PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 2, 1977

COHOMOLOGY OF HOMOMORPHISMS OF LIE ALGEBRAS AND LIE GROUPS

ROBERT F. BROWN

Given compact, connected Lie groups G_1 and G_2 and given $h: G_1 \to G_2$ a homomorphism with kernel K, let $Ph^*: PH^*(G_2) \to PH^*(G_1)$ be the homomorphism of the primitives in the real cohomology induced by h. We prove that if the rank of G_2 is greater than or equal to the rank of G_1, then the dimension of the kernel of Ph^* is greater than or equal to the rank of K. We discuss when the inequality is an equality and we use the inequality to study when the hypothesis that Ph^* is an isomorphism implies that h itself is an isomorphism.

1. **Introduction.** This paper was initially motivated by a quite specific question. Let G be a compact, connected Lie group and let $h: G \to G$ be an endomorphism of G, then h induces an endomorphism $h^*: H^*(G) \to H^*(G)$ of the real cohomology of G. The question is: if h^* is an automorphism, does it follow that h is an automorphism?

The answer to this question is "no" in general. Represent the circle S^1 as the complex numbers of norm one and define $h: S^1 \to S^1$ by $h(z) = z^2$. Then for $h^{*,1}: H^1(S^1) \to H^1(S^1) \cong \boldsymbol{R}$ (the reals) we have $h^{*,1}(x) = 2x$, so $h^*: H^*(S^1) \to H^*(S^1)$ is an automorphism even though the kernel of h contains two points. However, as we shall prove below (Corollary 5.1), if h^* is an automorphism then the differential dh of h is an automorphism of the Lie algebra of G. Consequently, the example illustrates the worst that can happen because if dh is an automorphism then h is onto and its kernel, though not necessarily trivial, is finite. (For this and other facts from Lie group theory, see [6].)

We wish to restate the relationship between h^* and h above in a form which will lead us in a natural way to a statement of the basic problem of this paper. We still have the endomorphism $h: G \to G$ that we assume induces an automorphism $h^*: H^*(G) \to H^*(G)$ of real cohomology. However, our results will be easier to describe if, instead of considering all of $H^*(G)$, we restrict our attention to elements which generate $H^*(G)$ as an algebra. Let $m: G \times G \to G$ be the group operation, then m induces $m^*: H^*(G) \to H^*(G) \otimes H^*(G)$. An element $\omega \in H^*(G)$ is *primitive* if $m^*(\omega) = 1 \otimes \omega + \omega \otimes 1$. Now let Ph^*: $PH^*(G) \to PH^*(G)$ be the restriction of h^* to the primitives in $H^*(G)$, then h^* is an automorphism if and only if Ph^* is an automorphism. Stating the hypothesis another way: the dimension of the kernel of Ph^* is zero. (The *dimension* of a graded vector space W, written $\dim W$, is the sum of the dimensions of the individual vector spaces that make up W.) The conclusion that the kernel of h is finite can be expressed by saying that the rank of the kernel of h is zero. (The *rank* of a compact Lie group G, written $rk(G)$, is the dimension of a maximal torus.) The basic problem of this paper is to determine the relationship between the dimension of $\ker Ph^*$, the kernel of Ph^*, and the rank of the kernel of h.

Since Corollary 5.1 implies that when the dimension of the kernel of Ph^* is zero the rank of the kernel of h is also zero, one might wonder whether the two numbers are always equal. Again represent S^1 as the complex numbers of norm one and let S^3 be the quaternions of norm one. Let $G = S^1 \times S^3$ and define $h: G \to G$ as follows. For a complex number $a + bi$ and a quaternion $t + ui + vj + wk$, set

$$h(a + bi, t + ui + vj + wk) = (1 + 0i, a + bi + 0j + 0k) .$$

Now h is homotopic to the constant map on G so the dimension of the kernel of Ph^* is two because that is the dimension of $PH^*(G)$. On the other hand, the kernel of h is isomorphic to S^3, which is of rank one. Therefore, equality does not hold in general.

