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A VARIANCE PROPERTY FOR ARITHMETIC FUNCTIONS

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A pivotal point for certain problems in probabilistic number theory is that there exists a positive constant c such that for every member f of the family of additive complex valued arithmetic functions

$$\sum_{m \leqq n} |f(m) - A(n)|^2 \leqq cnD^2(n)$$

where

$$A(n) = \sum_{p^\alpha \leqq n} f(p^\alpha)p^{-\alpha}(1 - p^{-1})$$

and

$$D^2(n) = \sum_{p^\alpha \leqq n} |f(p^\alpha)|^2 p^{-\alpha}(1 - p^{-1}) ,$$

p^α being a power of a prime number. This paper considers the extension of this property in two directions suggested by Harold N. Shapiro. First, an investigation is made of when this property holds for weight functions other than $w(m) \equiv 1$. Second, it is shown that this property can be extended to various nonadditive arithmetic functions.

1. Preliminaries. Let p and q represent prime numbers while m, n, ν, α and β represent positive integers. Nonnegative weight functions are represented by w and complex valued arithmetic functions by f. Then f is said to possess the Variance Property with respect to w if

$$\sum_{m \leqq n} w(m) \,|\, f(m) - A(n) \,|^2 \leqq cD^2(n) \sum_{m \leqq n} w(m) \tag{1.1}$$

for all n for some fixed constant $c(w, f)$, where (letting $\nu \,\|\, m$ denote $p^\alpha \mid m$ but $p^{\alpha+1} \nmid m$ for all p^α in the prime decomposition of ν)

$$\gamma(\nu) = \lim_{n \to \infty} \gamma(\nu, n) = \lim_{n \to \infty} \frac{\sum_{\substack{m \leqq n \\ \nu \| m}} w(m)}{\sum_{m \leqq n} w(m)} , \tag{1.2}$$

$$A(n) = \sum_{p^\alpha \leqq n} f(p^\alpha)\gamma(p^\alpha) , \tag{1.3}$$

and

$$D^2(n) = \sum_{p^\alpha \leqq n} |\, f(p^\alpha) \,|^2 \,\gamma(p^\alpha) . \tag{1.4}$$

For a given weight function $w = w(m)$, we let $\Phi(w)$ represent the set of arithmetic functions such that (1.1) holds. We associate with a given family of weight functions $\mathscr{W}$ the set of arithmetic functions $\Phi(\mathscr{W}) = \bigcap_{w \in \mathscr{W}} \Phi(w)$. Similarly for a given f let $\Omega(f)$ denote the set of weight functions w such that (1.1) holds, and for a collection of functions $\mathscr{F}$ define $\Omega(\mathscr{F}) = \bigcap_{f \in \mathscr{F}} \Omega(f)$. When asserting that (1.1) holds, the constant c may depend on both f and w (but not on n). A subset of $\Phi(\mathscr{W})$ is said to possess the Variance Property uniformly with respect to $\mathscr{W}$ if one constant c can be used for all the functions involved. Similarly, a subset of $\Omega(\mathscr{F})$ is said to possess the Variance Property uniformly with respect to $\mathscr{F}$ if one constant c can be used for all the functions involved. In this terminology the basic Variance Theorem cited in the introductory paragraph [1] concerning the set $\mathscr{A}$ of additive complex valued arithmetic functions (i.e., functions such that $f(mn) = f(m) + f(n)$ if $(m, n) = 1$) asserts that $\mathscr{A}$ possesses the Variance Property uniformly with respect to the weight function $w(m) \equiv 1$.

2. Subfamilies of $\Omega(\mathscr{A})$. Let

$$\mathscr{W}_0 = \{w(m) \colon \gamma(p^\alpha, n) \leqq c_0 \gamma(p^\alpha) \text{ for all } p^\alpha \text{ and } n, \text{ and for some } c_0(w)\} .$$

Theorem 2.1 gives two alternative sufficient conditions that will guarantee that a member of $\mathscr{W}_0$ possesses the Variance Property uniformly with respect to the set of additive functions; the remark at the end of this section shows that an averaging of one of these conditions is a necessary condition. Theorems 2.2 and 2.3 show that $w(m) = m^r$ (for any fixed real number r) and $w(m) = r^m$ (for some fixed r, $0 < r < 1$), possess the Variance Property uniformly with respect to $\mathscr{A}$.

