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THE GENERALIZED TRANSLATIONAL HULL OF A SEMIGROUP

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For a pair $(\mathscr{R}, \mathscr{L})$ consisting of a right quotient filter $\mathscr{R}$ and left quotient filter $\mathscr{L}$ on the semigroup S, a translational hull $\Omega(S: \mathscr{R}, \mathscr{L})$ is constructed. The results of Grillet and Petrich hold for $\Omega(S: \mathscr{R}, \mathscr{L})$.

Specializing $\mathscr{R}$ and $\mathscr{L}$ one obtains the usual translational hull $\Omega(S)$ of S and the semigroup of quotients $Q(S)$ of S due to Hinkle and McMorris. These results are applied to a weakly reductive semigroup S to show that $\Omega(S) = \Omega(S^n)$ for any positive integer n.

In recent years two seemingly unconnected developements have occurred in the theory of semigroups. Grillet and Petrich [4] have studied ideal extensions of a semigroup S by means of a universal extension $\Omega(S)$ of S, the translational hull of S. On the other hand, McMorris [7] and Hinkle [5] have developed a theory of one-sided semigroups of quotients of S using a maximal semigroup $Q(S)$ of quotients. Under certain conditions $Q(S)$ is essential over S while under similar conditions $\Omega(S)$ is a congruence dense extension of S. Berthiaume [1] showed that congruence dense extension and essential extension are the same concept. This similarity, along with many others, between $Q(S)$ and $\Omega(S)$ leads one to suspect the existence of a theory of semigroup extensions more general than the above mentioned theories. In this paper we offer a candidate for a general theory. In section three we show that our concept might reasonably be called a two-sided semigroup of quotients, having given our constructions in sections one and two. In section five we show that our construction yields an essential extension of S maximal in a certain category. In section four, we follow the lead of Grillet and Petrich [4] and examine extensions of S which are somewhat weaker than essential extensions. Along the way we obtain as corollaries some results of Hinkle [5] and Grillet and Petrich [4].

1. Basic definitions. In this paper, S will be a semigroup with zero, denoted by 0.

A left *S-set* ${}_SK$ is a set K, with a distinguished element $\mathscr{O}$, having a scalar multiplication $S \times K \to K$ satisfying for all $s, t \in S$ and $k \in K$, $(st)k = s(tk)$, and $0s = \mathscr{O}$ and $k\mathscr{O} = \mathscr{O}$.

Similarly one can define a right S-set K_S (with $\mathscr{O}$).

In this paper we will not distinguish between $\mathscr{O}$, the zero of K

and 0, the zero of S. The meaning of the symbol 0 will be clear from the context.

An (S, S)-*set* ${}_SK_S$ is a set K with scalar multiplications $S \times K \to K$ and $K \times S \to K$ such that ${}_SK$ is a left S-set, K_S is a right S-set, and for $s, t \in S$ and $k \in K$, $s(kt) = (sk)t$.

A *homomorphism* $\rho: {}_SK \to {}_SM$ of *left S-sets* is a mapping $\rho: K \to M$ satisfying $(sk)\rho = s(k\rho)$ for all $s \in S$ and $k \in K$.

Similarly one defines "homomorphism of right S-sets" and "homomorphism of (S, S)-sets". We write left S-homomorphisms on the right of their argument and right S-homomorphisms on the left.

${}_SK$ is a sub S-set of ${}_SM$ if $K \subseteq M$ and K is an S-set under the operation $S \times M \to M$.

DEFINITION (Hinkle [5]). A collection $\mathscr{R}$ of right ideals of S is a *right quotient* filter on S if

(1) if A, B are right ideals of S, $A \subseteq B$ and $A \in \mathscr{R}$, then $B \in \mathscr{R}$

(2) if $A, B \in \mathscr{R}$ and $\lambda: A \to S$ is a right S-set homomorphism, then $\lambda^{-1}(B) = \{a \in A \mid \lambda a \in B\} \in \mathscr{R}$.

Hinkle has shown that a right quotient filter is closed under finite intersections and if $A \in \mathscr{R}$ and $s \in S$, then

$$s^{-1}A = \{t \in S \mid st \in A\} \in \mathscr{R}\,.$$

An extension M_S of K_S is an $\mathscr{R}$-*extension* if for $m \in M$,

$$m^{-1}K = \{s \in S \mid ms \in K\} \in \mathscr{R}\,.$$

Dually one can define a left quotient filter $\mathscr{L}$ on S and "$\mathscr{L}$-extension."

An (S, S)-set ${}_SV_S$ is an $(\mathscr{R}, \mathscr{L})$-*extension* of ${}_SK_S$ if V_S is an $\mathscr{R}$-extension of K_S and ${}_SV$ is an $\mathscr{L}$-extension of ${}_SK$.

A right quotient filter $\mathscr{R}$ is *idempotent* if whenever $A \in \mathscr{R}$, I is a right ideal of S and $a^{-1}I \in \mathscr{R}$ for all $a \in A$, then $I \in \mathscr{R}$.

This condition is equivalent to the condition: if $A \in \mathscr{R}$ and for each $a \in A$ there is $R_a \in \mathscr{R}$, then

$$\bigcup_{a \in A} aR_a \in \mathscr{R}\,.$$

2. The construction. Let $\mathscr{L}$ be a left quotient filter on S, $\mathscr{R}$ be a right quotient filter on S, and K be an (S, S)-set. Consider all pairs (λ, ρ) where $\lambda: D_\lambda \to K$ is a right S-homomorphism with domain $D_\lambda \in \mathscr{R}$ and $\rho: D_\rho \to K$ is a left S-homomorphism with domain $D_\rho \in \mathscr{L}$.

DEFINITION 2.1. The pair (λ, ρ) is $(\mathscr{R}, \mathscr{L})$-*linked* if for all

$y \in D_\lambda$ and $x \in D_\rho$, $x(\lambda y) = (x\rho)y$.

Let $B(K: \mathscr{R}, \mathscr{L})$ be the collection of all such $(\mathscr{R}, \mathscr{L})$-linked pairs. Notice that for $k \in K$, the maps $\lambda_k: S \to K$ defined by $\lambda_k(s) = ks$ and $\rho_k: S \to K$ defined by $(s)\rho_k = sk$ given an $(\mathscr{R}, \mathscr{L})$-linked pair $(\lambda_k, \rho_k) \in B(K: \mathscr{R}, \mathscr{L})$. Moreover, $B(K: \mathscr{R}, \mathscr{L})$ is an (S, S)-set under the operation $s(\lambda, \rho) = (s\lambda, s\rho)$ where $s\lambda: D_\lambda \to K$ is given by $s\lambda(t) = s[\lambda(t)]$ and $s\rho: D_{s\rho} \to K$ is given by $(t)s\rho = (ts)\rho$ where $D_{s\rho} = (D_\rho)s^{-1} \in \mathscr{L}$. $(s\lambda, s\rho)$ is linked since for $y \in D_\lambda$ and $x \in D_{s\rho}$,

$$\begin{aligned} x((s\lambda)y) &= x[s(\lambda y)] = (xs)(\lambda y) = [(xs)\rho]y \\ &= [x(s\rho)]y \end{aligned}$$

since $xs \in D_\rho$ and (λ, ρ) is linked. The definition of ρs and λs and the multiplication $(\lambda, \rho)s = (\lambda s, \rho s)$ is similar.

Where $K = S$, then $B(K: \mathscr{R}, \mathscr{L})$ is a partial transformation semigroup.

Define a relation Θ on $B(K: \mathscr{R}, \mathscr{L})$ by $(\lambda, \rho)\Theta(\lambda', \rho')$ iff

(1) there is $R \in \mathscr{R}$ with $R \subseteq D_\lambda \cap D_{\lambda'}$ and $\lambda r = \lambda' r$ for all $r \in R$, and

(2) there is $L \in \mathscr{L}$ with $L \subseteq D_\rho \cap D_{\rho'}$ and $t\rho = t\rho'$ for all $t \in L$.