In the next section, we will show that rank the of the kernel K of an endomorphism $h: G \to G$ is always less than or equal to the dimension of the kernel of Ph^*. This inequality will follow from a corresponding inequality for homomorphisms between (possibly different) Lie groups. Section 3 examines how the one-dimensional cohomology contributes to the inequality. In § 4, we state sufficient conditions for the inequality to be an equality. Finally, in §5, we study homomorphisms $h: G_1 \to G_2$ with the property that $h^*: H^*(G_2) \to H^*(G_1)$ is an isomorphism.

The results of this paper are most naturally stated and proved in the context of Lie algebras. Consequently, §§2, 3, and 4 each begin with a theorem concerning homomorphisms of Lie algebras. The corresponding results for homormophisms of Lie groups follow as corollaries.

In order to avoid frequent repetitions of the same hypotheses, we adopt the following

conventions: (1) all Lie groups are compact and connected (2) all Lie algebras are the Lie algebras of compact Lie groups (3) all homomorphisms of Lie algebras are differentials of homomorphisms of Lie groups.

2. The main inequality. For a Lie algebra $\mathfrak{G}$, let $PH^*(\mathfrak{G})$ denote the primitives in the cohomology (see [5]). The *rank* of $\mathfrak{G}$

(written $rk(\mathfrak{G})$) is the dimension of a Cartan subalgebra [6, p. 264]. It follows from a theorem of Hopf [4] (or see [1]), via the de Rham theorem [3], that $rk(\mathfrak{G})$ is equal to the dimension of $PH^*(\mathfrak{G})$.

THEOREM 2.1. *Let* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *be Lie algebras such that* $rk(\mathfrak{G}_1) \leqq rk(\mathfrak{G}_2)$. *Let* $\eta: \mathfrak{G}_1 \to \mathfrak{G}_2$ *be a homomorphism with kernel* $\mathfrak{K}$ *inducing* $P\eta^*: PH^*(\mathfrak{G}_2) \to PH^*(\mathfrak{G}_1)$. *Then*

$$\dim \ker P\eta^* \geqq rk(\mathfrak{K}) .$$

Proof. Since $\mathfrak{K}$ is an ideal in $\mathfrak{G}_1$, it follows from [6, p. 213] that $\mathfrak{K}$ is a direct summand of $\mathfrak{G}_1$. Therefore, the homomorphism $P\iota^*: PH^*(\mathfrak{G}_1) \to PH^*(\mathfrak{K})$ induced by inclusion is onto. Let $m = \dim \ker P\eta^*$ and suppose that the dimension of $PH^*(\mathfrak{K})$ were greater than m. We can choose $\{z_1, z_2, \cdots, z_m, z_{m+1}\}$, a linearly independent set of elements of $PH^*(\mathfrak{K})$ - and then a linearly independent set $\{y_1, y_2, \cdots, y_m, y_{m+1}\}$ of elements of $PH^*(\mathfrak{G}_1)$ such that $\iota^*(y_j) = z_j$ for all j. Let V be the vector subspace of $PH^*(\mathfrak{G}_1)$ spanned by $\{y_1, \cdots, y_{m+1}\}$. Since $rk(\mathfrak{G}_1) \leqq rk(\mathfrak{G}_2)$, the dimension of $PH^*(\mathfrak{G}_1)$ is no larger than the dimension of $PH^*(\mathfrak{G}_2)$. Consequently, the dimension of the intersection of V and the image of $P\eta^*$ must be at least one. So there exists $y = \sum_{j=1}^{m+1} a_j y_j$ in the image of $P\eta^*$ with some a_j nonzero. Let $x \in PH^*(\mathfrak{G}_2)$ such that $\eta^*(x) = y$. Now the composition

$$\mathfrak{K} \xrightarrow{\iota} \mathfrak{G}_1 \xrightarrow{\eta} \mathfrak{G}_2$$

is trivial and yet

$$\iota^*\eta^*(x) = \sum_{j=1}^{m+1} a_j z_j \neq 0$$

which is a contradiction, so $rk(\mathfrak{K}) \leqq m$.

