact that Y is strictly convex shows that $T(\chi_E \operatorname{sgn} f_0)$ and $T(\chi_{E_0-E}\operatorname{sgn} f_0)$ are multiples of each other. Since $T(\chi_{E_0}\operatorname{sgn} f_0) = T(\chi_E \operatorname{sgn} f_0) + T(\chi_{E_0-E}\operatorname{sgn} f_0)$, $T(\chi_E \operatorname{sgn} f_0)$ is a scalar multiple of $\mathrm{T}(\chi_{E_0}\operatorname{sgn} f_0)$; i.e., $T(\chi_E \operatorname{sgn} f_0) = \gamma T(\chi_{E_0}\operatorname{sgn} f_0)$ for some scalar γ. On the other hand

$$\|T\|\,\mu(E) = y_0^* T(\chi_E \operatorname{sgn} f_0) = \gamma y_0^*(\chi_{E_0}\operatorname{sgn} f_0) = \gamma\,\|T\|\,\mu(E_0)\,;$$

thus $\gamma = \mu(E)/\mu(E_0)$. Therefore if $E \subset E_0$ and $\mu(E) > 0$,

$$\frac{T(\chi_E \operatorname{sgn} f_0)}{\mu(E)} = \frac{T(\chi_{E_0}\operatorname{sgn} f_0)}{\mu(E_0)} = y_0\,.$$

Now suppose $f \in L^1(\mu)$ is a simple function. Let $\varepsilon > 0$ and choose a simple function $\varphi \in L^1(\mu)$ such that $\|\overline{\operatorname{sgn}} f_0 - \varphi\|_\infty < \varepsilon$. (Here $\overline{\operatorname{sgn}} f_0$ is the complex conjugate of $\operatorname{sgn} f_0$.) Then $T(f) = T(f\,\overline{\operatorname{sgn}} f_0 \operatorname{sgn} f_0)$ and $\|T(f) - T(f\varphi \operatorname{sgn} f_0)\| \leqq \|T\|\,\|\overline{\operatorname{sgn}} f_0 \operatorname{sgn} f_0 - \varphi \operatorname{sgn} f_0\|_1 < \varepsilon\,\|T\|\,\mu\Omega$. Now select sets $A_1, \cdots, A_n \in \Sigma$ such that

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} \quad \text{and} \quad \varphi = \sum_{i=1}^n \beta_i \chi_{A_i}\,.$$

Then

$$\begin{aligned}T(f\varphi \operatorname{sgn} f_0 \chi_{E_0}) &= \sum_{i=1}^n \alpha_i\beta_i \frac{T(\chi_{A_i} \cap E_0 \operatorname{sgn} f_0)}{\mu(A_i \cap E_0)}\,\mu(A_i \cap E_0) \\ &= \sum_{i=1}^n \alpha_i\beta_i\mu(A_i \cap E_0)y_0 = \int_{E_0} f\varphi d\mu y_0\,.\end{aligned}$$

Letting ε go to zero reveals that

$$T(f\chi_{E_0}) = \int_{E_0} f\,\overline{\operatorname{sgn}} f_0 d\mu y_0\,.$$

Since simple functions are dense in $L^1(\mu)$, the equality

$$T(f\chi_{E_0}) = \int_{E_0} f\,\overline{\operatorname{sgn}} f_0 d\mu y_0$$

obtains for all $f \in L^1(\mu)$. This proves the first statement.

To prove the second statement, note that if Y is real, then $\operatorname{sgn} f_0$ takes on only the values $+1$ or -1. If $\operatorname{sgn} f_0 = 1$ on a set of positive measure E, in the support of f_0, take $E_0 = E$ and proceed as above. If $\operatorname{sgn} f_0 = -1$ almost everywhere in the support of f_0, multiply f_0 and y_0^* by -1 and proceed as in the last sentence.

With the help of Lemma 2, the main result becomes nothing but a straightforward exhaustion argument.