LEMMA 2.2. *Θ is an (S, S)-congruence on $B(K: \mathscr{R}, \mathscr{L})$.*

COROLLARY 2.3. *Θ is also a semigroup congruence on $B(S: \mathscr{R}, \mathscr{L})$.*

The straightforward proof of the above lemma and its corollary will be omitted.

The quotient (S, S)-set $B(K: \mathscr{R}, \mathscr{L})/\Theta$ will be denoted by $\Omega(K: \mathscr{R}, \mathscr{L})$ and is called the *$(\mathscr{R}, \mathscr{L})$-translational hull* of K.

We usually denote the class of (λ, ρ) in $\Omega(K: \mathscr{R}, \mathscr{L})$ by (λ, ρ), but when clarification is needed, we denote it by $[\lambda, \rho]$.

There is a canonical (S, S) homomorphism π of K into $\Omega(K: \mathscr{R}, \mathscr{L})$ given by $\pi(k) = (\lambda_k, \rho_k)$. When $K = S$, π is a semigroup homomorphism.

DEFINITION 2.4. If the homomorphism $\pi: K \to \Omega(K: \mathscr{R}, \mathscr{L})$ is injective, K is said to be *$(\mathscr{R}, \mathscr{L})$-reductive.*

REMARKS. (1) When $\mathscr{L}$ is the collection of all left ideals of S, then $\Omega(S: \mathscr{R}, \mathscr{L})$ is semigroup isomorphic to $Q_{\mathscr{R}}(S)$, the semigroup of right quotients of S developed by Hinkle [5].

Proof. The map $\sigma: \Omega(S: \mathscr{R}, \mathscr{L}) \to Q_{\mathscr{R}}(S)$ given by $\sigma[\lambda, \rho] = [\lambda]$ is the desired isomorphism. Since $\sigma[\lambda, \rho] = \sigma[\lambda', \rho']$ implies $[\lambda] = [\lambda']$, λ and λ' agree on some $R \in \mathscr{R}$ and so $[\lambda, \rho] = [\lambda', \rho']$ since ρ and ρ' agree on $(0) \in \mathscr{L}$ and so σ is injective. Moreover for $[\lambda] \in Q_{\mathscr{R}}(S)$, $\sigma[\lambda, 1_S] = [\lambda]$ where $1_S: S \to S$ is the identity map on S and σ is surjective.

To see that σ is a homomorphism, let $[\lambda, \rho]$, $[\lambda', \rho'] \in \Omega(S: \mathscr{R}, \mathscr{L})$. Then $\sigma([\lambda, \rho][\lambda', \rho']) = \sigma([\lambda\lambda', \rho\rho']) = [\lambda\lambda'] = [\lambda][\lambda'] = \sigma([\lambda, \rho])\sigma([\lambda', \rho'])$ where $\lambda\lambda': D'_\lambda \cap \lambda'^{-1}D_\lambda \to S$ and $\rho'\rho: D_{\rho'} \cap (\rho')^{-1}D_\rho \to S$. Thus σ is a semigroup isomorphism.

(2) Similarly, if $\mathscr{R}$ is the collection of all right ideals of S, the mapping $\beta: \Omega(S: \mathscr{R}, \mathscr{L}) \to Q_{\mathscr{L}}(S)$ given by $\beta([\lambda, \rho]) = [\rho]$ is a semigroup isomorphism from $\Omega(S: \mathscr{R}, \mathscr{L})$ onto the semigroup $Q_{\mathscr{L}}(S)$ of left quotients of S developed by Hinkle [5].

(3) If $\mathscr{R} = \mathscr{L} = \{S\}$, then $\Omega(S: \mathscr{R}, \mathscr{L}) = \Omega(S)$, the translational hull of S.

(4) Where $\mathscr{R}$ is the collection of all right ideals of S and $\mathscr{L}$ is the collection of all left ideals of S, then $\Omega(K: \mathscr{R}, \mathscr{L})$ is trivial since $(\lambda, \rho)\Theta(\lambda', \rho')$ for all (λ', ρ'), $(\lambda, \rho) \in B(K: \mathscr{R}, \mathscr{L})$ since λ and λ' agree on $(0) \in \mathscr{R}$ and ρ and ρ' agree on $(0) \in \mathscr{L}$.

(5) $\Omega(S: \mathscr{R}, \mathscr{L})$ always has an identity.

PROPOSITION 2.5. *$\Omega(K: \mathscr{R}, \mathscr{L})$ is an $(\mathscr{R}, \mathscr{L})$-extension of $\pi(K)$.*

Proof. We will show that $\Omega(K: \mathscr{R}, \mathscr{L})$ is an $\mathscr{L}$-extension of $\pi(K)$. Since the $\mathscr{R}$-extension part is similar, it will be left to the reader.

Let $[\lambda, \rho] \in \Omega(K: \mathscr{R}, \mathscr{L})$ with $D_\rho \in \mathscr{L}$. For $s \in D_\rho$, $s[\lambda, \rho] = [s\lambda, s\rho]$. Now $D_{s\rho} = D_\rho s^{-1} \in \mathscr{L}$ and for $t \in D_{s\rho}$, $t(s\rho) = (ts)\rho$ and since $s\rho \in K$, $s\rho = \rho_{s\rho}$. On the other hand, for $t \in D_{s\lambda} = D_\lambda$, $s\lambda(t) = s(\lambda t) = (s\rho)t = \lambda_{s\rho}(t)$. Since $s\rho \in K$, $s[\lambda, \rho] = [\lambda_{s\rho}, \rho_{s\rho}] \in \pi(K)$.

COROLLARY 2.6. *When K is $(\mathscr{R}, \mathscr{L})$-reductive, $\Omega(K: \mathscr{R}, \mathscr{L})$ is an $(\mathscr{R}, \mathscr{L})$-extension of K.*

When S is a semigroup, we are interested in the idealizer of $\pi(S)$ in $\Omega(S: \mathscr{R}, \mathscr{L})$.

PROPOSITION 2.6. *If S is $(\mathscr{R}, \mathscr{L})$-reductive, then the idealizer of $\pi(S)$ in $\Omega(S: \mathscr{R}, \mathscr{L})$ is*

$$T = \{[\lambda, \rho]: D_\lambda = D_\rho = S\} .$$

Proof. Note first that if $s \in D_\lambda$, and (λ, ρ) are linked, then for

$t \in s^{-1}D_\lambda$,

$$\lambda s(t) = \lambda(st) = (\lambda s)(t) = \lambda_{\lambda s}(t)$$

and for $t \in D_\rho$ we have

$$(t)(\rho s) = (t\rho)s = t(\lambda s) = (t)\rho_{\lambda s} .$$

Thus if $(\lambda, \rho) \in T$, $s \in D_\lambda$ we have

$$(\lambda, \rho)\pi(s) = (\lambda, \rho)(\lambda_s, \rho_s) = (\lambda s, \rho s) = (\lambda_{\lambda s}, \rho_{\lambda s}) = \pi(\lambda s) .$$

Similarly if (λ, ρ) are linked and $s \in D_\rho$, then $(t)s\rho = (t)\rho_{s\rho}$ and $s\lambda(t) = \lambda_{s\rho}(t)$.

Thus

$$\pi(s)(\lambda, \rho) = \pi(s\rho) \quad \text{for} \quad (\lambda, \rho) \in T .$$

Therefore, T is contained in the idealizer of $\pi(S)$.