The conclusion of Theorem 2.1 is false for the trivial homomorphism $\eta: \mathfrak{G}_1 \to \mathfrak{G}_2$ if $rk(\mathfrak{G}_1) > rk(\mathfrak{G}_2)$, so we do require a restriction on the ranks.

COROLLARY 2.2. *Let* G_1 *and* G_2 *be Lie groups such that* $rk(G_1) \leqq rk(G_2)$. *Let* $h: G_1 \to G_2$ *be a homomorphism with kernel* K *inducing* $Ph^*: PH^*(G_2) \to PH^*(G_1)$. *Then*

$$\dim \ker Ph^* \geqq rk(K) .$$

3. One-dimensional cohomology. For a Lie algebra $\mathfrak{G}$, we write $\mathfrak{G} \cong \mathfrak{Z} \oplus \mathscr{D}\mathfrak{G}$ where $\mathfrak{Z}$ is the center of $\mathfrak{G}$ and $\mathscr{D}\mathfrak{G}$ is semisimple.

LEMMA 3.1. *Let* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *be Lie algebras such that* $\dim(\mathfrak{Z}_1) =$

$\dim(\mathfrak{Z}_2)$ *and let* $\eta\colon \mathfrak{G}_1 \to \mathfrak{G}_2$ *be a homomorphism. The dimension of the kernel of the induced homomorphism* $\eta^{*,1}\colon H^1(\mathfrak{G}_2) \to H^1(\mathfrak{G}_1)$ *is equal to dimension of the Lie algebra* $\eta^{-1}(\mathscr{D}\mathfrak{G}_2) \cap \mathfrak{Z}_1$.

Proof. Define $\bar{\eta}\colon \mathfrak{Z}_1 \to \mathfrak{Z}_2$ to be the composition

$$\mathfrak{Z}_1 \xrightarrow{\iota} \mathfrak{G}_1 \xrightarrow{\eta} \mathfrak{G}_2 \xrightarrow{\pi} \mathfrak{Z}_2$$

where ι is inclusion and π is projection. Since $\iota^{*,1}$ and $\pi^{*,1}$ are isomorphisms, then $\ker \eta^{*,1} \cong \ker \bar{\eta}^{*,1}$. (See [5] for the cohomology of Lie algebras.) Since $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are abelian Lie algebras of the same dimension, it follows from the definitions that the dimension of the kernel of $\bar{\eta}^{*,1}$ is equal to the dimension of the kernel of $\bar{\eta}$. The fact that

$$\ker \bar{\eta} = \eta^{-1}(\mathscr{D}\mathfrak{G}_2) \cap \mathfrak{Z}_1$$

completes the argument.

THEOREM 3.2. *Let* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *be Lie algebras such that* $rk(\mathfrak{G}_1) \leqq rk(\mathfrak{G}_2)$ *and* $\dim(\mathfrak{Z}_1) = \dim(\mathfrak{Z}_2)$. *Let* $\eta\colon \mathfrak{G}_1 \to \mathfrak{G}_2$ *be a homomorphism with kernel* $\mathfrak{K}$. *If* $\dim \ker P\eta^* = rk(\mathfrak{K})$, *then* $\eta(\mathfrak{Z}_1) \subseteq \mathfrak{Z}_2$.

