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ANALYTIC EXTENSIONS OF VECTOR-VALUED FUNCTIONS

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Let Δ be the open unit disc in C, $\partial\Delta$ its boundary and $B \subset \partial\Delta$ a relatively open set. Let X be a complex Banach space. Denote by $H_B(\Delta, X)$ the set of all continuous functions from $\Delta \cup B$ to X which are analytic on Δ. A set $P \subset X$ is said to have the analytic extension property with respect to $H_B(\Delta, X)$ if for each relatively closed set $F \subset B$ of Lebesgue measure 0 and for each continuous function $f: F \to P$ there exists $g \in H_B(\Delta, X)$ with $g \mid F = f$ and $g(\Delta \cup B) \subset P$.

THEOREM. **Let $P \subset X$ be an open set. Then P has the analytic extension property with respect to $H_B(\Delta, X)$ for every relatively open $B \subset \partial\Delta$ if and only if P is connected.**

By a result of E. A. Heard and J. H. Wells any closed disc in C has the analytic extension property with respect to $H_B(\Delta, C)$ for every relatively open $B \subset \partial\Delta$ (see [9]). The special case $B = \partial\Delta$ is the well known Rudin-Carleson theorem (see [4], [10], [12]). This result was generalized to the vector case by proving that every closed ball in X has the analytic extension property with respect to $H_B(\Delta, X)$ for every relatively open $B \subset \partial\Delta$ (see [6]), the special case $B = \partial\Delta$ is the Rudin-Carleson theorem for vector-valued functions (see [5], [11], [14]).

It is a natural question whether the balls above can be replaced by some other sets:

Problem (see [8]). Obtain a (geometrical, topological) characterization of the sets having the analytic extension property with respect to $H_B(\Delta, X)$ for every relatively open $B \subset \partial\Delta$.

It seems that this problem is not solved even for the subsets of C.

Taking $B = \partial\Delta$, $F = \{-1, 1\}$ it is trivial to see that every set having the analytic extension property with respect to $H_B(\Delta, X)$ for every relatively open $B \subset \partial\Delta$, is pathwise connected. The converse is not true in general as shown by taking $P = \{t: 0 \leqq t \leqq 1\}$. However, the converse turns out to be true for open sets and this is the main result of the present paper.

Throughout, we denote by $\bar{\Delta}$ the closure of Δ. Given $r > 0$ we denote by $B_r(X)$ the open ball in X of radius r, centered at the origin. If K is a compact Hausdorff space we denote by $C(K, X)$

the space of all continuous functions from K to X. By $A(\Delta, X)$ we denote the Banach space of all continuous functions from $\bar{\Delta}$ to X, analytic on Δ, with sup norm, and we write $A = A(\Delta, C)$. We write $I = \{t: 0 \leqq t \leqq 1\}$ and we denote the set of all positive integers by N.

For the proof of theorem we shall need four lemmas.

LEMMA 1. *Suppose that G is a closed subset of $\partial\Delta$ of Lebesgue measure 0 and let $U(G) \subset \bar{\Delta}$ be a neighbourhood of G. Let $p: I \to X$ be a path in a complex Banach space X and let $\varepsilon > 0$ be arbitrary. There exists $\phi \in A(\Delta, X)$ having the following properties:*

(i) $\|\phi(z) - p(1)\| < \varepsilon$ $(z \in G)$

(ii) $\|\phi(z) - p(0)\| < \varepsilon$ $(z \in \bar{\Delta} - U(G))$

(iii) $\phi(\bar{\Delta}) \subset p(I) + B_\varepsilon(X)$.