Conversely, let (λ, ρ) belong the idealizer of $\pi(S)$ in $\Omega(S\colon \mathscr{R}, \mathscr{L})$. Then for all $s \in S$, $(\lambda, \rho)\pi(s)$ and $\pi(s)(\lambda, \rho)$ belong to $\pi(S)$. We consider $(\lambda, \rho)\pi(s)$ since the other case is similar. Now $(\lambda, \rho)\pi(s) = \pi(t)$ for some $t \in S$. Define $\lambda'\colon S \to S$ by $\lambda'(s) = t$ if $(\lambda, \rho)\pi(s) = \pi(t)$. Note that λ' is well defined since π is injective. Moreover λ' agrees with λ on D_λ since if $s \in D_\lambda$, $(\lambda, \rho)\pi(s) = \pi(\lambda s)$. It remains to show that λ' is a right S-homomorphism for then $(\lambda, \rho)\Theta(\lambda', \rho)$. Similarly define $\rho'\colon S \to S$ by $(s)\rho' = t$ iff $\pi(s)(\lambda', \rho) = \pi(s)(\lambda, \rho) = \pi(t)$. Then $(\lambda, \rho)\Theta(\lambda', \rho')$ and $(\lambda', \rho') \in T$.

Suppose $\lambda'(s) = t$ and $\lambda'(sx) = a$. We consider two cases. First, if $s \in D_\lambda$, then $\lambda'(s) = \lambda s = t$ and $a = \lambda'(sx) = \lambda(sx) = (\lambda s)x = tx$ and so $\lambda'(s)x = \lambda'(sx)$. Next, if $s \notin D_\lambda$, then for $z \in x^{-1}s^{-1}D_\lambda = (sx)^{-1}D_\lambda$,

$$az = (\lambda sx)[z] = \lambda[s(xz)] = \lambda s[xz] = t[xz] = (tx)z .$$

Moreover, for $y \in D_\rho$

$$ya = y(\rho sx) = [(y\rho)s]x = [y\rho s]x = (yt)x = y(tx)$$

thus $\pi(a) = \pi(tx)$ or $a = tx$ or $\lambda'(sx) = \lambda'(s)x$.

COROLLARY 2.8. *When $\mathscr{L} = \mathscr{R} = \{S\}$, and S is $(\mathscr{R}, \mathscr{L})$ reductive then $\pi(S)$ is an ideal of $\Omega(S\colon \mathscr{R}, \mathscr{L})$.*

3. Two sided semigroup of quotients. In the last section we show that $\Omega(S\colon \mathscr{R}, \mathscr{L})$ can be the translational hull or a semigroup of quotients of S. In this section we show that $\Omega(S\colon \mathscr{R}, \mathscr{L})$ can naturally be considered as a two-sided semigroup of quotients of S.

DEFINITION 3.1. Let V be an $(\mathscr{R}, \mathscr{L})$ extension of K. For each

$a \in V$, define $\lambda^a, \rho^a, \tau^a = [\lambda^a, \rho^a]$ where $\lambda^a: a^{-1}K \to K$ is given by $\lambda^a(d) = ad$ and $\rho^a: Ka^{-1} \to K$ is given by $(d)\rho^a = da$.

THEOREM 3.2. *If V is an $(\mathscr{R}, \mathscr{L})$ extension of K, the mapping $\tau: a \to \tau^a$ is a canonical (S, S)-homomorphism of V into $\Omega(K: \mathscr{R}, \mathscr{L})$ which extends the canonical homomorphism π of K into $\Omega(K: \mathscr{R}, \mathscr{L})$.*

The proof of the above theorem is straight forward and will be omitted. When there is any danger of confusion, we will denote $\tau: V \to \Omega(K: \mathscr{R}, \mathscr{L})$ by $\tau(V: K)$.

DEFINITION 3.3. The $(\mathscr{R}, \mathscr{L})$*-congruence* on an (S, S)-set M is denoted by η_M and defined by

$$\eta_M = \{(m, n) \mid tm = tn \text{ for all } t \text{ in some } L \in \mathscr{L} \text{ and } mr = nr \text{ for all } r \text{ in some } R \in \mathscr{R}\} .$$

When the filters $\mathscr{R}$ and $\mathscr{L}$ are to be stressed, we write

$$\eta_M = \eta(M: \mathscr{R}, \mathscr{L}) .$$

LEMMA 3.4. *The $(\mathscr{R}, \mathscr{L})$ congruence on K is $\pi \circ \pi^{-1}$ where π is the canonical homomorphism of K into $\Omega(K: \mathscr{R}, \mathscr{L})$.*

COROLLARY 3.5. *K is $(\mathscr{R}, \mathscr{L})$-reductive iff η_K is identity congruence.*

In order to determine when τ is the unique homomorphism, extending π we use the following item.

LEMMA 3.6. *If K is $(\mathscr{R}, \mathscr{L})$-reductive or both $\mathscr{R}$ and $\mathscr{L}$ are idempotent, then the $(\mathscr{R}, \mathscr{L})$-congruence on K is the identity.*

Proof. Suppose $\omega, \omega' \in \Omega(K: \mathscr{R}, \mathscr{L})$ and $(\omega, \omega') \in \eta_{\Omega(K: \mathscr{R}, \mathscr{L})}$ where $\omega = [\lambda, \rho]$ and $\omega' = [\lambda', \rho']$. Then for all $x \in D_\lambda \cap D_{\lambda'} \cap D$ and $s \in S$, (where $\omega x = \omega' x$ for all $x \in D \in \mathscr{R}$),

$$\lambda(xs) = \lambda\lambda_x(s) = \lambda'\lambda_x(s) = \lambda'(xs)$$

and for all $y \in D_\rho \cap D_{\rho'} \cap D'$ and $s \in S$, (where $d\omega = d\omega'$ for $d \in D' \in \mathscr{L}$)

$$(sy)\rho = (s)\rho_y\rho = (s)\rho_y\rho' = (sy)\rho' .$$

If $\mathscr{L}$ and $\mathscr{R}$ are idempotent, there is $B^2 \in \mathscr{L}$, $A^2 \in \mathscr{R}$ with $A^2 \subseteqq D_\lambda \cap D_{\lambda'} \cap D$ and $B^2 \subseteqq D_\rho \cap D_{\rho'} \cap D'$ and so λ and λ' agree on A^2 while ρ and ρ' agree on B^2, thus $(\lambda, \rho)\Theta(\lambda', \rho')$ and $\omega = \omega'$ in $\Omega(K: \mathscr{R}, \mathscr{L})$.

On the other hand if $x \in D_\lambda \cap D_{\lambda'} \cap D$ and $y \in D_\rho \cap D_{\rho'} \cap D'$, then

$$y(\lambda x) = (y\rho)x = (y)\rho\rho_x = (y)\rho'\rho_x = (y\rho')x = y(\lambda' x)$$

and similarly $(y\rho)x = (y\rho')x$. Consequently if K is $(\mathscr{R}, \mathscr{L})$-reductive, $y\rho = y\rho'$ and $\lambda x = \lambda' x$ for all $y \in D_\rho \cap D_{\rho'}$ and $x \in D_\lambda \cap D_{\lambda'}$ and so $(\lambda, \rho)\Theta(\lambda', \rho')$ and $\omega = \omega'$.

PROPOSITION 3.7. *If K is $(\mathscr{R}, \mathscr{L})$ reductive or both $\mathscr{R}$ and $\mathscr{L}$ are idempotent, then $\tau(V: K)$ is the unique (S, S)-homomorphism of V into $\Omega(K: \mathscr{R}, \mathscr{L})$ extending π.*

The proof of this proposition is a simple modification of the proof of Proposition 1.3 of [4] and so will be omitted.

PROPOSITION 3.8. *If V and V' are $(\mathscr{R}, \mathscr{L})$-extensions of K and ϕ is an (S, S)-homomorphism of V into V' which is the identity on K, then*

$$\tau(V: K) = \tau(V': K) \circ \phi .$$

Proof. Let $v \in V$, $x \in v^{-1}K$ and $y \in Kv^{-1}$, then $\phi(v)x = \phi(vx) = vx$ and $y\phi(v) = \phi(yv) = yv$. Thus $\rho_{\phi(v)}$ and ρ_v agree on $Kv^{-1} \in \mathscr{L}$ and $\lambda_{\rho(v)}$ and λ_v agree on $v^{-1}K \in \mathscr{R}$ and so the conclusion follows.