LEMMA. *Let $w \in \mathscr{W}_0$ and $f \in \mathscr{A}$. Then $f \in \Phi(w)$ if and only if*

$$\Big| \sum_{\substack{p^\alpha \leqq n,\, q^\beta \leqq n \\ p \neq q}} f(p^\alpha)\overline{f(q^\beta)}\, T(p^\alpha, q^\beta, n) \Big| \leqq c_1 D^2(n) \tag{2.1}$$

for all n and some fixed $c_1(f)$, where

$$\begin{aligned} T(p^\alpha, q^\beta, n) = \gamma(p^\alpha q^\beta, n) &- \gamma(p^\alpha)\gamma(q^\beta, n) - \gamma(q^\beta)\gamma(p^\alpha, n) \\ &+ \gamma(p^\alpha)\gamma(q^\beta) . \end{aligned} \tag{2.2}$$

If (2.1) holds for all $f \in \mathscr{A}$, then $\mathscr{A}$ possesses the Variance Property uniformly with respect to w if and only if c_1 can be chosen independent of $f \in \mathscr{A}$.

Proof. $\sum_{m\leqq n} w(m)\,|\,f(m)-A(n)\,|^2=(R_1+R_2+R_3)\sum_{m\leqq n} w(m)$ where

$$R_1 = \sum_{p^\alpha\leqq n} |\,f(p^\alpha)\,|^2\,\gamma(p^\alpha, n)\ ,$$

$$R_2 = \sum_{\substack{p^\alpha\leqq n\\ \beta\leqq\left[\frac{\log n}{\log p}\right]}} f(p^\alpha)\overline{f(p^\beta)}(\gamma(p^\alpha)\gamma(p^\beta) - \gamma(p^\alpha)\gamma(p^\beta, n) - \gamma(p^\beta)\gamma(p^\alpha, n))\ ,$$

and

$$R_3 = \sum_{\substack{p^\alpha\leqq n,\ q^\beta\leqq n\\ p\neq q}} f(p^\alpha)\overline{f(q^\beta)}T(p^\alpha, q^\beta, n)\ .$$

Now $w\in\mathscr{W}_0$ implies $|\,R_1\,|\leqq c_0D^2(n)$ and (by using Schwarz's inequality)

$$\begin{aligned}|\,R_2\,| &\leqq \max{(1,\, 2c_0)}\sum_{p^\alpha\leqq n} |\,f(p^\alpha)\,|^2\,\gamma(p^\alpha)\sum_{\beta\leqq\left[\frac{\log n}{\log p}\right]}\gamma(p^\beta)\\ &\leqq \max{(1,\, 2c_0)}D^2(n)\ ,\end{aligned}$$

which completes the proof.

THEOREM 2.1. *Let $w\in\mathscr{W}_0$ be such that either*

$$\sum_{\substack{q^\beta\leqq n\\ q\neq p}} |\,T(p^\alpha, q^\beta, n)\,| \leqq c_2\gamma(p^\alpha) \tag{2.3}$$

for all n and all $p^\alpha\leqq n$, for some constant c_2, or

$$\sum_{\substack{p^\alpha\leqq n,\ q^\beta\leqq n\\ \gamma(p^\alpha)\neq 0,\ \gamma(q^\beta)\neq 0\\ p\neq q}} \frac{T^2(p^\alpha, q^\beta, n)}{\gamma(p^\alpha)\gamma(q^\beta)} \leqq c_2^2 \tag{2.4}$$

for all n. Then w possesses the Variance Property uniformly with respect to $\mathscr{A}$.

The proof follows immediately by applying Schwarz's inequality in two different ways to the left hand side of (2.1).