Proof. Let $\mathscr{D}\eta\colon \mathscr{D}\mathfrak{G}_1 \to \mathscr{D}\mathfrak{G}_2$ denote the restriction of η. For $\mathfrak{G} = \mathfrak{G}_1$ or $\mathfrak{G}_2$, define

$$\hat{P}H^*(\mathfrak{G}) = \sum_{j \geqq 2} PH^j(\mathfrak{G})$$

so $PH^*(\mathfrak{G}) = H^1(\mathfrak{G}) \oplus \hat{P}H^*(\mathfrak{G})$. Then η induces $\eta^{*,1}\colon H^1(\mathfrak{G}_2) \to H^1(\mathfrak{G}_1)$ and $\hat{P}\eta^*\colon \hat{P}H^*(\mathfrak{G}_2) \to \hat{P}H^*(\mathfrak{G}_1)$. Furthermore,

$$\dim \ker P\eta^* = \dim \ker \eta^{*,1} + \dim \ker \hat{P}\eta^* .$$

The inclusion of $\mathscr{D}\mathfrak{G}_2$ into $\mathfrak{G}_2$ induces on isomorphism between $\hat{P}H^*(\mathfrak{G}_2)$ and $PH^*(\mathscr{D}\mathfrak{G}_2)$, so since $H^1(\mathscr{D}\mathfrak{G}_2) = 0$,

$$\dim \ker \hat{P}\eta^* = \dim \ker \hat{P}(\mathscr{D}\eta)^* = \dim \ker P(\mathscr{D}\eta)^* .$$

Lemma 3.1 then implies that

$$\dim \ker P\eta^* = \dim(\eta^{-1}(\mathscr{D}\mathfrak{G}_2) \cap \mathfrak{Z}_1) + \dim \ker P(\mathscr{D}\eta)^* .$$

Since $\mathfrak{K} \cap \mathscr{D}\mathfrak{G}_1 = \ker(\mathscr{D}\eta)$, Theorem 2.1 states that

$$rk(\mathfrak{K} \cap \mathscr{D}\mathfrak{G}_1) \leqq \dim \ker P(\mathscr{D}\eta)^* .$$

If $\eta(\mathfrak{Z}_1) \cap \mathscr{D}\mathfrak{G}_2 \neq 0$ then $\mathfrak{K} \cap \mathfrak{Z}_1 \subsetneqq \eta^{-1}(\mathscr{D}\mathfrak{G}_2) \cap \mathfrak{Z}_1$ and so

$$\begin{aligned} \dim \ker P\eta^* &= \dim(\eta^{-1}(\mathscr{D}\mathfrak{G}_2 \cap \mathfrak{Z}_1)) + \dim \ker P(\mathscr{D}\eta)^* \\ &\geqq \dim(\eta^{-1}(\mathscr{D}\mathfrak{G}_2 \cap \mathfrak{Z}_1)) + rk(\mathfrak{K} \cap \mathscr{D}\mathfrak{G}_1) \\ &> \dim(\mathfrak{K} \cap \mathfrak{Z}_1) + rk(\mathfrak{K} \cap \mathscr{D}\mathfrak{G}_1) = rk(\mathfrak{K}) . \end{aligned}$$

Denote the identity component of the center of a Lie group G by Z^0.

COROLLARY 3.3. *Let G_1 and G_2 be Lie groups such that $rk(G_1) \leqq rk(G_2)$ and* $\dim Z_1^0 = \dim Z_2^0$. *Let $h: G_1 \to G_2$ be a homomorphism with kernel K. If* $\dim \ker Ph^* = rk(K)$, *then $h(Z_1^0) \subseteqq Z_2^0$.*

Corollary 3.3 explains why, in the example $h: G \to G = S^1 \times S^3$ of § 1, we found that $\dim \ker Ph^* > rk(K)$. The reason is that the center S^1 of G was not mapped into itself by h. There can be other reasons why $\dim \ker Ph^* > rk(K)$, as the following examples show.