THEOREM 3. *Let Y be a strictly convex Banach space. If the norm attaining members of $B(L^1[0, 1], Y)$ are dense in $B(L^1[0, 1], Y)$, then Y has the Radon-Nikodým property.*

Proof. Let $T \in B(L^1[0, 1], Y)$ and $\varepsilon > 0$ be given. Define a class of Lebesgue measurable sets $\mathscr{M}$ by agreeing that $E \in \mathscr{M}$ if there exists an essentially bounded Bochner integrable $g(=g(E, \varepsilon))$: $[0, 1] \to Y$ such that

$$\left\|T(f\chi_E) - \int_E fgd\mu\right\| \leqq \varepsilon \, \|f\chi_E\|_1 \, .$$

Note that if A is Lebesgue measurable and $A \subset E \in \mathscr{M}$ then

$$\begin{aligned}\left\|T(f\chi_A) - \int_A fg((E, \varepsilon)d\mu)\right\| &= \left\|T((f\chi_A)\chi_E) - \int_E (f\chi_A)gd\mu\right\| \\ &\leqq \|f\chi_A\chi_E\|_1 = \varepsilon \, \|f\chi_A\|_1 \, .\end{aligned}$$

Therefore, if $E \in \mathscr{M}$, every measurable subset of E belongs to $\mathscr{M}$. Now let $\alpha = \sup \{\mu(E) : E \in \mathscr{M}\}$ and let $(E_n) \subset \mathscr{M}$ be a sequence such that $\lim_n \mu(E_n) = \alpha$. Write $A_1 = E_1$, $A_2 = E_2 - E_1$, $\cdots$, $A_n = E_n - \bigcup_{i=1}^{n-1} E_i$. Then the $A_n^1 s$ are disjoint, $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n$ and $\mu(\bigcup_{n=1}^{\infty} A_n) \geqq \alpha$. $A_n \subset E_n$ and $E_n \in \mathscr{M}$, $A_n \in \mathscr{M}$ and there exists a sequence of essentially bounded functions $g_n : [0, 1] \to Y$, $n = 1, 2, \cdots$, such that for all $f \in L^1[0, 1]$,

$$\left\|T(f\chi_{A_n}) - \int_{A_n} fg_n d\mu\right\| \leqq \varepsilon \, \|f\chi_{A_n}\|_1 \, .$$

Accordingly,

$$\left\|\int_{A_n} fg_n d\mu\right\| \leqq \|T(f\chi_{A_n})\| + \varepsilon \|f\chi_{A_n}\|_1 \leqq (\|T\| + \varepsilon)\|f\|_1 \, .$$

By Lemma A,

$$\operatorname{ess\,sup} \|g_n\chi_{A_n}\| - \sup_{\|f\|_1 \leqq 1} \left\|\int_{A_n} fg_n d\mu\right\| \leqq \|T\| + \varepsilon \, .$$

Therefore $\sup_n \operatorname{ess\,sup} \|g_n\| \leqq \|T\| + \varepsilon$. Now define $g : [0, 1] \to Y$ by

$$g(t) = \begin{cases} g_n(t) & \text{for} \quad t \in A_n \\ 0 & \text{for} \quad t \notin \bigcup_{n=1}^{\infty} A_n \, . \end{cases}$$

Then $\operatorname{ess\,sup} \| g \| \leqq \| T \| + \varepsilon$ and if $f \in L^1[0, 1]$,

$$\begin{aligned} &\Big\| T(f\chi_{\bigcup_n A})_n - \int_{\bigcup_n A_n} fg\,d\mu \Big\| \\ &\leqq \sum_{n=1}^{\infty} \Big\| T(f\chi_{A_n}) - \int_{A_n} fg_n\,d\mu \Big\| \\ &\leqq \sum_{n=1}^{\infty} \varepsilon \, \| f\chi_{A_n} \|_1 \leqq \varepsilon \, \| f \|_1 . \end{aligned}$$

Therefore $\bigcup_n A_n \in \mathscr{M}$. Next we shall see that $\mu(\bigcup_n A_n) = 1$. For, if $\mu(\bigcup_n A_n) < 1$, then $\mu(\bigcup_n E_n) \leqq 1$ and $\alpha < 1$. Set $B_0 = [0, 1] - \bigcup_m A_m$ and recall that $L^1(B_0)$ (Lebesgue integrable functions supported on B_0) is isometric to $L^1[0, 1]$. Define $T_1 \colon L^1(B_0) \to Y$ by $T_1(f) = T(f\chi_{B_0})$ for $f \in L^1(E)$. Since $L^1(B_0)$ is isometric to $L^1[0, 1]$, there exists an operator $T_2 \colon L^1(B_0) \to Y$ that attains its norm such that $\| T_1 - T_2 \| \leqq \varepsilon$.