THEOREM 2.2. *The weight function $w(m) = m^r$, where r is any fixed real number, possesses the Variance Property uniformly with respect to $\mathscr{A}$. For this weight function*

$$\gamma(p^\alpha) = \begin{cases} p^{\alpha r}(1 - p^r) & \textit{if}\quad r\leqq -1\\ p^{-\alpha}(1 - p^{-1}) & \textit{if}\quad r\geqq -1\ .\end{cases} \tag{2.5}$$

Proof. Let

$$\lambda(\nu, n) = \frac{\sum_{\substack{k\leqq n\\ \nu\mid k}} w(k)}{\sum_{k\leqq n} w(k)} = \nu^r\frac{\sum_{k\leqq n/\nu} k^r}{\sum_{k\leqq n} k^r} \tag{2.6}$$

so that

$$\gamma(p^\alpha, n) = \lambda(p^\alpha, n) - \lambda(p^{\alpha+1}, n) \tag{2.7}$$

and (for $p \neq q$)

$$\begin{aligned} \gamma(p^\alpha q^\beta, n) = \lambda(p^\alpha q^\beta, n) &- \lambda(p^{\alpha+1}q^\beta, n) - \lambda(p^\alpha q^{\beta+1}, n) \\ &+ \lambda(p^{\alpha+1}q^{\beta+1}, n)\,. \end{aligned} \tag{2.8}$$

Using elementary calculus we obtain the following inequalities:

$$\left.\begin{matrix} 1 \\ \log x \\ \dfrac{x^{r+1}}{r+1} - \dfrac{1}{r+1} \\ \dfrac{x^{r+1}}{r+1} - 2^r x^r \end{matrix}\right\} \leqq \sum_{k \leqq x} k^r \leqq \begin{cases} r/(r+1) & \text{for } r < -1 \\ 1 + \log x & \text{for } r = -1 \\ \dfrac{x^{r+1}}{r+1} & \text{for } -1 < r < 0 \\ \dfrac{x^{r+1}}{r+1} + x^r & \text{for } r \geqq 0 \end{cases} \tag{2.9}$$

(where the term $-2^r x^r$ may be omitted in the $r \geqq 0$ case if x is an integer). For $r \geqq -1$ this result provides upper and lower bounds for $\lambda(\nu, n)$ which show that

$$\lambda(\nu) = \lim_{n\to\infty} \lambda(\nu, n) = \begin{cases} \nu^r & \text{for } r \leqq -1\,, \\ \nu^{-1} & \text{for } r \geqq -1\,, \end{cases} \tag{2.10}$$

where, for $r < -1$, the result follows from the convergence of $\sum k^r$. (2.10) and (2.7) now yield (2.5). Also, the upper bounds on $\lambda(\nu, n)$ show that

$$\gamma(\nu, n) \leqq \lambda(\nu, n) \leqq \frac{1}{2} c_0 \lambda(\nu) \tag{2.11}$$

for some $c_0 = c_0(r)$. Thus we see that $w \in \mathscr{W}_0$ since

$$\lambda(p^\alpha) \leqq 2\gamma(p^\alpha)\,. \tag{2.12}$$

We note that $\lambda(p^\alpha q^\beta) = \lambda(p^\alpha)\lambda(q^\beta)$, and hence as a result of (2.2), (2.11) and (2.12)

$$\begin{aligned} \sum_{\substack{p^\alpha \leqq n,\, q^\beta \leqq n \\ p^\alpha q^\beta > n \\ p \neq q}} \frac{T^2(p^\alpha, q^\beta, n)}{\gamma(p^\alpha)\gamma(q^\beta)} &\leqq 4(1 + c_0)^2 \sum_{\substack{p^\alpha \leqq n,\, q^\beta \leqq n \\ p^\alpha q^\beta > n}} \lambda(p^\alpha)\lambda(q^\beta) \\ &\leqq 4(1 + c_0)^2 \sum_{\substack{p^\alpha \leqq n,\, q^\beta \leqq n \\ p^\alpha q^\beta > n}} \frac{1}{p^\alpha q^\beta} \end{aligned}$$

which is known to be bounded [1]. All that remains to verify (2.4) is to show

$$S = \sum_{\substack{p^\alpha q^\beta \leqq n \\ p \neq q}} \frac{T^2(p^\alpha, q^\beta, n)}{\gamma(p^\alpha)\gamma(q^\beta)} \tag{2.13}$$

