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PACKING SPHERES IN ORLICZ SPACES

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A collection of open balls of radius r can be packed in the unit ball U of a Banach space provided each ball is a subset of U and the intersection of any two is empty. In an infinite dimensional Banach space, it is possible to find a largest number Λ so that if $r \leqq \Lambda$ then an infinite number of spheres of radius r can be packed in U. In this paper, upper and lower bounds are found for this number in Orlicz spaces.

For the space l_2, this number was found by Rankin [7] to be $1/(1+\sqrt{2})$ and this result was extended in [1] to show that the number in $l_p(1 \leqq p < \infty)$ is $1/(1+2^{1-1/p})$. In 1970 Kottman [4] showed that $1/3 \leqq \Lambda \leqq 1/2$ for any Banach space. More recently, Wells and Williams [10] used a generalized Riesz-Thorin interpolation theorem to obtain the exact value of Λ in the $L^p(\mu)$ $(1 \leqq p < \infty)$ spaces with some restrictions on the measure space when $2 < p < \infty$. The results in this paper include all the above and also show that all restrictions can be removed in the L^p case. Recent results have demonstrated that the structure of Orlicz spaces is quite different from L^p spaces and very little seems to be known in the Orlicz case. The packing criteria lead to some results on isometric embeddings of subspaces and to notions of noncompactness.

2. Preliminaries. An Orlicz function M will be a continuous convex nondecreasing function defined for $x \geqq 0$ and such that $M(0)=0$, $M(\infty)=\infty$ and $M(x)>0$ for $x>0$. The Orlicz space $L_M(X, \mathscr{A}, \mu)(=L_M)$ is the set of measurable scalar-valued functions defined on the measure space $(X, \mathscr{A}, \mu)$ such that $f \varepsilon L_M$ if and only if $\|f\|' < \infty$ where

$$\|f\|'_M = \inf\left\{k>0: \int_X M\left(\frac{|f|}{k}\right)d\mu \leqq 1\right\}.$$

For each Orlicz function M, a complementary function N is defined by

$$N(x) = \sup\{xy\text{-}M(y): 0 < y < \infty\}.$$

If $M(x) = \int_0^x p(t)dt$ where p is a right continuous nondecreasing function, then $N(p(x)) = xp(x) - M(x)$ (cf [5]). Using this function, another norm can be defined on L_M

$$||f||_M = \sup\left\{\int_X |fg|\, d\mu \colon ||g||'_N \leqq 1\right\}.$$

These norms are equivalent if every set of positive μ-measure contains a subset of positive finite μ-measure and in this paper the latter will be used. In the case of $M(x) = x^p$, $p > 1$, it follows that $||f||_p = ||f||'_M = K||f||_M$ where K is independent of f (cf [11]). It will be assumed in the remainder of the paper that M is chosen so that the simple functions are dense in L_M.

If M_1 and M_2 are two Orlicz functions then M_s will denote the inverse of $M_s^{-1} = (M_1^{-1})^{1-s}(M_2^{-1})^s$ for $0 \leqq s \leqq 1$, where M^{-1} is the unique inverse of the Orlicz function M. The function M_s is an Orlicz function and satisfies most of properties of M_1 and M_2 including the fact that the simple functions are dense in L_{M_s} if the same is true in L_{M_1} and L_{M_2}. The complementary function to M_s is not always the same as the inverse of $N_s^{-1} = (N_1^{-1})^{1-s}(N_2^{-1})^s$ where N_1 and N_2 are the respective complements of M_1 and M_2. However, the complement of M_s and the inverse of N_s^{-1} qenerate the same Orlicz space with equivalent norms (cf [8]). Since the complementary function is the one of interest in this paper, N_s will denote the complement of M_s.

One condition which guarantees the separability of L_M is the Δ_2-condition. An Orlicz function is said to satisfy the Δ_2-condition at ∞ if $\lim_{x\to\infty} \sup M(2x)/M(x) < \infty$. In the case of sequence spaces, separability occurs if and only if the Δ_2-condition holds at 0. A necessary and sufficient condition that M satisfy the Δ_2-condition is that $\lim_{x\to\infty} \sup xM'(x)/M(x) = \alpha < \infty$ where $M'(x)$ is the derivative of M (cf [5], p. 24). If M' and N' are both continuous where N is the complement of M, then this condition is equivalent to

$$\liminf_{x\to\infty} \frac{xN'(x)}{N(x)} > \alpha/\alpha - 1 .$$

This and elementary calculus lead to a lemma that will be useful in later sections.