Proof. By the Mergelyan theorem for analytic functions into a Banach space (see [3]) there exists a polynomial $f: C \to X$ satisfying $\|f(z) - p(z)\| < \varepsilon$ $(z \in I)$. By the continuity of f there exists an open neighbourhood V of I such that $f(V) \subset p(I) + B_\varepsilon(X)$. Let $W \subset V$ be an open set, bounded by a Jordan curve, containing the point 1 in its boundary and satisfying $I - \{1\} \subset W$, $\bar{W} \subset V$. Let $T \subset W$ be a neighbourhood of the point 0 in W such that $\|f(z) - p(0)\| < \varepsilon$ $(z \in T)$. Assume for a moment that $\alpha \in A$ satisfies $\alpha(\bar{\Delta}) \subset \bar{W}$, $\alpha(G) = \{1\}$ and $\alpha(\bar{\Delta} - U(G)) \subset T$. Then it is easy to check that $\phi = f \circ \alpha$ has all the required properties. It remains to prove the existence of such an α. By the Riemann mapping theorem (see [13]) there exists a homeomorphism β from $\bar{\Delta}$ onto $\bar{W}$, analytic on Δ and satisfying $\beta(0) = 0$, $\beta(1) = 1$. Let $S \subset \Delta$ be a neighbourhood of 0 such that $\beta(S) \subset T$. By the Rudin-Carleson theorem (see [12]) there exists $\gamma \in A$ satisfying $\gamma(\bar{\Delta}) \subset \bar{\Delta}$, $\gamma(G) = \{1\}$. Also (see [15], p. 205) there exists $\psi \in A$ satisfying $\psi(\bar{\Delta}) \subset \bar{\Delta}$, $\psi(G) = \{1\}$, $|\psi(z)| < 1$ $(z \in \bar{\Delta} - G)$. Let $U_1 \subset U(G)$ be an open subset of $\bar{\Delta}$ containing G. Now $\bar{\Delta} - U_1$ is a compact set disjoint from G and it follows that for sufficiently large $n \in N$ we have $\psi^n(z) \cdot \gamma(z) \in S$ $(z \in \bar{\Delta} - U_1)$. Now putting $\alpha(z) = \beta[\psi^n(z) \cdot \gamma(z)]$ $(z \in \bar{\Delta})$ it is easy to see that α has all the required properties.

LEMMA 2. *Let X be a complex Banach space and let Q be an open connected subset of X. Given a compact subset K of Q and a point $x \in K$ there exists $\delta_0 > 0$ such that for every $\delta: 0 < \delta < \delta_0$ there exists a path $p: I \to X$ satisfying*

(i) $p(0) = x$

(ii) $K \subset p(I) + B_\delta(X)$

(iii) $p(I) + B_{6\delta}(X) \subset Q$.

Proof. By the compactness of K there exists an $\varepsilon > 0$ such that $K + B_{7\varepsilon}(X) \subset Q$. Cover K by a finite number of balls, say by $B_1, B_2, \cdots, B_n$ of radii ε whose centers lie in K. With no loss of generality assume that the center of B_1 is x. By the connectedness of Q there exists a path $q: I \to X$, satisfying $q(I) \subset Q$, $q(0) = x$, and connecting the centers of all B_i. By the compactness of $q(I)$ there exists δ_0: $0 < \delta_0 < \varepsilon$ such that $q(I) + B_{6\delta_0}(X) \subset Q$. Let δ satisfy $0 < \delta < \delta_0$ and cover K by a finite number of balls $D_1, D_2, \cdots, D_m$ of radii δ whose centers lie in K. Let $1 \leqq i \leqq n$. Consider those balls D_k whose centers lie in B_i. Connect all these centers by a path p_i starting and ending at the center of B_i and satisfying $p_i(I) \subset B_i$. Having done this for all i, denote by q_i $(1 \leqq i \leqq n - 1)$ the part of the path q between the centers of B_i, B_{i+1}. Now define p as the sum of the paths

$$p = \sum_{i=1}^{n-1} (p_i + q_i) + p_n .$$

If $s \in I$ is such that $p(s)$ is in none of the balls B_i $(1 \leqq i \leqq n)$ then $p(s) \in q(I)$ and consequently $p(s) + B_{6\delta}(X) \subset q(I) + B_{6\delta_0}(X) \subset Q$. If $s \in I$ is such that $p(s)$ is in some B_i then $p(s) + B_{6\delta}(X) \subset B_i + B_{6\delta_0}(X) \subset K + B_{7\varepsilon}(X) \subset Q$. On the other hand, if $y \in K$ then $y \in D_k$ for some ball D_k whose center is contained in $p(I)$ which means that $y \in p(I) + B_\delta(X)$.