Let $j: \mathrm{SO}(8) \to \mathrm{SO}(9)$ be defined as follows. For A a matrix in SO(8), let

$$j(A) = \begin{bmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Then j is one-to-one, but there is an element of $H^{15}(\mathrm{SO}(9))$ in $PH^*(\mathrm{SO}(9))$ while $H^{15}(\mathrm{SO}(8)) = 0$. For an example where $G_1 = G_2$, let $G = \mathrm{SO}(8) \times \mathrm{SO}(9)$ and define $h: G \to G$ to be the composition

$$\mathrm{SO}(8) \times \mathrm{SO}(9) \xrightarrow{\pi} \mathrm{SO}(8) \xrightarrow{j} \mathrm{SO}(9) \xrightarrow{i} \mathrm{SO}(8) \times \mathrm{SO}(9)$$

where π is projection and i is inclusion. Now the kernel of Ph^* contains $PH^*(\mathrm{SO}(8))$ and $H^{15}(\mathrm{SO}(9))$ which implies that it is of dimension at least 5. But the kernel of h is isomorphic to SO(9), a group of rank 4.

4. Sufficient conditions for equality. The previous section suggests that strong hypotheses will be required in order for the inequality of § 2 to be an equality. The next results employ such hypotheses.

THEOREM 4.1. *Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be Lie algebras such that $rk(\mathfrak{G}_1) = rk(\mathfrak{G}_2)$. Let $\eta: \mathfrak{G}_1 \to \mathfrak{G}_2$ be a homomorphism with kernel $\mathfrak{K}$. If $\eta(\mathfrak{G}_1)$ is an ideal of $\mathfrak{G}_2$, then*

$$\dim \ker (P\eta^*) = rk(\mathfrak{K}) .$$

Proof. Let $\mathfrak{H} = \eta(\mathfrak{G}_1)$ then by hypothesis $\mathfrak{H}$ is an ideal of $\mathfrak{G}_2$ so the kernel of the homomorphism $P\iota^*: PH^*(\mathfrak{G}_2) \to PH^*(\mathfrak{H})$ induced by inclusion is isomorphic to $PH^*(\mathfrak{G}_2/\mathfrak{H})$. Let $\bar{\eta}: \mathfrak{G}_1/\mathfrak{K} \to \mathfrak{H}$ be the isomorphism induced by η. Then the diagram

$$\begin{tikzcd} \mathfrak{G}_1 \arrow[r, "\eta"] \arrow[d, "q"'] & \mathfrak{G}_2 \\ \mathfrak{G}_1/\mathfrak{K} \arrow[r, "\bar{\eta}"'] & \mathfrak{H} \arrow[u, "\iota"'] \end{tikzcd}$$

where q is the quotient homomorphism, can be used to show that

$$\ker P\eta^* \cong PH^*(\mathfrak{G}_2/\mathfrak{H}) .$$

Since $rk(\mathfrak{G}_1) = rk(\mathfrak{G}_2)$, it follows that $rk(\mathfrak{K}) = rk(\mathfrak{G}_2/\mathfrak{H})$ and that completes the proof.

COROLLARY 4.2. *Let G_1 and G_2 be Lie groups such that $rk(G_1) = rk(G_2)$. Let $h: G_1 \to G_2$ be a homomorphism with kernel K. If $h(G_1)$ is a normal subgroup of G_2, then*

$$\dim \ker (Ph^*) = rk(K) .$$

If we mimic the example at the end of § 3 with $SU(2) \times SU(3)$ in place of $SO(8) \times SO(9)$ then $h(G) = ij\pi(G)$ is not a normal subgroup of G and yet $\dim \ker (Ph^*) = rk(K) = 2$. Consequently, the sufficient conditions of this section are not necessary conditions for equality.

5. Homomorphisms that induce isomorphisms.

We come now to the result promised in the introductory section.

COROLLARY 5.1. *If $h: G_1 \to G_2$ is a homomorphism of Lie groups such that $h^*: H^*(G_2) \to H^*(G_1)$ is an isomorphism, then the differential of h is an isomorphism of Lie algebras.*

Proof. Let K be the kernel of h. Since h^* is an isomorphism, so also is $Ph^*: PH^*(G_2) \to PH^*(G_1)$ and, by Corollary 2.2, the rank of K is equal to zero. Let $dh: \mathfrak{G}_1 \to \mathfrak{G}_2$ be the differential of h, then $\mathfrak{K}$, the kernel of dh, is trivial because it is of rank zero. We conclude that dh is an isomorphism.