LEMMA 2.1. *Let M and N be complementary functions with M' and N' continuous. If*

$$\alpha = \limsup_{x\to\infty} \frac{xM'(x)}{M(x)}$$

then

$$\liminf_{x\to\infty} \frac{N^{-1}(x)}{N^{-1}(2x)} \geqq \frac{1}{2^{(\alpha-1)/\alpha}} .$$

3. Interpolation. In this section a generalized interpolation theorem is described and then applied to obtain inequalities that will be useful in next section. This theorem generalizes Theorem 1 in [8] and follows the development in [10] of the L_p case.

Let (X_1, μ_1), (X_2, μ_2), $\cdots$, (X_n, μ_n) be measure spaces and $M = (M_1, M_2, \cdots, M_n)$ be an n-tuple of Orlicz functions. Define the direct sum $\bigoplus L_{M_k}(\mu_k)$ by

$\bigoplus L_{M_k}(\mu_k) = \{f = (f_1, f_2, \cdots, f_n) \mid f_k \varepsilon L_{M_k}(\mu_k),\ k = 1, 2, \cdots, n\}$ with usual addition and scalar multiplication. For each r, $1 \leqq r \leqq \infty$ and each n-tuple $\lambda = (\lambda_1, \cdots, \lambda_n)$ of positive weights, introduce the following norm on $\bigoplus L_{M_k}(\mu_k)$,

$$||f||_{M,r} = \begin{cases} \left\{ \sum_{k=1}^{n} ||f_k||_{M_k}^r \lambda_k \right\}^{1/r} & 1 \leqq r < \infty \\ \max_{1 \leqq k \leqq n} ||f_k||_{M_k} & r = \infty . \end{cases}$$

The space of all f such that $||f||_{M,r} < \infty$ is a Banach space and will be denoted by $L_M^r(\lambda)$.

For two n-tuples $M_1 = (M_{11}, M_{12}, \cdots, M_{1n})$ and $M_2 = (M_{21}, M_{22}, \cdots, M_{2n})$ define $M_s = (M_{s1}, M_{s2} \cdots, M_{sn})$, $0 \leqq s \leqq 1$, where M_{sk} is the inverse of the function $M_{sk}^{-1} = (M_{1k}^{-1})^{1-s}(M_{2k}^{-1})^s$, $k = 1, 2, \cdots, n$.

Now let (Y_1, ν_1), (Y_2, ν_2), $\cdots$, (Y_m, ν_m) be another collection of measure spaces, $\eta = (\eta_1, \cdots, \eta_m)$ and define m-tuples Q_1, Q_2 in the same manner as M_1 and M_2. Letting $X = (X_1, X_2, \cdots, X_n)$ and $Y = (Y_1, Y_2, \cdots, Y_m)$, the following interpolation theorem was proved in [2].

THEOREM 3.1. *Let $1 \leqq r_i, t_i \leqq \infty$, $i = 1, 2$, $0 \leqq s \leqq 1$ with $1/r = 1 - s/r_1 + s/r_2$, $1/t = 1 - s/t_1 + s/t_2$ and suppose M_i and Q_i, $i = 1, 2$, are defined on X and Y respectively. If T is a linear transformation from $L_{M_i}^{t_i}(\lambda)$ into $L_{Q_i}^{r_i}(\eta)$, $i = 1, 2$, with bounds K_1 and K_2 respectively, then T takes $L_{M_s}^t$ into $L_{Q_s}^r$ and*

$$||Tf||_{Q_s, r} \leqq K_1^{1-s} K_2^s ||f||_{M_s, t} .$$

This result is quite useful in establishing inequalities as the following theorem demonstrates.