is bounded. Then the desired result will follow from Theorem 2.1. For $r < -1$ we note that

$$S \leqq 4(1 + c_0)^2 \sum_{\substack{p^\alpha q^\beta \leqq n \\ p \neq q}} \lambda(p^\alpha)\lambda(q^\beta) \leqq 8(1 + c_0)^2 \sum_{k \leqq n} k^r$$

which is bounded. In view of (2.6), (2.7) and (2.8), for $r \geqq -1$ and $p \neq q$ we have

$$\begin{aligned} U &= T(p^\alpha, q^\beta, n) \sum_{k \leqq n} k^r \\ &= p^{r\alpha}q^{r\beta} \sum_{k \leqq n/(p^\alpha q^\beta)} k^r - p^{r(\alpha+1)}q^{r\beta} \sum_{k \leqq n/(p^{\alpha+1}q^\beta)} k^r \\ &\quad - p^{r\alpha}q^{r(\beta+1)} \sum_{k \leqq n/(p^\alpha q^{\beta+1})} k^r + p^{r(\alpha+1)}q^{r(\beta+1)} \sum_{k \leqq n/(p^{\alpha+1}q^{\beta+1})} k^r \\ &\quad - p^{\alpha}(1 - p^{-1})q^{r\beta} \sum_{k \leqq n/q^\beta} k^r + p^{-\alpha}(1 - p^{-1})q^{r(\beta+1)} \sum_{k \leqq n/(q^{\beta+1})} k^r \\ &\quad - q^{-\beta}(1 - q^{-1})p^{r\alpha} \sum_{k \leqq n/p^\alpha} k^r + q^{-\beta}(1 - q^{-1})p^{r(\alpha+1)} \sum_{k \leqq n/(p^{\alpha+1})} k^r \\ &\quad + p^{-\alpha}q^{-\beta}(1 - p^{-1})(1 - q^{-1}) \sum_{k \leqq n} k^r . \end{aligned} \tag{2.14}$$

Using the inequalities of (2.9) on (2.14) to find upper and lower bounds for U, and then applying (2.9) to the factor $\sum k^r$ of U in order to obtain upper and lower bounds for $T(p^\alpha, q^\beta, n)$, we find after lengthy but straightforward calculation that, for $n \geqq 2$ and $p \neq q$

$$|T(p^\alpha, q^\beta, n)| \leqq \begin{cases} \dfrac{c_3}{p^\alpha q^\beta \log n} & \text{for } r = -1 \\ \dfrac{c_3(p^\alpha q^\beta)^r}{n^{r+1}} & \text{for } -1 < r < 0 \\ c_3/n & \text{for } r \geqq 0 \end{cases} \tag{2.15}$$

for some $c_3(\mathrm{r})$. Actually, for the case where $r = -1$, the calculations leading to (2.15) assume $p^{\alpha+1}q^{\beta+1} \leqq n$, since the appropriate inequality of (2.9) used in bounding U assumes $1 + \log n/p^{\alpha+1}q^{\beta+1} \geqq 0$. But we may only assume $p^\alpha q^\beta \leqq n$ in showing S is bounded. However, if $p^\alpha q^\beta \leqq n < p^{\alpha+1}q^{\beta+1}$, then

$$\begin{aligned} &\left| \frac{1}{p^{\alpha+1}q^{\beta+1}}\left(1 + \log \frac{n}{p^{\alpha+1}q^{\beta+1}}\right) \right| \\ &\qquad \leqq \frac{1}{p^{\alpha+1}q^{\beta+1}} \left| 1 - \log \frac{p^\alpha q^\beta}{n} pq \right| \leqq \frac{1}{p^\alpha q^\beta} . \end{aligned}$$

Similarly, all the other special cases do not affect the result given by (2.15).

For $r = -1$, (2.13) and (2.15) yield

$$S \leqq \frac{4c_3^2}{\log^2 n} \sum_{p^\alpha q^\beta \leqq n} \frac{1}{p^\alpha q^\beta} \leqq \frac{4c_3^2}{\log^2 n}\Big(\sum_{p^\alpha \leqq n} p^{-\alpha}\Big)^2 = \frac{O(\log\log n)^2}{\log^2 n}$$

which is bounded.