A homomorphism $h: G_1 \to G_2$ induces a homomorphism $h_\pi: \pi_1(G_1) \to \pi_1(G_2)$ of the fundamental groups.

COROLLARY 5.2. *If $h: G_1 \to G_2$ is a homomorphism of Lie groups such that $h^*: H^*(G_2) \to H^*(G_1)$ is an isomorphism. Then*

(i) *$h_\pi: \pi_1(G_1) \to \pi_1(G_2)$ is one-to-one*

(ii) *h is an isomorphism if and only if h_π is an isomorphism.*

Proof. Corollary 5.1 implies that h is a local homeomorphism so $h_1: G_1 \to G_2$ is a covering space with fiber K, the kernel of h. Thus we have the exact sequence

$$\pi_1(K) \longrightarrow \pi_1(G_1) \xrightarrow{h_\pi} \pi_1(G_2) \longrightarrow \pi_0(K) \longrightarrow \pi_0(G_1)$$

and observe that h_π is one-to-one. The homomorphism h is an isomorphism if and only if K is trivial, that is, if and only if $\pi_0(K) = 1$. By exactness, $\pi_0(K) = 1$ if and only if h_π is onto.

COROLLARY 5.3. *Let G be a semisimple Lie group and let $h\colon G \to G$ be an endomorphism such that $h^*\colon H^*(G) \to H^*(G)$ is an isomorphism, then h is an isomorphism.*

Proof. Since G is semisimple, $\pi_1(G)$ is finite, so the result is a consequence of Corollary 5.2.

An inner automorphism $h\colon G \to G$, defined by $h(x) = axa^{-1}$ for some $a \in G$, is homotopic to the identity map because G is pathwise connected, so h induces the identity isomorphism on $H^*(G)$. Our final result shows that the converse is also true.

COROLLARY 5.4. *Let G be a compact, connected Lie group and let $h\colon G \to G$ be an endomorphism such that $h^*\colon H^*(G) \to H^*(G)$ is the identity isomorphism, then h is an inner automorphism of G.*

Proof. We write $\pi_1(G) \cong \boldsymbol{Z}^m \oplus T$ where $\boldsymbol{Z}^m$ is free abelian and T is finite. By Corollary 5.2, $h_\pi\colon \boldsymbol{Z}^m \oplus T \to \boldsymbol{Z}^m \oplus T$ is one-to-one and therefore its restriction to T must be an isomorphism of T to itself. The fact that $h^{*,1}\colon H^1(G) \to H^1(G)$ is the identity isomorphism, together with the Universal Coefficient Theorem and the Hurewicz Homomorphism Theorem, imply that the restriction of h_π to $\boldsymbol{Z}^m$ is the identity transformation from $\boldsymbol{Z}^m$ to itself. Therefore, by Corollary 5.2, h is an isomorphism of G inducing the identity isomorphism on $H^*(G)$. Proposition 4 of [2] thus implies that h is an inner automorphism.

REFERENCES

1. R. Brown, *Fixed points of automorphisms of compact Lie groups*, Pacific J. Math., **63** (1976), 79-87.

2. R. Brown, *On the power map in compact groups, II*, Rocky Mountain J. Math., **3** (1973), 9-10.

3. C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc., **63** (1948), 85-124.

4. H. Hopf, *Über den Rang geschlossener Liescher Gruppen*, Comment. Math. Helv., **13** (1940), 119-143.

5. J. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France, **78** (1950), 65-127.

6. V. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations,* Prentice-Hall, 1974.

Received October 25, 1976. This work was supported under NSF Grant No. MCS 76-05971.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES
LOS ANGELES, CA 90024