THEOREM 3.2. *Let M be an Orlicz function, $M_0(x) = x^2$ and $M_s^{-1} = (M^{-1})^{1-s}(M_0^{-1})^s$, $0 \leqq s \leqq 1$. Then for any collection of positive numbers $c_1, c_2, \cdots, c_n$ such that $\sum_{i=1}^n c_i = 1$, the inequality*

$$\sum_{i,j=1}^{n} c_i c_j ||f_i - f_j||_{M_s}^{2/(2-s)} \leqq 2\gamma^{2(1-s)/(2-s)} \sum_{i=1}^{n} c_i ||f_i||_{M_s}^{2/(2-s)}$$

holds wherever $f_1, f_2, \cdots, f_n \varepsilon L_{M_s}$ *and* $\gamma = \max_{1\leq i\leq n}(1 - c_i)$.

Proof. Let M_i, $i = 1, 2$ be the constant n-tuple with each component M and Q_i, $i = 1, 2$, the constant n^2-tuple with each component M. Setting $t_1 = r_1 = 1$, $t_2 = r_2 = 2$, $c = (c_1, c_2, \cdots, c_n)$ and $c^2 = (c_i c_j)_{i,j=1}^n$ define T from $L_{M_i}^{t_i}(c)$ into $L_{Q_i}^{r_i}(c^2)$ by $T(f_1, f_2, \cdots, f_n) = (f_i - f_j)_{i,j=1}^n$. Now

$$\begin{aligned} \|Tf\|_{M,1} &= \sum_{i,j=1}^n c_i c_j \|f_i - f_j\|_M \\ &\leqq \sum_{i,j=1}^n c_i c_j (\|f_i\|_M + \|f_j\|_M) - 2\sum_{i=1}^n c_i^2 \|f_i\|_M \\ &= 2\sum_{i=1}^n \|f_i\|_M (1 - c_i) c_i \leqq 2\gamma \sum_{i=1}^n c_i \|f_i\|_M = 2\gamma \|f\|_{M,1} . \end{aligned}$$

It follows from properties of Hilbert space that $\|Tf\|_{M_0,2} \leqq \sqrt{2}\|f\|_{M_0,2}$. According to Theorem 3.1, T takes $L_{M_s}^{2/(2-s)}(c^2)$ into $L_{M_s}^{2/(2-s)}(c^2)$ and

$$\|Tf\|_{M_s,2/(2-s)} \leqq (2\gamma)^{1-s}(\sqrt{2})^s \|f\|_{M_s,2/(2-s)} .$$

This says

$$\left\{\sum_{i,j=1}^n c_i c_j \|f_i - f_j\|_{M_s}^{2/(2-s)}\right\}^{(2-s)/2} \leqq (2\gamma)^{1-s}(\sqrt{2})^s \left\{\sum_{i=1}^n c_i \|f_i\|_{M_s}^{2(2-s)}\right\}^{(2-s)/2} .$$

Raising both sides to the $2/(2 - s)$ power, the desired inequality is obtained.

The above theorem reduces to the results found in [10] for the L_p case.

COROLLARY 3.3. *Let* $1 < p < \infty$ *and* $c_1, c_2, \cdots, c_n$ *be any collection of positive numbers such that* $\sum_{i=1}^n c_i = 1$. *Then for any* $f_1, f_2, \cdots, f_n$ *in* L_p,

(i) $\sum_{i,j=1}^n c_i c_j \|f_i - f_j\|_p^p \leqq 2\gamma^{2-p} \sum_{i=1}^n c_i \|f_i\|_p^p$, $1 \leqq p \leqq 2$

and

(ii) $\sum_{i,j=1}^n c_i c_j \|f_i - f_j\|_p^{p'} \leqq 2\gamma^{2-p'} \sum_{i=1}^n c_i \|f_i\|_{p'}^{p'}$ $2 < p < \infty$

where $p' = p/p - 1$.

Proof. To prove (i), choose l so that $1 < l < p \leqq 2$ and let $M(x) = x^l$. If we set $s = 2/p((p - l)/(p - 2))$, $M_s(x) = x^p$ and let $l \to 1$, then $2/(2 - s)$ approaches p. Similarly one can show (ii) by choosing $l > p$ and allowing $l \to \infty$.

4. **Packing.** The main object of this section is to find bounds on the number Λ_M where Λ_M satisfies the property that for $r \leqq \Lambda_M$, infinite packing is possible and for $r > \Lambda_M$, only a finite number of balls of radius r can be packed in the unit ball of the Orlicz space L_M. It has been shown by Kottman [4] that $1/3 \leqq \Lambda_M \leqq 1/2$. These bounds are improved below in the spaces $L_M[0, 1]$ but it is clear that the techniques apply to a wider class of spaces.