An appeal to Lemma 2 produces a $y_1 \in Y$ and set $B_1 \subset B_0$ with $\mu(B_1) > 0$ such that

$$T_2(f) = \int_{B_1} f\,d\mu y_1$$

for all $f \in L^1(B_0)$. Set $g' = y_1\chi_{B_1}$. Then

$$\begin{aligned} \Big\| T(f\chi_{B_1}) - \int_{B_1} fg'\,d\mu \Big\| &= \| T_1(f\chi_{B_1}) - T_2(f\chi_{B1}) \| \\ &\leqq \| T_1 - T_2 \| \, \| f\chi_{B_1} \|_1 \leqq \varepsilon \, \| f\chi_{B_1} \|_1 . \end{aligned}$$

Therefore $B_1 \in \mathscr{M}$. Now set $\tilde{g} = g + g'$. If $f \in L^1([0, 1])$,

$$\begin{aligned} &\Big\| T(f\chi_{\bigcup_{n=1}^{\infty} A_n \cup B_1}) - \int_{\bigcup_{n=1}^{\infty} A_n \cup B_1} f\tilde{g}\,d\mu \Big\| \\ &\leqq \sum_{n=1}^{\infty} \Big\| T(f\chi_{A_n}) - \int_{A_n} fg_n\,d\mu \Big\| + \Big\| T(f\chi_{B_1}) - \int_{B_1} fg'd \Big\| \\ &\leqq \varepsilon \sum_{n=1}^{\infty} \| f\chi_{A_n} \| + \varepsilon \| f\chi_{B_1} \| = \| f\chi_{\bigcup_{n=1}^{\infty} A_n \cup B_1} \| . \end{aligned}$$

Therefore $\bigcup_n A_n \cup B_1 = \bigcup_n E_n \cup B_1 \in \mathscr{M}$. But

$$\begin{aligned} \mu\Big(\bigcup_n E_n \cup B_1\Big) &= \mu\Big(\bigcup_n E_n\Big) + \mu(B_1) \\ &\geqq \lim_n \mu(E_n) + \mu(B_1) = \alpha + \mu(B_1) > \alpha \end{aligned}$$

contradicting the definition of α. Thus $\mu(\bigcup_n A_n) = 1$ and

$$\Big\| T(f) - \int_{[0,1]} fg\,d\mu \Big\| \leqq \varepsilon \, \| f \|_1 \text{ for all } f \in L^1[0, 1] .$$

Finally, to check that Y has the Radon-Nikodým property, let

$g_n\colon [0, 1] \to Y$ be a sequence of Bochner integrable essentially bounded functions such that for all $f \in L^1[0, 1]$

$$\left\| T(f) - \int_{[0,1]} f g_n d\mu \right\| \leqq 1/n \, \| f \|_1$$

for all n. An appeal to Lemma 1 shows that $\lim_{n,m} \operatorname{ess\,sup} \| g_n - g_m \|_1$ Hence there exists a Bochner integrable essentially bounded $g\colon [0, 1] \to Y$ with $\lim_n \operatorname{ess\,sup} \| g_n - g \| = 0$. If $f \in L^1[0, 1]$, the dominated convergence theorem guarantees that

$$T(f) - \lim_n \int_{[0,1]} f g_n d\mu = \int_{[0,1]} f g d\mu \, .$$

Thus Y has the Radon-Nikodým property by Lemma B.

The role of strict convexity seems to be crucial in Theorem 3: for by perturbing co-ordinate functions it is seen easily that norm attaining operators are dense in $B(L^1[0, 1], c_0)$, $B(L^1[0, 1], l^\infty)$ or for that matter $B(X, l^\infty)$ for any Banach space X. See [8, Prop. 3].

On the other hand, the role of strict convexity could be made even more palatable by an affirmative answer to an old question of Diestel's: Does every Banach space with the Radon-Nikodým property have an equivalent strictly convex norm?

COROLLARY 4. *If X is a strictly convex renorming of $L^1[0, 1]$, then the norm attaining operators are not dense in $B(L^1[0, 1], X)$.*

Proof. Evidently X lacks the Radon-Nikodým property.

This leaves unsolved the question of whether the norm attaining operators are dense in $B(L^1[0, 1], L^1[0, 1])$.

Finally say that a Banach space X has property B if for every Banach space Y the norm attaining operators are dense in $B(Y, X)$. Lindenstrauss [8, Proposition 4] has observed that if there is a non-compact operator in $B(c_0, X)$ and X is strictly convex, then X lacks property B. It is not difficult to see that if X has the Radon-Nikodým property, then every operator in $B(c_0, X)$ is compact and that the converse in false. Thus Theorem 3 is a better test for Property B than [8, Proposition 4]. Of course this brings up a question that is well beyond the scope of this note. If X is a strictly convex Banach space, does X has property B if and only if X has the Radon-Nikodým property?

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