For a right S-set M and *rqf* $\mathscr{R}$, Hinkle [5] defined the $\mathscr{R}$-singular congruence $\eta(\mathscr{R})$ on M by

$$\eta(\mathscr{R}) = \{(m, n) \mid mr = nr \text{ for all } r \text{ in some } R \in \mathscr{R}\} = \eta(M: R)$$

Similarly for a left S-set N and *lqf* $\mathscr{L}$, there is an $\mathscr{L}$-singular congruence $\eta(\mathscr{L})$ on N. $\eta(\mathscr{R})$ is a right S-congruence and $\eta(\mathscr{L})$ is a left S-congruence.

LEMMA 3.9. *For an (S, S) set M, rqf $\mathscr{R}$ and lqf $\mathscr{L}$,*

$$\eta(M: \mathscr{R}, \mathscr{L}) = \eta(\mathscr{R}) \cap \eta(\mathscr{L}) .$$

The proof is straightforward as is the proof of the next lemma and hence both proofs are omitted.

LEMMA 3.10. *If one of $\eta(\mathscr{R})$ or $\eta(\mathscr{L})$ is the identity congruence, then M is $(\mathscr{R}, \mathscr{L})$-reductive.*

Given a right S-set K and a *rqf* $\mathscr{R}$, Hinkle [5] constructed a maximal right S-set of quotients $Q(K: \mathscr{R})$ of K as the S-set of all right S-homomorphisms with domain a member of $\mathscr{R}$ and codomain

K factored by the congruence $\lambda\Theta\lambda'$ iff $\lambda s = \lambda' s$ for all s in some $R \subseteqq D_\lambda \cap D_{\lambda'}$ where $R \in \mathscr{R}$. An S-homomorphism $\lambda: K \to Q(K: \mathscr{R})$ can be defined by $\lambda(k) = [\lambda_k]$ where $\lambda_k: S \to K$ is given by $s \to ks$. Then $\lambda \circ \lambda^{-1} = \eta(K: \mathscr{R})$ and λ is an injection of K into $Q(K: \mathscr{R})$ when $\eta(K: \mathscr{R})$ is the identity. Analogous results hold for a left S-set M and lqf $\mathscr{L}$. The maximal left S-set of quotients of M is denoted by $Q(M: \mathscr{L})$.

Now let S be a semigroup with zero, $\mathscr{L}$ be a lqf on S, $\mathscr{R}$ be a rqf on S and both $\eta(S: \mathscr{L})$ and $\eta(S: \mathscr{R})$ be the identity. Note that S is a right S-set, left S-set and (S, S)-set with respect to the semigroup multiplication. Since both $\eta(S: \mathscr{L})$ and $\eta(S: \mathscr{R})$ are the identity, S is $(\mathscr{R}, \mathscr{L})$ reductive, and so we identify S with $\pi(S) \subseteqq \Omega(S: \mathscr{R}, \mathscr{L})$, $\lambda(S) \subseteqq Q(S: \mathscr{R})$ and $\rho(S) \subseteqq Q(S: \mathscr{L})$.

Now $Q(S: \mathscr{R})$ is a semigroup under the multiplication $[\lambda_1][\lambda_2] = [\lambda_1 \circ \lambda_2]$ where $\lambda_1 \circ \lambda_2: \lambda_2^{-1}(D_{\lambda_1}) \to S$ is the composition map. Moreover the canonical map $\lambda: S \to Q(S: \mathscr{R})$ is a semigroup monomorphism. Let $V = \{q \in Q(S: \mathscr{R}) \mid Sq^{-1} \in \mathscr{L}\}$. Then V is the maximal subsemigroup of $Q(S: \mathscr{R})$ which is an $(\mathscr{R}, \mathscr{L})$ extension of S. Define $\phi: V \to Q(S: \mathscr{L})$ by $\phi(q) = q' = [\rho^q]$. Thus ϕ is a semigroup homomorphism which is the identity on S since $\rho^{q_1 q_2}$ agrees with $\rho^{q_1} \circ \rho^{q_2}$ on $Sq_2^{-1}q_1^{-1} \in \mathscr{L}$. Since ϕ is the identity on S, $\phi(sq) = s\phi(q) = sq'$ and so $\phi(V) = \{q' \in Q(S: \mathscr{L}): (q')^{-1}S \in \mathscr{R}\}$. Since ϕ is a monomorphism, we identify V with $\phi(V)$ and so $V = Q(S: \mathscr{R}) \cap Q(S: \mathscr{L})$.

Now $\tau(V: S) = \tau(V': S) \circ \phi$ by Proposition 3.8. Moreover, $\tau(V: S)$ is injective for if $\tau^{q_1} = \tau^{q_2}$, then $(\lambda^{q_1}, \rho^{q_1})\Theta(\lambda^{q_2}, \rho^{q_2})$ so there is $R \in \mathscr{R}$ with $q_1 r = q_2 r$ for all $r \in R$, thus $q_1 = q_2$. Recall $\tau(V: S)$ is the identity on S.

Finally we show that $\tau(V: S)$ is surjective. Let $[\lambda, \rho] \in \Omega(S: \mathscr{R}, \mathscr{L})$, then $q = [\lambda] \in Q(S: \mathscr{R})$ and $q' = [\rho] \in Q(S: \mathscr{L})$. It suffices to show that $q \in V$. To this end let $t \in D_\rho$, and $s \in D_\lambda$, then

$$(t\rho)s = t(\lambda s) = t(qs) = (tq)s \in S .$$

Thus $\lambda^{(tq)} = \lambda_{(t\rho)}$ on $D_\lambda \in \mathscr{R}$, and since $t\rho \in S$ for $t \in D_\rho$, $[\lambda_{t\rho}] = [\lambda^{(tq)}] = tq \in S$ for all $t \in D_\rho$. Thus $q \in V$. Clearly $\tau^q = [\lambda^q, \rho^q] = [\lambda, \rho]$ and so $\tau(V: S)$ is surjective.

We have proven the following result.

THEOREM 3.11. *When both $\eta(S: \mathscr{R})$ and $\eta(S: \mathscr{L})$ are the identity congruence, $\Omega(S: \mathscr{R}, \mathscr{L})$ is a semigroup isomorphic over S to $V' = \{q \in Q(S: \mathscr{L}) \mid q^{-1}S \in \mathscr{R}\}$ and $\{q \in Q(S: \mathscr{R}) \mid Sq^{-1} \in \mathscr{L}\} = V$.*

If we identify V with V' and $\Omega(S: \mathscr{R}, \mathscr{L})$, the above result shows that $\Omega(S: \mathscr{R}, \mathscr{L}) = Q(S: \mathscr{R}) \cap Q(S: \mathscr{L})$.

4. Strict and pure extensions. In the remaining sections essential extensions will play a large role. However, we would like information on $(\mathscr{R}, \mathscr{L})$-extensions V of K which fail to be essential. We can classify such extensions by their image under $\tau(V: K)$.

DEFINITION 4.1. The *type* of an $(\mathscr{R}, \mathscr{L})$-extension V of K is the image $T(V: K)$ of V under $\tau(V: K)$.

When K is $(\mathscr{R}, \mathscr{L})$-reductive, it is easily seen that the types of extensions correspond to the subsets T of $\Omega(K: \mathscr{R}, \mathscr{L})$ which are (S, S)-sets containing K. We first discuss the $(\mathscr{R}, \mathscr{L})$-extensions V of K which are, in some sense, as bad as possible—that is, for $v \in V$, there is $k \in K$ for which $xk = xv$ for all x in some member L of $\mathscr{L}$ and $ky = vy$ for all y in some $R \in \mathscr{R}$.