DEFINITION 4.1. A family of balls $\{B_r(f_j)\}_{j\in I}$ of radius r and centers $\{f_j\}_{j\in I}$ can be packed in the unit ball B_1 of L_M provided

(i) $B_r(f_j) \subset B_1$ for each $j \in I$

(ii) $\operatorname{int}(B_r(f_j)) \cap \operatorname{int}(B_r(f_k)) = \phi$, $j \neq k$.

If a family of balls $\{B_r(f_j)\}_{j\in I}$ can be packed in B_1 then it is clear that

$$\|f_j\| \leqq 1 - r, \quad j \in I \tag{4.1}$$

$$\|f_j - f_k\| \geqq 2r, \quad j \neq k \tag{4.2}$$

must be satisfied. Thus to find an example to serve as lower bound one needs to find vectors $f_1, f_2, \cdots$, satisfying these inequalities.

Given an Orlicz function M with complement N, choose a sequence of disjoint measurable sets $\{E_j\}_{j=1}^{\infty}$ in $[0, 1]$ and define

$$g_k = \frac{1}{N^{-1}\left(\frac{1}{\mu(E_k)}\right)\mu(E_k)}\chi_{E_k}, \qquad k = 1, 2, \cdots .$$

Each g_k has the property that $\|g_k\|_M = 1$ (cf [5]). To compute the norm of the difference of two of these, consider the function

$$h \equiv N^{-1}\left(\frac{1}{2\mu(E_k)}\right)\chi_{E_k} + N^{-1}\left(\frac{1}{2\mu(E_n)}\right)\chi_{E_n} .$$

Then

$$\int_X N(h) = \int_{E_k} \frac{1}{2\mu(E_k)}\chi_{E_k} + \int_{E_n} \frac{1}{2\mu(E_n)}\chi_{E_n} = 1$$

and hence $\|h\|'_N \leqq 1$. Now

$$\|g_k - g_n\|_M = \sup_{\|f\|'_N \leqq 1} \int_X |g_k - g_n|\,|f|\,d\mu$$

$$\geqq \int_X |g_k - g_h|\,|h| = \frac{N^{-1}\left(\frac{1}{2\mu(E_k)}\right)}{N^{-1}\left(\frac{1}{\mu(E_k)}\right)} + \frac{N^{-1}\left(\frac{1}{2\mu(E_n)}\right)}{N^{-2}\left(\frac{1}{\mu(E_n)}\right)} .$$

By choosing a subsequence we obtain

$$||g_k - g_n||_M \geqq 2 \liminf_{x\to\infty} \frac{N^{-1}(x)}{N^{-1}(2x)} .$$

Putting $f_k = (1 - r)g_k$, $k = 1, 2, \cdots$, it follows that $||f_k|| = 1 - r$ and

$$||f_k - f_n|| \geqq (1 - r)2 \liminf_{x\to\infty} \frac{N^{-1}(x)}{N^{-1}(2x)} .$$

Setting

$$\beta = \liminf_{x\to\infty} \frac{N^{-1}(x)}{N^{-1}(2x)} ,$$

the inequalities (4.1) and (4.2) will be satisfied provided $(1-r)2\beta \geqq 2r$ or $r \leqq 1/(1 + 1/\beta)$. This example shows that $\Lambda_M \geqq 1/(1 + 1/\beta)$ and leads to the following theorem.

THEOREM 4.2. *$L_M[0, 1]$ be an Orlicz space with N the complement of M and set $M_s^{-1} = (M^{-1})^{1-s}(M_0^{-1})^s$ where $M_0(x) = x^2$, $0 \leqq s \leqq 1$. Then with*

$$\beta = \liminf_{x\to\infty} \frac{N_s^{-1}(x)}{N_s^{-1}(2x)} ,$$

$$\frac{1}{1 + 1/\beta} \leqq \Lambda_{M_s} \leqq \frac{1}{1 + 2^{s/2}} . \tag{4.3}$$

Furthermore, if $1/(1 + 2^{s/2}) < r < 1$ then at most a finite number $\Gamma_{M_s}(r)$ of balls of radius r can be packed in B_1 and that number satisfies

$$\Gamma_{M_s}(r) \leqq \left[1 - 1/2\left(\frac{1 - r}{r}\right)^{2/s} \right]^{-1} . \tag{4.4}$$