LEMMA 3. *Let $F \subset \partial\Delta$ be a closed set of Lebesgue measure 0 and let $U(F) \subset \bar{\Delta}$ be a neighbourhood of F. Suppose that Q is an open connected set in a complex Banach space X containing the point* 0. *Let $\varepsilon > 0$ be arbitrary. Given $f \in C(F, X)$ satisfying $f(F) \subset Q$ there exists $\tilde{f} \in A(\Delta, X)$ satisfying*

(i) $\tilde{f} \mid F = f$

(ii) $\tilde{f}(\bar{\Delta}) \subset Q$

(iii) $\| \tilde{f}(z) \| < \varepsilon$ $(z \in \bar{\Delta} - U(F))$.

Proof. $f(F) \cup \{0\}$ is a compact set contained in Q. By Lemma 2 there exists δ: $0 < \delta < \varepsilon/5$ and a path $p: I \to X$ satisfying $f(F) \subset p(I) + B_\delta(X)$, $p(I) + B_{6\delta}(X) \subset Q$ and $p(0) = 0$. Since F is a compact set the function f is uniformly continuous on F. By the assumption F is nowhere dense on $\partial\Delta$. It follows that

$$F = \bigcup_{i=1}^{n} F_i$$

where $F_i \subset \partial\Delta$ are disjoint closed sets such that

$$\| f(\eta) - f(\zeta) \| < \delta \quad (\eta, \zeta \in F_i;\ 1 \leqq i \leqq n) .$$

Let U_i $(1 \leqq i \leqq n)$ be disjoint open subsets of $\bar{\Delta}$ satisfying $F_i \subset U_i \subset U(F)$ $(1 \leqq i \leqq n)$. Since $f(F) \subset p(I) + B_\delta(X)$ there exist $t_i \in I$ and $z_i \in F_i$ $(1 \leqq i \leqq n)$ such that

$$\| p(t_i) - f(z_i) \| < \delta \quad (1 \leqq i \leqq n) .$$

Applying Lemma 1 to the paths $t \mapsto p(t_i t)$ $(1 \leqq i \leqq n)$ there exist functions $\phi_i \in A(\Delta, X)$ $(1 \leqq i \leqq n)$ satisfying

$$\begin{aligned} &\| \phi_i(z) - p(t_i) \| < \delta \quad (z \in F_i) \\ &\| \phi_i(z) \| < \delta/n \quad (z \in \bar{\Delta} - U_i) \\ &\phi_i(\bar{\Delta}) \subset p(I) + B_\delta(X) . \end{aligned}$$

Now define $\Psi \in A(\Delta, X)$ by

$$\Psi = \sum_{i=1}^{n} \phi_i .$$

If $z \in \Delta - \bigcup_{i=1}^{n} U_i$ then

$$\| \Psi(z) \| \leqq \sum_{i=1}^{n} \| \phi_i(z) \| < n \,.\, \delta/n = \delta .$$

If $z \in U_i$ for some i then $z \notin U_j$ $(i \neq j)$ and

$$\Psi(z) = \phi_i(z) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \phi_j(z) \in p(I) + B_\delta(X) + B_\delta(X) \subset p(I) + B_{2\delta}(X) .$$

Consequently $\Psi(\bar{\Delta}) \subset p(I) + B_{2\delta}(X)$. Now define $\Theta \in C(F, X)$ by $\Theta(z) = \Psi(z) - f(z)$ $(z \in F)$. If $z \in F$ then $z \in F_i$ for some i and consequently

$$\begin{aligned} \| \Theta(z) \| &\leqq \| \Psi(z) - p(t_i) \| + \| p(t_i) - f(z_i) \| + \| f(z_i) - f(z) \| \\ &\leqq \sum_{\substack{j=1 \\ j \neq i}}^{n} \| \phi_i(z) \| + \| \phi_i(z) - p(t_i) \| + 2\delta . \\ &< 4\delta . \end{aligned}$$

By the Rudin-Carleson theorem for vector valued functions there exists $\tilde{\Theta} \in A(\Delta, X)$ satisfying $\| \tilde{\Theta} \| < 4\delta$, $\tilde{\Theta} \mid F = \Theta$. Finally, define $\tilde{f}(z) = \Psi(z) - \tilde{\Theta}(z)$ $(z \in \bar{\Delta})$. Clearly $\tilde{f} \in A(\Delta, X)$. Further, $\tilde{f}(\bar{\Delta}) \subset p(I) + B_{2\delta}(X) + B_{4\delta}(X) \subset p(I) + B_{6\delta}(X) \subset Q$. Clearly $\tilde{f} \mid F = f$. Also, if $z \in \bar{\Delta} - U(F)$ then $z \in \bar{\Delta} - \bigcup_{i=1}^{n} U_i$ hence $\| \tilde{f}(z) \| \leqq \| \Psi(z) \| + \| \tilde{\Theta}(z) \| < \delta + 4\delta < \varepsilon$.