For $-1 < r < 0$, (2.13) and (2.15) show

$$S \leqq O(n^{-2r-2}) \sum_{\substack{p^\alpha q^\beta \leqq n \\ p \neq q}} (p^\alpha q^\beta)^{1+2r} \leqq O(n^{-2r-2}) \sum_{k \leqq n} k^{1+2r} ,$$

which is bounded ((2.9) yields this for $-1 < r < -1/2$, and $k^{1+2r} \leqq n^{1+2r}$ implies the bound when $-1/2 \leqq r < 0$).

For $r \geqq 0$, (2.13) and (2.15) show

$$S \leqq O(n^{-2}) \sum_{\substack{p^\alpha q^\beta \leqq n \\ p \neq q}} p^\alpha q^\beta \leqq O(n^{-1}) \sum_{k \leqq n} 1 = O(1) .$$

Thus S is bounded for all values of r, which finishes the proof of the theorem.

THEOREM 2.3. *The weight function* $w(m) = r^m$, *where* r *is a fixed number,* $0 < r < 1$, *possesses the Variance Property uniformly with respect to* $\mathscr{A}$. *For this weight function,*

$$\gamma(p^\alpha) = r^{p^\alpha - 1} \frac{1-r}{1-r^{p^\alpha}} \frac{1 - r^{p^{\alpha+1}} - p^\alpha}{1 - r^{p^{\alpha+1}}} . \tag{2.16}$$

Proof. Let

$$\lambda(\nu, n) = \frac{\sum_{\substack{m \leqq n \\ \nu \mid m}} w(m)}{\sum_{m \leqq n} w(m)} = \frac{\sum_{k \leqq n/\nu} (r^\nu)^k}{\sum_{m \leqq n} r^m} = r^{\nu - 1} \frac{1-r}{1-r^\nu} \frac{1 - r^{\nu[n/\nu]}}{1 - r^n} .$$

Now $\gamma(p^\alpha) = \lim_{n\to\infty} (\lambda(p^\alpha, n) - \lambda(p^{\alpha+1}, n))$ yields (2.16). Also $\gamma(\nu, n) \leqq \lambda(\nu, n) \leqq r^{\nu-1}$ and (2.16) imply

$$(1-r)\gamma(p^\alpha, n) \leqq (1-r) r^{p^\alpha - 1} \leqq \gamma(p^\alpha) \leqq r^{p^\alpha - 1} ,$$

which shows that $w \in \mathscr{W}_0$ and (cf. (2.2)) $|T(p^\alpha, q^\beta, n)| \leqq 4r^{p^\alpha + q^\beta - 2}$. Thus

$$\sum_{\substack{p^\alpha \leqq n, q^\beta \leqq n \\ p \neq q}} \frac{T^2(p^\alpha, q^\beta, n)}{\gamma(p^\alpha)\gamma(q^\beta)} = O\Big(\sum_{p^\alpha \leqq n} r^{p^\alpha}\Big)^2 = O(1) .$$

The desired result now follows from Theorem 2.1.

REMARK. Assume $w \in \mathscr{W}_0$. Then a necessary condition for w

to belong to $\Omega(\mathscr{A})$ is

$$\left| \sum_{p^\alpha \leqq n,\, q^\beta \leqq n} T(p^\alpha, q^\beta, n) \right| \leqq c_1 \sum_{p^\alpha \leqq n} \gamma(p^\alpha) . \tag{2.17}$$

This follows from the lemma by use of the additive function determined by $f(p^\alpha) = 1$ for all p^α.

3. The Variance Property for nonadditive functions. In this section attention is restricted to weight functions which possess the Variance Property with respect to $\mathscr{A}$. The reason for doing so is the fact that any arithmetic function g can be written as $g = f + h$ where

$$f(m) = \sum_{p^\alpha \| m} g(p^\alpha) ; \tag{3.1}$$

then $A(n)$ and $D^2(n)$ are the same for g and f, and f is additive. Thus, the results of § 2 apply to f. In the rest of this section f will always represent the purely "additive" part of g while h will represent the purely "nonadditive" part of g.