Proof. It remains to show (4.4) and the right hand side of (4.3). Suppose there are n disjoint balls of radius r with centers $f_1, f_2, \cdots, f_n$ packed in B_1. Then by Theorem 3.2,

$$\sum_{i,j=1}^{n} c_i c_j ||f_i - f_j||_{M_s}^{2/(2-s)} \leqq 2\gamma \frac{2(1 - s)}{2 - s} \sum_{i=1}^{n} c_i ||f_i||_{M_s}^{2/(2-s)} \tag{4.5}$$

for any collection $c_1, c_2, \cdots, c_n$ of positive numbers such that $\sum_{i=1}^{n} c_i = 1$. In particular, if $c_i = 1/n$, $i = 1, 2, \cdots, n$ then $\gamma = 1 - 1/n$ and (4.5) reduces to

$$(4.6) \qquad \sum_{i,j=1}^{n} \frac{1}{n^2} \|f_i - f_j\|_{M_s}^{2/(2-s)} \leqq 2\left(1 - \frac{1}{n}\right)^{2(1-s)/(2-s)} \sum_{i=1}^{n} \frac{1}{n} \|f_i\|_{M_s}^{2/(2-s)}$$

since the balls are disjoint, $\|f_i - f_j\| \geqq 2r$, $i \neq j$, and $\|f_i\| \leqq 1 - r$. Hence (4.6) implies

$$\frac{1}{n^2} n(n-1)(2r)^{2/(2-s)} \leqq 2\left(\frac{n-1}{n}\right)^{2(1-s)/(2-s)} \frac{1}{n} \cdot n(1-r)^{2/(2-s)} .$$

This inequality then reduces to

$$(4.7) \qquad r \leqq \frac{1}{1 + 2^{s/2}\left(\dfrac{n-1}{n}\right)^{s/2}} .$$

If we allow $n \to \infty$, the right hand side of (4.3) is obtained. The inequality (4.4) follows by solving (4.7) for n.

In the case when M and N have continuous derivatives, proposition 2.1 gives a lower bound in terms of M_s.

COROLLARY 4.3. *Let M and N be complementary Orlicz functions with M satisfying the Δ_2-condition. If M and N have continuous derivatives and $M_s^{-1} = (M^{-1})^{1-s}(M_0^{-1})^s$, $0 \leqq s \leqq 1$, then*

$$(4.8) \qquad \frac{1}{1 + 2^{(\alpha-1)/\alpha}} \leqq \Lambda_{M_s} \leqq \frac{1}{1 + 2^{s/2}}$$

where

$$\alpha = \limsup_{x \to \infty} \frac{x M_s'(x)}{M_s(x)} .$$

If we set $M(x) = x^p$ and use a proof similar to Corollary 3.3, the exact value $\Lambda_M \equiv \Lambda_p$ is obtained for L^p, $1 \leqq p \leqq 2$.

COROLLARY 4.4. *Let $1 \leqq p \leqq 2$. Then $\Lambda_p = 1/(1 + 2^{1-1/p})$ for the space $L_p(\mu)$.*

This holds for any measure space because, for $M(x) = x^p$ in the example preceding Theorem 4.2, $N^{-1}(2x)/N^{-1}(x) = 2^{1-1/p}$ for all x.

The upper bounds are independent of the measure space but not the lower. Corollary 4.3 does not give the exact number for $2 < p < \infty$ but gives a lower bound which was shown in [1] to be exact for l_p. However, it is demonstrated in [10] that the number in $L_p[0, 1]$, $2 < p < \infty$ is $1/(1 + 2^{1/p})$. A simple generalization of this gives us new lower bounds in Orlicz spaces.