LEMMA 4. *Let E be closed subset of $\partial\Delta$ and let $G \subset \partial\Delta - E$ be a relatively closed set of Lebesgue measure* 0. *Let $H \subset \partial\Delta - E$ be a compact set of Lebesgue measure* 0, *disjoint from G. Let Q be an open connected set in a complex Banach space X containing the point* 0 *and suppose that $f \in C(H, X)$ satisfies $f(H) \subset Q$.*

There exists $\delta_0 > 0$ such that for every $\delta: 0 < \delta < \delta_0$ and for *every* $\varepsilon: 0 < \varepsilon < \delta$ *and for every neighbourhood* $U \subset \bar{\Delta} - E$ *of* H *there exists a continuous function* $\tilde{f}: \bar{\Delta} - E \to X$, *analytic on* Δ *and satisfying*

(i) $\tilde{f} \mid H = f$

(ii) $\tilde{f} \mid G = 0$

(iii) $\| \tilde{f}(z) \| < \varepsilon \ (z \in (\bar{\Delta} - E) - E) - U)$

(iv) $f(\bar{\Delta} - E) + B_\delta(X) \subset Q$.

Proof. With no loss of generality we may assume that $U \cap G = \varnothing$. By Lemma 2 there exists $\delta_0 > 0$ such that for every $\delta: 0 < \delta < \delta_0$ there exists a path $p: I \to X$ satisfying $p(0) = 0$, $f(H) \subset p(I) + B_\delta(X)$, $p(I) + B_{6\delta}(X) \subset Q$. Let $0 < \delta < \delta_0$ and $0 < \varepsilon < \delta$. Applying Lemma 3 to the function f and to the (open connected) set $p(I) + B_\delta(X)$ there exists $\tilde{f}_1 \in A(\Delta, X)$ satisfying

$$\tilde{f}_1 \mid H = f$$
$$\tilde{f}_1(\bar{\Delta}) + B_{3\delta}(X) \subset Q$$
$$\| \tilde{f}_1(z) \| < \varepsilon/2 \ (z \in \bar{\Delta} - U) .$$

Define

$$f_2(s) = \begin{cases} -\tilde{f}_1(s) & (s \in G) \\ 0 & (s \in H) . \end{cases}$$

Then f_2 is continuous on $G \cup H$ and satisfies $\| f_2(s) \| < \varepsilon/2 \ (s \in G \cup H)$. By Theorem 2 in [6] there exists a continuous function $\tilde{f}_2: \bar{\Delta} - E \to X$, analytic on Δ, satisfying $\tilde{f}_2 \mid G \cup H = f_2$ and $\| \tilde{f}_2(z) \| \leqq \varepsilon/2 \ (z \in \bar{\Delta} - E)$. Put $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$. It is easy to check that $\tilde{f}$ has all the required properties.

Proof of theorem. Let Q be an open connected subset of a complex Banach space X. Let $E \subset \partial\Delta$ be a closed set and let $F \subset \partial\Delta - E$ be a relatively closed set of Lebesgue measure 0. Suppose that $f: F \to X$ is a continuous function satisfying $f(F) \subset Q$. We will prove that there exists a continuous extension $\tilde{f}: \bar{\Delta} - E \to X$, $\tilde{f} \mid F = f$, which is analytic on Δ and which satisfies $f(\bar{\Delta} - E) \subset Q$.

If E is empty then the statement of the theorem is proved by Lemma 3. So assume that E is not empty. With no loss of generality assume that $0 \in Q$. As in [6] write $F = \bigcup_{n=1}^{\infty} F_n$ where $F_n \subset \bar{\Delta} - E$ are compact sets such that there exist disjoint open sets $U_n \subset \bar{\Delta} - E$ satisfying $F_n \subset U_n$ for all n.