Theorem 3.1 gives a necessary and sufficient global condition (in the sense that it involves the average value of $|h|$) for h to satisfy in order to have g possess the Variance Property with respect to a weight function w. Its corollary provides a local sufficient condition (in the sense that it involves the specific values of h). Remark 3.1 shows that in general this local sufficient condition cannot be substantially improved upon, and that in general there is a "gap" between the local sufficient condition and a local necessary condition. However, Remark 3.2 shows that in specific cases the local sufficient condition of the corollary can be improved, and the gap between it and a local necessary condition can be reduced almost to the difference between a small "o" relation and a big "O" relation.

THEOREM 3.1. *Assume $w \in \Omega(\mathscr{A})$. Then $g \in \Phi(w)$ if and only if*

$$\sum_{m \leqq n} w(m) \,|\, h(m) \,|^2 = O(D^2(n) \sum_{m \leqq n} w(m)) . \tag{3.2}$$

The proof follows directly from (1.1).

COROLLARY. *Assume $w \in \Omega(\mathscr{A})$. Then $g \in \Phi(w)$ if*

$$h(m) = O(D(m)) . \tag{3.3}$$

REMARK 3.1. Consider the weight function $w(m) \equiv 1$. Then $|\, h(m) \,| = O(\sqrt{m}\, D(m))$ is a necessary condition for $g = f + h$ to belong to $\Phi(w)$. For if there were a sequence of integers $m_k \to \infty$ such that

$$\frac{\sum_{m \leqq m_k} |h(m)|^2}{m_k D^2(m_k)} \geqq \frac{|h(m_k)|^2}{m_k D^2(m_k)} \longrightarrow \infty ,$$

this would contradict the fact that $g \in \Phi(w)$.

Now if we examine the function

$$g(m) = \begin{cases} 4 & \text{if } m = 2 \\ \sqrt{m} & \text{if } m = 15^k \\ 0 & \text{otherwise} \end{cases}$$

we find that $A(n) = 1$, $D^2(n) = 4$, and $g \in \Phi(w)$. Thus the necessary condition $|h(m)| = O(\sqrt{m}\, D(m))$ cannot be improved.

On the other hand, if we let f be the additive function determined by setting $f(p^\alpha) = p^{-\alpha/2}$ so that $1/8 \leqq D^2(n) = O(1)$, and define

$$h(m) = \begin{cases} 0 & \text{for } m = p^\alpha \\ m^a D(m) & \text{otherwise} \end{cases}$$

for $a > 0$ fixed, then it follows from Theorem 3.1 that $g = f + h$ does not belong to $\Phi(w)$. To see this, note that

$$\sum_{m \leqq n} |h(m)|^2 \geqq \frac{1}{8} \sum_{\substack{m \leqq n \\ m \neq p^\alpha}} m^{2a} \geqq \delta n^{1+2a}$$

for some fixed constant $\delta > 0$. Thus we see the sufficient condition (3.3) cannot be substantially improved either.

REMARK 3.2. Consider the weight function $w(m) = r^m$, $0 < r < 1$. Then $|h(m)| = O(r^{-m/2} D(m))$ is a necessary condition for g to belong to $\Phi(w)$ since Theorem 3.1 and the existence of a sequence of integers $m_k \to \infty$ such that

$$\frac{\sum_{m \leqq m_k} r^m |h(m)|^2}{D^2(m_k) \sum_{m \leqq m_k} r^m} \geqq \frac{(1-r) r^{m_k} |h(m_k)|^2}{D^2(m_k)} \longrightarrow \infty$$

would imply that g did not belong to $\Phi(w)$.

On the other hand, $|h(m)| = O(a(m) r^{-m/2} D(m))$, where $a(m) \geqq 0$ and $\sum a(m) < \infty$, is a sufficient condition for g to belong to $\Phi(w)$, since in this case

$$\sum_{m \leqq n} r^m |h(m)|^2 = O(D^2(n) \sum_{m \leqq n} a(m)) = O(D^2(n)) .$$

Thus we see that for these weight functions the sufficient condition (3.3) can be improved, and the difference between the necessary condition and the sufficient condition is almost the difference between a small "*o*" relation and a big "*O*" relation.

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