DEFINITION 4.2. An $(\mathscr{R}, \mathscr{L})$-extension V of K is *strict* if it has type $\pi(K)$.

When K is $(\mathscr{R}, \mathscr{L})$-reductive, we will characterize the strict $(\mathscr{R}, \mathscr{L})$-extensions of K by means of (partial) homomorphisms.

REMARK. In this and the following actions we pretty much follow the approach of Grillet and Petrich [4]. The proofs of many of the results are easy modifications of the proofs in [4] and so will be omitted.

DEFINITION 4.3. An (S, S)-set T is $(\mathscr{R}, \mathscr{L})$-trivial if for each $t \in T$, $0t^{-1} \in \mathscr{L}$ and $t^{-1}0 \in \mathscr{R}$.

An $(\mathscr{R}, \mathscr{L})$-extension V of K is called an $(\mathscr{R}, \mathscr{L})$-extension of K by T if T is (S, S) isomorphic to the factor (S, S)-set V/K. Notice that in this case T is $(\mathscr{R}, \mathscr{L})$-trivial.

Strict $(\mathscr{R}, \mathscr{L})$-extensions of $(\mathscr{R}, \mathscr{L})$-reductive S-sets by T may be characterized in terms of partial homomorphisms of T in the following sense:

DEFINITION 4.3. Let V be an $(\mathscr{R}, \mathscr{L})$-extension of K by T. The extension V is said to be determined by a partial homomorphism $\pi: T\backslash\{0\} V$ if for nonzero $a \in T$ and all $s \in S$,

$$a \circ s = \begin{cases} as & \text{if } as \neq 0 \\ \pi(a)s & \text{if } as = 0 \end{cases}$$

while

$$s \circ a = \begin{cases} sa & \text{if } sa \neq 0 \\ s\pi(a) & \text{if } sa = 0 \end{cases}$$

where $\circ$ is the scalar multiplication in V.

PROPOSITION 4.4. *An* $(\mathscr{R}, \mathscr{L})$*-extension* V *of* K *by* T *is determined by a partial homomorphism iff* K *is an* (S, S)*-retract of* V.

The proof of Petrich ([9], Proposition 2, p. 51) carries over verbati to this case.

PROPOSITION 4.5. *Each* $(\mathscr{R}, \mathscr{L})$*-extension determined by a partial homomorphism is strict.*

There is a converse to this proposition when K is $(\mathscr{R}, \mathscr{L})$-reductive.

THEOREM 4.6. *Let* K *be* $(\mathscr{R}, \mathscr{L})$*-reductive. Then each strict* $(\mathscr{R}, \mathscr{L})$*-extension of* K *is determined by a partial homomorphism.*

Proof. Let $\tau = \tau(V: K)$ where V is a strict extension of K. Then $\tau: V \rightarrow \pi(K)$ and since π is an isomorphism, $\pi^{-1} \circ \tau: V \rightarrow K$ is an (S, S)-homomorphism whose restriction to K is the identity.

COROLLARY 4.7. *Let* S *be an* $(\mathscr{R}, \mathscr{L})$*-reductive semigroup and* Q *be an* $(\mathscr{R}, \mathscr{L})$*-trivial semigroup. Then there is a strict* $(\mathscr{R}, \mathscr{L})$*-extension of* S *by* Q *iff there is a partial homomorphism of* $Q\backslash\{0\}$ *into* S.

Strict $(\mathscr{R}, \mathscr{L})$-extensions of K can be characterized as follows

PROPOSITION 4.8. *Let* V *be an* $(\mathscr{R}, \mathscr{L})$*-extension of* K. *If any* (S, S)*-homomorphism of* K *into another* (S, S)*-set can be extended to* V, *then* V *is a strict extension of* K. *The converse holds if* K *is* $(\mathscr{R}, \mathscr{L})$*-reductive.*

Proof. The identity map $\mathrm{id}: K \rightarrow K$ is an (S, S)-homomorphism and so has an extension $f: V \rightarrow K$. Thus K is a retract of V and so V is a strict extension of K.

Conversely, let K be $(\mathscr{R}, \mathscr{L})$-reductive and K be a retract of V. Let $\alpha: K \rightarrow T$ be an (S, S)-homomorphism. Then if $r: V \rightarrow K$ is the retraction, $r \circ \alpha: V \rightarrow T$ is the desired extension.

Finally, strict extensions shed some light on the structure of S.

PROPOSITION 4.9. *If every* $(\mathscr{R}, \mathscr{L})$*-extension of* S *is strict, then* S *has an identity.*

Proof. The extension S^1 obtained by adjoining an identity to S is an $(\mathscr{R}, \mathscr{L})$-extension since $S \in \mathscr{R} \cap \mathscr{L}$ and so is strict. Thus for some $c \in S$, $\tau^1 = \pi c$. Thus for $x \in S$,

$$1x = cx = x = xc = xe$$

since $D_\lambda 1 = D_\rho 1 = S$. Thus c is an identity for S.

At the opposite end of the spectrum from the strict extensions, we have the pure extensions.

DEFINITION 4.10. An $(\mathscr{R}, \mathscr{L})$-extension V of K is *pure* if the canonical homomorphism $v: V/K \to T(V: K)/\pi(K)$ satisfies $v^{-1}(0) = \{0\}$ where v is induced by $\tau(V: K)$.

LEMMA 4.11. *An $(\mathscr{R}, \mathscr{L})$-extension V of K is pure iff for any $a \in V$, $\tau^a \in \pi(K)$ implies $a \in K$.*

Lemma 4.11 says that pure extensions are "best" in the sense that no element of V agrees with some element of K on a member of $\mathscr{L}$ and on a member of $\mathscr{R}$.

We have the following result which determines all pure $(\mathscr{R}, \mathscr{L})$-extensions of S.

DEFINITION 4.12. An (S, S) homomorphism between (S, S)-sets with zero $f: K \to Q$ is *pure* if $f^{-1}(0) = \{0\}$.

THEOREM 4.13. *Let K be $(\mathscr{R}, \mathscr{L})$-reductive and Q be an $(\mathscr{R}, \mathscr{L})$-trivial (S, S)-set with zero. Every pure homomorphism of Q onto the (S, S)-set $T/\pi(K)$, where T is a type of $(\mathscr{R}, \mathscr{L})$-extension of K, determines a pure $(\mathscr{R}, \mathscr{L})$-extension of K by Q of type T, whose scalar multiplication $*$ is given by the following formula (where $Q^* = Q\backslash\{0\}$ and $\Theta(a) = \Theta^a = [\lambda^a, \rho^a] \in T\backslash\pi(K)$ for $a \in Q^*$):*

$$a^*b = \begin{cases} ab & a \in K,\ b \in S \text{ or } b \in K,\ a \in S \\ \Theta(ab) = \Theta^a b & a \in Q^*,\ b \in S \\ \Theta(ab) = a\Theta^b & a \in S,\ b \in Q^* . \end{cases}$$

Conversely, every pure extension of K can be constructed in this fashion.

COROLLARY 4.14. *When Q^* is a semigroup, Θ is a semigroup homomorphism and $K = S$, then the above result shows that each pure $(\mathscr{R}, \mathscr{L})$-extension can be given a semigroup multiplication by defining for $a, b \in Q^*$, $a^*b = ab$ if $ab \neq 0$ and $a^*b = s \in S$ if $ab = 0$ and $\Theta^a\Theta^b = \pi s \in \pi(S)$.*

The proof of the above results is an easy modification of the proof of Theorem 2.11 of [4].

The reason for considering strict and pure extensions is evident by the next theorem.