Now we define inductively a sequence $\{D_n\}$ of open subsets of $\bar{\Delta} - E$ satisfying $F_n \subset D_n \subset U_n$ for all n, a decreasing sequence $\{\delta_n\}$ of positive numbers and a sequence $\{\phi_n\}$ of functions from $\bar{\Delta} - E$ to X having the following properties:

(i) for each $i \in N$, ϕ_i is continuous on $\bar{\Delta} - E$ and analytic on Δ

(ii) $\phi_i \mid F_j = 0$ $(i \neq j;\ i, j \in N)$

(iii) $\phi_i \mid F_i = f \mid F_i$ $(i \in N)$

(iv) $\phi_i(\bar{\Delta} - E) + B_{\delta_i}(X) \subset Q$ $(i \in N)$

(v) $\| \phi_i(z) \| < \delta_i/2^{i+1}$ $(z \in (\bar{\Delta} - E) - D_i;\ i \in N)$

(vi) $\| \sum_{j=1}^{i} \phi_j(z) \| < \delta_{i+1}/2$ $(z \in D_{i+1};\ i \in N)$.

If $i = 1$, put $D_1 = U_1$ and apply Lemma 4 to the function $f \mid F_1$ to obtain δ_1 satisfying $B_{\delta_1}(X) \subset Q$ and ϕ_1 which satisfies (i)-(v) above for $i = 1$. Now assume that δ_i, D_i, ϕ_i $(1 \leqq i \leqq n)$ are given satisfying (i)-(v) for $1 \leqq i \leqq n$ and (vi) $1 \leqq i \leqq n - 1$. Applying Lemma 4 to the function $f \mid F_{n+1}$ there exists δ_{n+1}: $0 < \delta_{n+1} < \delta_n$ such that Lemma 4 holds for $\delta = \delta_{n+1}$. Since the function

$$z \longmapsto \sum_{j=1}^{n} \phi_j(z)$$

is continuous on $\bar{\Delta} - E$ and equal 0 on F_{n+1} there exists a neighbourhood $D_{n+1} \subset \bar{\Delta} - E$ of F_{n+1} satisfying $D_{n+1} \subset U_{n+1}$ and such that (vi) is satisfied for $i = n$. Now, by Lemma 4 there exists ϕ_{n+1} satisfying (i)-(v) for $i = n + 1$.

Define

$$\tilde{f}(z) = \sum_{i=1}^{\infty} \phi_i(z) \quad (z \in \bar{\Delta} - E) .$$

If $z \in (\bar{\Delta} - E) - \bigcup_{j=1}^{\infty} D_j$ then $\| \phi_i(z) \| < \delta_i/2^{i+1} < \delta_1/2^{i+1}$. Consequently the series converges uniformly for all such z. By

$$\sum_{i=1}^{\infty} \| \phi_i(z) \| < \delta_1/2$$

and by $B_{\delta_1}(X) \subset Q$ we have $f(z) \in Q$ for all such z. Suppose that $z \in D_k$ for some k. Then $z \notin D_j$ for $j \neq k$ and by the above argument the series converges uniformly on D_k. Further, by (v) and (vi) we have

$$\Big\| \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \phi_j(z) \Big\| \leqq \Big\| \sum_{j=1}^{k-1} \phi_j(z) \Big\| + \sum_{j=k+1}^{\infty} \| \phi_j(z) \| < \delta_k/2 + \delta_k/2 = \delta_k .$$

Consequently by (iv) $f(z) \in \phi_k(\bar{\Delta} - E) + B_{\delta_k}(X) \subset Q$. Since each compact subset of $\bar{\Delta} - E$ misses all but a finite number of the sets D_i the series converges uniformly on compact subsets of $\bar{\Delta} - E$. Consequently $\tilde{f}$ is continuous on $\bar{\Delta} - E$, analytic on Δ and, as shown above, satisfies $\tilde{f}(\bar{\Delta} - E) \subset Q$. By the properties of ϕ_i we have also $\tilde{f} \mid F = f$.

COROLLARY (see [7]). *Given any open connected subset Q of a* *separable complex Banach space X there exists an analytic function $\tilde{f}: \Delta \rightarrow X$ whose range is contained and dense in Q.*

Proof. Put $E = \{1\}$ and let $F = \{z_n\} \subset \partial\Delta - \{1\}$ be an injective sequence converging to 1. Let $f(z_n) = w_n$ where $\{w_n\} \subset Q$ is a sequence dense in Q and then apply theorem.

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