For each positive integer n and each integer j, $0 < j \leqq 2^n$, define $E_{nj} = ((j-1)/2^n, \ j/2^n)$. Now for each integer n, define the function g_n by

$$g_n = \frac{1}{N^{-1}(1)} \sum_{k=1}^{2^n} (-1)^{k+1} \chi_{E_{nk}}$$

where N is the complementary function of the Orlicz function M and $\chi_{E_{nk}}$ is the characteristic function of the set E_{nk}. Then $\|g_n\|_M = 1$ for each n and $\|g_n - g_m\| = N^{-1}(2)/N^{-1}(1)$, $n \neq m$. Consider the spheres $S_r(f_j)$, $j = 1, 2, \cdots$ with centers $f_j = (1-r)g_j$. Thus $\|f_i\| = 1 - r$ and $\|f_j - f_k\| = (1-r)N^{-1}(2)/N^{-1}(1)$. The inequalities (4.1) and (4.2) will be satisfied provided $(1-r)N^{-1}(2)/N^{-1}(1) \geqq 2r$ or $r \leqq 1/(1 + 2N^{-1}(1)/N^{-1}/N^{-1}(2))$.

THEOREM 4.4. *Let $L_M[0,1]$ be an Orlicz space and set $M_s^{-1} = (M^{-1})^{1-s}(M_s^{-1})^s$ where $\phi_0(x) = x^2$ and $0 \leqq s \leqq 1$. If N_s is the complementary function to M_s, then*

$$\frac{1}{1 + 2\dfrac{N_s^{-1}(1)}{N_s^{-1}(2)}} \leqq \Lambda_{M_s} \leqq \frac{1}{1 + 2^{s/2}} \, .$$

The example constructed above does not depend on $[0, 1]$ but rather on being able to find sets E_{nj} with the same properties. However, for the L_p spaces the construction on $[0, 1]$ is enough and theorem 16.2 in [10] generalizes to the following.

COROLLARY 4.5. *Let $2 \leqq p < \infty$ and μ be any measure which is not purely atomic. Then for $L_p(\mu)$,*

$$\Lambda_p(\mu) = \frac{1}{1 + 2^{1/p}} \, .$$

Proof. For the space $L_p[0, 1]$, the usual argument gives the result. It is known ([3] or [9]) that if μ is not purely atomic, $L_p(\mu)$ has a subspace isometric to $L_p[0, 1]$. Suppose there are infinitely many balls of radius r in $L_p[0, 1]$ then there is a sequence of points satisfying inequalities (4.1) and (4.2) in the subspace and hence in $L_p(\mu)$. Thus the lower bound for Λ in $L_p(\mu)$ is greater than or equal to Λ_p and since the upper bound is independent of the measure, the result follows.

The problem of embedding $L_p[0, 1]$ into $L_r[0, 1]$ has been studied extensively and it has been shown [cf 3] that for $1 \leqq r \leqq p < 2$, $L_p[0, 1]$ is isometric to a subspace of $L_r[0, 1]$. More recently Nielsen [6] has given conditions under which $L_M(0, \infty)$ is isomorphic to a subspace $L_p[0, 1]$. Also the Khintchin inequality implies l_2 is isomorphic to a subspace of $L_M[0, 1]$ for every Orlicz function M and furthermore l_2 is actually isometric to a subspace of $L_p[0, 1]$ for every $p, 1 \leqq p < \infty$. Consistent with these results is the following.

THEOREM 4.6. *Let M_1 and M_2 be Orlicz functions and suppose L_{M_2} is isometric to a subspace of L_{M_1}. Then $\Lambda_{M_2} \geqq \Lambda_{M_1}$. In particular if l_2 is isometric to a subspace of L_M then $1/(1 + \sqrt{2}) \leqq \Lambda_M \leqq 1/2$.*

A converse to this theorem would be of interest. A reasonable conjecture might be to try to show that if $[\alpha_{M_1}, \alpha_{M_2}] < [\alpha_{M_1}, \alpha_{M_2}] < 2$ (see [3] for definitions) and $\Lambda_{M_2} \geqq \Lambda_{M_1}$ then $L_{M_1}[0, 1]$ is isometric to a subspace of $L_{M_2}[0, 1]$.

For the sequence case, the situation is different. Using the example preceding Theorem 4.2 with each E_k a singleton, it follows that $\lambda_M \geqq 1/(1 + (N^{-1}(1)/N^{-1}(1/2))$. The proof in [1] for l_p depends on the strong property

$$M^{-1}\left(\sum_{j=1}^{\infty} M(x_j)\right) = \|X\|_M \tag{4.9}$$

where $M(x) = x^p/p$. If we mimic their proof the following is obtained.