THEOREM 4.15. *Let V be an $(\mathscr{R}, \mathscr{L})$-extension of K. The complete inverse image U of $\pi(K)$ under $\tau(V: K)$ is the greatest strict $(\mathscr{R}, \mathscr{L})$-extension of K in V and V is a pure $(\mathscr{R}, \mathscr{L})$-extension of U.*

Proof. Since $\tau(V: K)$ is a homomorphism and $T(V: K)$ contains $\pi(K)$, U is an (S, S)-subset of V and so is an $(\mathscr{R}, \mathscr{L})$-subset of V. Since $\tau(V: K)$ maps K into $\pi(K)$, we must have $K \subseteqq U$. And since $U \subseteqq V$, U is an $(\mathscr{R}, \mathscr{L})$-extension of K. Clearly U is a strict extension of K. Moreover if U' is an $(\mathscr{R}, \mathscr{L})$-extension of K in V, then $\tau(U': K)$ is the restriction of $\tau(V: K)$ to U'. Hence if U' is strict, then $U' \subseteqq U$ and so U is the greatest strict $(\mathscr{R}, \mathscr{L})$-extension of K in V.

Now let $v \in V$, and suppose that $\tau^v(V: U) \in \pi(U)$. Then for some $u \in U$, $vs = us$ for all $s \in R' \in \mathscr{R}$ and $tv = tu$ for all $t \in L' \in \mathscr{L}$. But $u \in U$ implies that for some $k \in K$, $kx = ux$ for all x in some $R'' \in \mathscr{R}$ and $yk = yu$ for $y \in$ some $L'' \in \mathscr{L}$. Let $R = R' \cap R'' \in \mathscr{R}$ and $L = L' \cap L'' \in \mathscr{L}$, then for all $x \in L$ and all $y \in R$, $xv = xk$ and $vy = ky$ and so $v \in U$.

5. Congruence dense extensions.

In this section we will show for $(\mathscr{R}, \mathscr{L})$-reductive K, that $\Omega(K: \mathscr{R}, \mathscr{L})$ is the maximal essential $(\mathscr{R}, \mathscr{L})$-extension of K and so is unique up to isomorphism.

In the remainder of this section, V will be an $(\mathscr{R}, \mathscr{L})$-extension of K.

An (S, S)-congruence on V whose restriction to K is the identity is called a *K-congruence*.

V is a *congruence dense extension* of K if the identity is the only K-congruence on V.

V is an essential extension of K if each (S, S)-homomorphism $f: V \rightarrow T$, T any (S, S)-set, whose restriction to K is injective is an injection.

Berthiaume [1] has shown that congruence dense extensions coincide with essential extensions.

LEMMA 5.1. *$\eta(V: \mathscr{R}, \mathscr{L}) = \tau \circ \tau^{-1}$ where $\tau = \tau(V: K)$.*

Proof. $(x, y) \in \eta(V: \mathscr{R}, \mathscr{L})$ iff there is $R \in \mathscr{R}$, $L \in \mathscr{L}$ with $xr = yr$ for $r \in \mathscr{R}$ and $tx = ty$ for $t \in L$. Let $L' = Kx^{-1} \cap Kv^{-1} \cap$

$L \in \mathscr{L}$ and $R' = x^{-1}K \cap y^{-1}K \cap R \in \mathscr{R}$. Then $tx = ty$ for $t \in L'$ and $xr = yr$ for $r \in R'$, thus $\tau^x = [\lambda x, \rho x] = [\lambda y, \rho y] = \tau^y$.

Conversely if $\tau^x = \tau^y$, then $[\lambda^x, \rho^x] = [\lambda^y, \rho^y]$ so there is $L \in \mathscr{L}$ and $R \in \mathscr{R}$ with $xr = yr$ for $r \in \mathscr{R}$ and $tx = ty$ for $t \in L$. Thus $(x, y) \in \eta(V: \mathscr{R}, \mathscr{L})$.

THEOREM 5.2. *Every K-congruence on V is contained in* $\eta(V: \mathscr{R}, \mathscr{L})$. *Moreover if K is* $(\mathscr{R}, \mathscr{L})$*-reductive,* $\eta(V: \mathscr{R}, \mathscr{L})$ *is the largest K-congruence on V. In any case,* $\eta(K) = \{v \in V \mid v\eta k$ for some $k \in K\}$ *is the largest strict subextension of V.*

Proof. Let $\mathscr{C}$ be a K-congruence on V. Then $a\mathscr{C}b$ implies $as = bs$ for all $s \in a^{-1}K \cap b^{-1}K \in \mathscr{R}$. Likewise $ta = tb$ for all $t \in Ka^{-1} \cap Kb^{-1} \in \mathscr{L}$. Thus $(a, b) \in \eta(V: \mathscr{R}, \mathscr{L})$.

If K is $(\mathscr{R}, \mathscr{L})$-reductive, then $\eta(V: \mathscr{R}, \mathscr{L})|_K = \eta(K: \mathscr{R}, \mathscr{L})$ which is the identity thus $\eta(V: \mathscr{R}, \mathscr{L})$ is a K-congruence.

If $a \in \eta(K)$, then $\tau^a = \tau^b$ for some $k \in K$ and so $\eta(K)$ is the largest strict subextension of V.

COROLLARY 5.3. *If K is* $(\mathscr{R}, \mathscr{L})$*-reductive, then V is a pure extension iff* $\mathscr{C}(K) = K$ *for every K-congruence on V.*

When K is $(\mathscr{R}, \mathscr{L})$-reductive, the following theorem characterizes strict $(\mathscr{R}, \mathscr{L})$-extensions by means of extensions of (S, S)-congruences on K. The proof is modelled after that of [4, Proposition 3.3].

THEOREM 5.4. *Let V be an* $(\mathscr{R}, \mathscr{L})$*-extension of K. If each* (S, S)*-congruence on K is the restriction of some* (S, S)*-congruence* $\overline{\mathscr{C}}$ *on V such that* $\overline{\mathscr{C}}(K) = V$*, then V is a strict extension of K. The converse holds if K is* $(\mathscr{R}, \mathscr{L})$*-reductive.*

Proof. Let $\mathscr{C}$ be the identity congruence on K, then $\overline{\mathscr{C}}(K) = V$ and so for $v \in V$, there is a unique $k \in K$ with $v\overline{\mathscr{C}}k$. Now if $s \in Kv^{-1}$, $sv\overline{\mathscr{C}}sk$ and so $sv = sk$ on Kv^{-1} and similarly, if $t \in v^{-1}K$, $vt\overline{\mathscr{C}}kt$ and so $vt = kt$ on $v^{-1}K$. These equations imply that $\tau^v = \tau^k$ and thus the extension is strict.

If K is $(\mathscr{R}, \mathscr{L})$-reductive, then K is a retract of V iff V is a strict extension of K. Then given an (S, S)-congruence $\mathscr{C}$ on K, extend $\mathscr{C}$ to $\overline{\mathscr{C}}$ on V by $\omega\overline{\mathscr{C}}v$ iff $r(\omega)\mathscr{C}r(v)$ where $r: V \to K$ is the retraction. Then since $\omega\overline{\mathscr{C}}r(\omega)$, $\overline{\mathscr{C}}(K) = V$.

REMARK. This result may be used to give a different proof of Proposition 4.8.

Theorem 5.2 characterizes congruence dense extensions of $(\mathscr{R}, \mathscr{L})$-reductive (S, S)-sets as follows:

THEOREM 5.5. *Let K be $(\mathscr{R}, \mathscr{L})$-reductive, then V is congruence dense* (essential) *over K iff V is $(\mathscr{R}, \mathscr{L})$-reductive. Thus congruence dense extensions are pure.*

When $\mathscr{L}$ is the lattice of left ideals of S, we have the following corollary due to Hinkle [5, Corollary 4.13].

COROLLARY 5.6. *When S is $\mathscr{R}$-torsion free, then an $\mathscr{R}$-extension V of S is essential iff V is $\mathscr{R}$-torsion free.*

When $\mathscr{R} = \mathscr{L} = \{S\}$, we have the following corollary due to Petrich and Grillet [4, Theorem 3.7].