THEOREM 4.7. *Let M and N be complementary functions both satisfying the Δ_2-condition at 0, and M satisfies (4.9). Then for the space l_M,*

$$\frac{1}{1 + \dfrac{N^{-1}(1)}{N^{-1}\left(\dfrac{1}{2}\right)}} \leqq \lambda_M \leqq \frac{1}{1 + \dfrac{2}{M^{-1}(2M(1))}}\,.$$

Furthermore, if $N^{-1}(1/2) \leqq 1/2\, N^{-1}(1)M^{-1}(2M(1))$, then for

$$\frac{1}{1 + \dfrac{2}{M^{-1}(2M(1))}} < r \leqq \frac{1}{1 + 2\dfrac{N^{-1}(1)}{N^{-1}(2)}}\,,$$

any finite number of spheres of radius r can be packed in the unit ball of l_M but not an infinite number.

Proof. The Δ_2-condition on both M and N is equivalent to reflexivity. Now suppose there are an infinite number of balls of radius r in l_M with centers $y_1, y_2, \cdots$ satisfying inequalities (4.1) and (4.2). Assume y is the weak limit point of $\{y_j\}$, then $y \in l_M$ and $\|y\| \leqq 1 - r$. Let $\varepsilon > 0$ and fix a positive integer n. Then there exists N such that $\|\tilde{y}_k\| < \varepsilon(1 - r)$ where

$$\tilde{y}_{kj} = \begin{Bmatrix} 0 & j \leqq N \\ y_{kj} & j > N \end{Bmatrix} .$$

Then

$$M\left(\frac{2r}{1-r}\right) \leqq M\left(\frac{\|y_n - y_m\|}{1-r}\right) = \sum_{j=1}^{\infty} M\left(\frac{|y_{nj} - y_{mj}|}{1-r}\right)$$
$$= \sum_{j=1}^{N} M\left(\frac{|y_{nj} - y_{mj}|}{1-r}\right) + \sum_{j>N} M\left(\frac{|y_{nj} - y_{mj}|}{1-r}\right) .$$

Now

$$M^{-1}\left(\sum_{j>N} M\left(\frac{|y_{nj} - y_{mj}|}{1-r}\right)\right) = \left\|\frac{y_n - \tilde{y}_m}{1-r}\right\|$$
$$\leqq \frac{\|y_m\|}{1-r} + \frac{\|\tilde{y}_n\|}{1-r} \leqq 1 + \varepsilon .$$

Thus

$$M\left(\frac{2r}{1-r}\right) \leqq \sum_{j=1}^{N} M\left(\frac{|y_{nj} - y_{mj}|}{1-r}\right) + M(1 + \varepsilon) .$$

This argument is independent of m and hence

$$M\left(\frac{2r}{1-r}\right) - M(1 + \varepsilon) \leqq \sum_{j=1}^{N} M\left(\frac{|y_{nj} - y_j|}{1-r}\right) .$$

Letting $N \to \infty$ and $\varepsilon \to 0$, this becomes

$$M\left(\frac{2r}{1-r}\right) - M(1) \leqq \sum_{j=1}^{\infty} M\left(\frac{|y_{nj} - y_j|}{1-r}\right) .$$

Repeat the argument using y in place of y_n obtaining

$$M\left(\frac{2r}{1-r}\right) - M(1) \leqq \sum_{j=1}^{N'} M\left(\frac{|y_{nj} - y_j|}{1-r}\right) + M(1 + \varepsilon) .$$

Now, letting $n \to \infty$ and $\varepsilon \to 0$, it follows that

$$M\left(\frac{2r}{1-r}\right) \leqq 2M(1)$$

and

$$r \leqq \frac{1}{1 + \dfrac{2}{M^{-1}(2M(1))}}.$$

The last statement follows by constructing the example preceding Theorem 4.4 on the set $(1, 2, \cdots, 2^n)$ in place of $[0, 1]$.

COROLLARY 4.8. *For* $1 \leqq p \leqq \infty$, $\lambda_p = 1/(1 + 2^{1-1/p})$ *for the spaces* l_p. *Furthermore if* $2 \leqq p < \infty$ *then for*

$$\frac{1}{1 + 2^{1-1/p}} < r \leqq \frac{1}{1 + 2^{1/p}},$$

any finite number of spheres of radius r *can be packed in* l_p *but not an infinite number.*

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