COROLLARY 5.7. *Let S be weakly reductive. Then V is a congruence dense extension of S iff $\tau(V: S)$ is injective.*

Returning to the general case we have the

COROLLARY 5.8. *When K is $(\mathscr{R}, \mathscr{L})$-reductive, V is a congruence dense* (essential) *extension of K iff there is a monomorphism of V over K into $\Omega(K: \mathscr{R}, \mathscr{L})$.*

Since $\tau(\Omega: K)$ is the identity on $\Omega(K: \mathscr{R}, \mathscr{L})$ where K is $(\mathscr{R}, \mathscr{L})$-reductive, then $\Omega(K: \mathscr{R}, \mathscr{L})$ is congruence dense over K. Hence when $\mathscr{L}$ is the lattice of left ideals of S and S is $\mathscr{R}$-torsion free, then [5, Corollary 5.6] $Q_{\mathscr{R}}(S)$ is essential over S.

Finally, we characterize pure $(\mathscr{R}, \mathscr{L})$-extensions of $(\mathscr{R}, \mathscr{L})$-reductive (S, S)-sets by means of congruence dense extensions as follows:

COROLLARY 5.9. *Let K be $(\mathscr{R}, \mathscr{L})$-reductive. Then V is a pure extension iff there is an (S, S)-homomorphism ϕ over K of V into a congruence dense extension D of K with $\phi^{-1}(K) = K$.*

Proof. Let $T = T(V: K)$. Then by Corollaries 5.7 and 5.8, there is a dense extension D of K of type T. Then $\phi = \tau(D: K)^{-1} \circ \tau(V: K)$ is an (S, S)-homomorphism over K of V into D. Since V is pure over K, $\phi^{-1}(K) = \tau(V: K)^{-1}(\pi(K)) = K$.

Conversely, let ϕ be the given homomorphism, then $\tau(V: K) = \tau(D: K) \circ \phi$ by 3.8 and $\tau(V: K)^{-1}(\pi(K)) = \phi^{-1}(K) = K$ since $\tau(D: K)$ is injective. Thus V is pure.

We now prove the main result of this section.

THEOREM 5.10. *When K is $(\mathscr{R}, \mathscr{L})$-reductive, $\Omega(K: \mathscr{R}, \mathscr{L})$ is the maximal essential $(\mathscr{R}, \mathscr{L})$-extension of K.*

Proof. By the remark after Corollary 5.8, $\Omega = \Omega(K: \mathscr{R}, \mathscr{L})$ is essential over K. Now suppose $V \supseteqq \Omega$ is an essential $(\mathscr{R}, \mathscr{L})$-extension of K. Thus $\tau(V: K)$ is injective and $\tau(V: K)|_{\Omega} = \tau(\Omega: K)$ is the identity. Moreover if $v \notin \Omega$, then $\tau(V: K)(v) = x \in \Omega$, and $\tau(V: K)$ is injective. For $s \in \Omega v^{-1} \in \mathscr{L}$ and $t \in v^{-1}\Omega \in \mathscr{R}$, we have $xs = \tau(V: K)(vs) = vs$ and $tx = \tau(V: K)(tv) = tv$. Thus $(x, v) \in \eta(V: \mathscr{R}, \mathscr{L})$ which is the identity. Thus $x = v$ and $V = \Omega(K: \mathscr{R}, \mathscr{L})$.

THEOREM 5.11. *$\Omega(K: \mathscr{R}, \mathscr{L})$ is unique up to isomorphism over K, when K is $(\mathscr{R}, \mathscr{L})$-reductive.*

Proof. Let V be any other maximal essential $(\mathscr{R}, \mathscr{L})$-extension of K. Then $\tau(V: K)$ is injective by Theorem 5.5. If $T(V: K) = T \subseteqq \Omega(K: \mathscr{R}, \mathscr{L})$, then $\tau(V: K)$ can be extended to an (S, S)-isomorphism of an $(\mathscr{R}, \mathscr{L})$-extension $V' \supseteqq V$ onto $\Omega(K: \mathscr{R}, \mathscr{L})$. Consequently, V' is congruence dense over K by Theorem 5.5. Thus $V' = V$ and so $\tau(V: K)$ is an isomorphism.

When $\mathscr{R} = \mathscr{L} = \{S\}$, then we have as a corollary the following theorem of Gluskin [3]:

THEOREM 5.12. *Let S be weakly reductive, then S is a densely embedded ideal of V iff there is an isomorphism over S of V onto $\Omega(S)$.*

Notice also that Theorem 5.10 says that $\Omega(S: \mathscr{R}, \mathscr{L})$ is not only the maximal congruence dense $(\mathscr{R}, \mathscr{L})$-semigroup extension but is also maximal among congruence dense $(\mathscr{R}, \mathscr{L})$-extensions as an (S, S)-set.

6. The injectivity of $\Omega(K: \mathscr{R}, \mathscr{L})$. In this section we show that $\Omega(K: \mathscr{R}, \mathscr{L})$ is the $(\mathscr{R}, \mathscr{L})$-injective hull of K. First we prove that an $(\mathscr{R}, \mathscr{L})$-injective hull of K exists.

DEFINITION 6.1. A bi-S-set ${}_SK_S$ is *$(\mathscr{R}, \mathscr{L})$-injective* iff each (S, S)-homomorphism $f: {}_ST_S \rightarrow {}_SK_S$ has for any $(\mathscr{R}, \mathscr{L})$-extension ${}_SN_S$ of T an (S, S)-extension $\bar{f}: N \rightarrow K$. In particular, ${}_SK_S$ is injective when $\mathscr{R}$ consists of all right ideals of S and $\mathscr{L}$ consists of all left ideals.

Let K be any bi-S-set and let K^{S^1} denote the set of all mappings from S^1 to K. K^{S^1} is a bi-S-set under the mutiplication $(sf)(x) = s(f(x))$ and $(ft)(x) = f(tx)$ for all $x \in S^1$. Consider K as a subset of K^{S^1} by $k: S^1 \to K$ by $k(x) = kx$ for all $x \in S^1$. That K^{S^1} is an injective (S, S)-set follows by noting in Theorem 6 of [1] that the constructed extension is an (S, S)-homomorphism.

PROPOSITION 6.2. *For each (S, S)-set K, there is an injective (S, S)-set I_S containing K.*

We require the following lemma from [1].

LEMMA 6.3. *Let A, B and C be (S, S)-sets with $A \subseteqq B \subseteqq C$. Then A is essential in C iff A is essential in B and B is essential in C.*

Now let ${}_SK_S$ be given. Following Berthiaume, we see that K has a maximal essential extension $\hat{K}$ which is also the minimal injective extension of K. Moreover $\hat{K}$ is unique up to isomorphism over K. For any injective extension I of K, $\hat{K}$ is the maximal essential extension of K in I. Let E be a maximal $(\mathscr{R}, \mathscr{L})$-extension of K in $\hat{K}$ which exists by Zorn's lemma.

LEMMA 6.4. *E is $(\mathscr{R}, \mathscr{L})$-injective when both $\mathscr{R}$ and $\mathscr{L}$ are idempotent.*

Proof. Let ${}_SM_S$ be an $(\mathscr{R}, \mathscr{L})$-extension of ${}_SN_S$ and $\phi: N \to E$ be an (S, S)-homomorphism. Let $\bar{\phi}: M \to \hat{K}$ be an extension of ϕ to M. Consider $W = \bar{\phi}(M) \cup E \subseteqq \hat{K}$. It suffice to show that W is an $(\mathscr{R}, \mathscr{L})$-extension of E, for then W is an $(\mathscr{R}, \mathscr{L})$-extension of K since $\mathscr{R}$ and $\mathscr{L}$ are idempotent; thus $W = E$ and we are done.

Therefore, let $\bar{\phi}(t) \notin E$. Then there is $R \in \mathscr{R}$ and $L \in \mathscr{L}$ with $tR \subseteqq N$ and $Lt \subseteqq N$. Thus $\bar{\phi}(t)R = \bar{\phi}(tR) \subseteqq E$ and likewise $L\bar{\phi}(t) \subseteqq E$. Hence W is an $(\mathscr{R}, \mathscr{L})$-extension of E and we are done.

THEOREM 6.5. *$E(K) = E$ is the maximal $(\mathscr{R}, \mathscr{L})$-essential extension of K.*

Proof. Let $T \supseteqq E$ be an $(\mathscr{R}, \mathscr{L})$-essential extension of K. Then without loss of generality, $T \subseteqq K$, the maximal essential extension of K. Thus $T = E$.

THEOREM 6.6. *$E(K) = E$ is the minimal $(\mathscr{R}, \mathscr{L})$-injective extension of K, when both $\mathscr{R}$ and $\mathscr{L}$ are idempotent.*

Proof. Let $K \subseteqq T \subseteqq E$ and T be an $(\mathscr{R}, \mathscr{L})$-injective extension of K. Then there is an extension $\phi: E \to T$ of the identity map $1: T \to T$. By Lemma 6.3, T is essential in E and so ϕ is one-to-one. Hence $T = E$.

By Theorem 5.10 when K is $(\mathscr{R}, \mathscr{L})$-reductive, $\Omega(K: \mathscr{R}, \mathscr{L})$ is the maximal essential $(\mathscr{R}, \mathscr{L})$-extension of K. By Theorem 6.5 and Theorem 5.11, $\Omega(K: \mathscr{R}, \mathscr{L})$ is isomorphic over K to E. Consequently, when $\mathscr{R}$ and $\mathscr{L}$ are idempotent, $\Omega(K: \mathscr{R}, \mathscr{L})$ is the injective hull of K by Theorem 6.6. Thus we have proved the following theorem.

THEOREM 6.7. *When K is $(\mathscr{R}, \mathscr{L})$-reductive and both $\mathscr{R}$ and $\mathscr{L}$ are idempotent, the $\Omega(K: \mathscr{R}, \mathscr{L})$ is the $(\mathscr{R}, \mathscr{L})$-injective hull of K.*

COROLLARY 6.8. *When S is idempotent and weakly reductive, $\Omega(S)$ is $(\{S\}, \{S\})$-injective.*

Proof. Since $\mathscr{R} = \mathscr{L} = \{S\}$ and $S^2 = S$, $\mathscr{R}$ and $\mathscr{L}$ are idempotent. The weak reductivity of S implies that $\eta_{\Omega(S:\{S\},\{S\})}$ is the identity and so S is $(\mathscr{R}, \mathscr{L})$-reductive. The result now follows from Theorem 6.7.

7. An application. In this section, we apply our theory to show that when S is weakly reductive, then $\Omega(S) = \Omega(S^n)$ for all n positive.

Let $\mathscr{R} = \mathscr{L} = \{S\}$, and write $\Omega(K: \mathscr{R}, \mathscr{L}) = \Omega(K: S, S)$.

LEMMA 7.1. *If $\eta(S: S, S) = \mathrm{id}$, then $\eta(S^n: S^n, S^n) = i$ for all n.*

Proof. Let $x \neq y$ in S^n. Then $x \neq y$ in S so there is $s_1, t_1 \in S$ with $s_1x \neq s_1y$ and $xt_1 \neq yt_1$. Now suppose we have $s_1, \cdots, s_{n-1}, t_1, \cdots, t_{n-1} \in S$ with $s_{n-1} \cdots s_1x \neq s_{n-1} \cdots \dot{s}_1y$ and $xt_1 \cdots t_{n-1} \neq yt_1 \cdots t_{m-1}$. Then there is $s_n, t_n \in S$ with $s_ns_{n-1} \cdots s_1x \neq s_ns_{n-1} \cdots s_1y$ and $xt_1 \cdots t_{n-1} \neq yt_1 \cdots t_{n-1}t_n$. Since $t_1 \cdots t_n, s_n \cdots s_1 \in S^n$, the result follows.

COROLLARY 7.2. *If $\eta(S: S, S) = \mathrm{id}$, then $\eta(S^n: S, S) = \mathrm{id}$.*

DEFINITION 7.3. ${}_SK_S$ is strictly essential in ${}_SN_S$ if for all $m \neq n$ in N, there are $s \in m^{-1}K \cap n^{-1}K$ and $t \in Km^{-1} \cap Kn^{-1}$ with $ms \neq ns$ and $tm \neq tn$.

LEMMA 7.4. [1]. *If ${}_SK_S$ is strictly essential in ${}_SN_S$, then K is essential in N.*

LEMMA 7.5. *For all positive n, ${}_SS^n_S$ is strictly essential in ${}_SS_S$ when $\eta(S: S, S) = \mathrm{id}$.*

The proof of this lemma is contained in the proof of Lemma 7.1.

Now if $\eta(S: S, S) = \mathrm{id}$, i.e., if S is weakly reductive, then ${}_SS_S$ is essential over ${}_SS^n_S$ and so $\tau(S, S^n)$ is injective. Thus without loss of generality $S^n \subseteqq S \subseteqq \Omega(S^n: S, S)$. Since $\Omega(S^n: S, S)$ is congruence dense over S, $S \subseteqq \Omega(S^n: S, S) \subseteqq \Omega(S: S, S)$ without loss of generality. By Lemma 6.3, S^n is congruence dense in $\Omega(S: S, S)$ but since $\Omega(S^n: S, S)$ is the maximal congruence dense extension of S^n, $\Omega(S^n: S, S) = \Omega(S: S, S)$.

THEOREM 7.6. *Let S be weakly reductive, then for $n > 0$, $\Omega(S^n: S, S) = \Omega(S: S, S)$.*

Next notice that $\Omega(S^n: S^n, S^n)$ is strictly (S^n, S^n)-essential over S^n since $\eta(S^n: S^n, S^n) = \mathrm{id}$. Thus $\Omega(S^n: S^n, S^n)$ is strictly (S, S)-essential over S^n since $S^n \subseteqq S$. Thus we may suppose $S^n \subseteqq \Omega(S^n: S^n, S^n) \subseteqq \Omega(S^n: S, S)$. However for $q_1 \neq q_2$ in $\Omega(S^n: S, S)$, there are $s, t \in S$ with $sq_1 \neq sq_2$, $q_1t \neq q_2t$ and $sq_1, sq_2, q_1t, q_2t \in S^n$. Since $\eta(S^n: S^n, S^n) = \mathrm{id}$, there is $s^1, t^1 \in S^n$ with $s^1sq_1 \neq s^1sq_2$ and $q_1tt^1 \neq q_2tt^1$. Since $tt^1, s^1s \in S^{n+1} \subseteqq S^n$, we see that $\Omega(S^n: S, S)$ is strictly (S^n, S^n)-essential over S^n. Hence it is (S^n, S^n)-essential and so (S^n, S^n)-congruence dense over S^n. Thus

$$\Omega(S^n: S^n, S^n) = \Omega(S^n: S, S) .$$

THEOREM 7.7. *Let S be weakly reductive. Then for all $n > 0$, $\Omega(S^n) = \Omega(S)$.*

Proof. The result follows from the above discussion upon noting that $\Omega(S^n) = \Omega(S^n: S^n, S^n) = \Omega(S^n: S, S) = \Omega(S: S, S) = \Omega(S)$.

REMARK. When S is not weakly reductive, the above result is false. To see this let S be a semigroup with zero satisfying $S^n = 0$, $S^{n-1} \neq 0$. Then for $x \in S^{n-1}$, $\lambda_x: S \to S^{n-1}$ and $\rho_x: S \to S^{n-1}$ are the zero maps. Thus $\Omega(S^{n-1}: S, S) = 0$, but $\Omega(S) \neq 0$ for there is $x \in S^{n-2}$ with $xS = 0$.

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