PACIFIC JOURNAL OF MATHEMATICS
Vol. 72, No. 2, 1977

ALGEBRAIC AUTOMORPHISMS OF ALGEBRAIC GROUPS WITH STABLE MAXIMAL TORI

SARAH J. GOTTLIEB

Let T_1 and T_2 be maximal tori of a connected linear algebraic group $G \subseteq GL(n, \kappa)$, and suppose some (algebraic group) automorphism σ of G stabilizes both T_1 and T_2. Suppose further that σ also stabilizes two Borel subgroups, B_1 and B_2, of G. This paper is about the following natural questions:

(1) Are T_1 and T_2 conjugate by a σ-fixed point of G?

(2) Are B_1 and B_2 conjugate by a σ-fixed point of G?

(3) If $T_i \subseteq B_i$, $(i = 1, 2)$, are the T_i and B_i respectively conjugate by a single σ-fixed point of G?

(4) Are at least T_1 and T_2 described in (3) above conjugate by a σ-fixed point of G?

In this paper is treated the case in which σ is an algebraic automorphism. If either $p = \text{char } \kappa = 0$ or σ is semisimple, then the answer to (4) above is yes; but there are counterexamples for (1), (2), and (3). (See below, Counterexamples A-1 and B.) Moreover, if both $p > 0$ and σ is not semisimple, then there is also a counterexample for question (4). (See below, Counterexample C.)

Incidental in the proofs is the simple result that when σ is algebraic, a σ-stable maximal torus is pointwise fixed by some finite power of σ, and by σ itself for $p = 0$, σ unipotent (Theorem 1).

Robert Steinberg has studied the questions above in [3], for the case that σ has finite fixed-point set in G, finding that the answers to questions (2), (3), and (4) are all yes. There is a counterexample for question (1) in the finite fixed-point set case, when the σ-stable maximal tori are not respectively contained in σ-stable Borel subgroups. (See below, Counterexample A-2.)

When σ is an algebraic automorphism of a general algebraic group G, its fixed-point set may be infinite. In fact, Steinberg shows (by [3], 10.10) that if σ is algebraic with finite fixed-point set, then G is necessarily solvable.

Throughout the paper the (now standard) terminology and basic results of Borel ([1] and [2]) are used, including the name *Borel subgroup* for a maximal solvable connected subgroup. In addition the mnemonic *clag* is used for a connected linear algebraic group, and the expression "the pair $T \subseteqq B$" for a maximal torus T and a Borel subgroup B containing T.

In all of the following theorems, G is a clag and σ an algebraic automorphism of G.

THEOREM 1. *If G has a σ-stable maximal torus T, then T is pointwise-fixed by some power σ^n of σ. If $p = 0$ and σ is unipotent, then T is pointwise fixed by σ.*

Proof. Since σ is an algebraic automorphism of G, there is a closed linear algebraic group $\mathscr{G}$ with $G \Delta \mathscr{G}$ and an element $s \in \mathscr{G}$ such that $\sigma(g) = sgs^{-1}$ for each $g \in G$. (In fact this may be taken as the definition for an algebraic automorphism of G.)

Form the algebraic group generated by T and s, $\mathscr{A}(T, s) = \mathscr{A}$ in $\mathscr{G}$ (see [1], §3). T is normalized by s, so T is normal in $\mathscr{A}$. Moreover, T is a torus in $\mathscr{A}_0$, and so is contained in a maximal torus of $\mathscr{A}_0$. Thus T is contained in every maximal torus of $\mathscr{A}_0$, hence is contained centrally in every Borel subgroup of $\mathscr{A}_0$ by ([1], §18, 18.1). T is therefore central in $\mathscr{A}_0$ by ([1], §18, 18.5).

Now $s \in \mathscr{A} \Rightarrow$ some power s^n of s is in $\mathscr{A}_0$, whence s^n centralizes T. Equivalently, σ^n fixes T pointwise.

Suppose now that $p = 0$ and σ is unipotent. Since s^n centralizes T; so does $\mathscr{A}(s^n)$. Now σ unipotent $\Rightarrow s$ unipotent; and for $p = 0$, $\mathscr{A}(s^n) = \mathscr{A}(s)$ (see [1], 8.2). Thus s also centralizes T, i.e., σ fixes T pointwise.

THEOREM 2. *Let G be solvable, and let either $p = 0$ or σ be semisimple. Then two σ-stable maximal tori T_i of G ($i = 1, 2$) are conjugate by a σ-fixed point of G.*

Proof. (1) Since σ has finite order, say n (n is prime to p when $p > 0$), on T_1 and T_2, it may be assumed without change in hypothesis that σ has such finite order on all of G, by replacing G with $(G_{\sigma^n})_0$, the connected component of the set of σ^n-fixed points in G.

(2) Let U be the unipotent part of G, and let V be the unipotent part of $C(T_1)$, the Cartan subgroup of T_1. There exists $u \in U$ such that $uT_1u^{-1} = T_2$, and for any such u, $u^{-1}\cdot\sigma(u) \in V$. Therefore it suffices to show that whenever there is a u with $u^{-1}\cdot\sigma(u) \in V$, then there must exist $v \in V$ with $u^{-1}\cdot\sigma(u) = v^{-1}\cdot\sigma(v)$. For in that case, uv^{-1} is σ-fixed with $uv^{-1}T_1vu^{-1} = T_2$.

In view of (1) and (2) it suffices to prove the following lemma:

LEMMA 3. *Let G be a unipotent clag with automorphism σ of finite order n* (prime to p when $p > 0$). *If G has a σ-stable subclag H, and an element $g \in G$ such that $g^{-1}\cdot\sigma(g) \in H$, then $\exists h \in H$ such that $g^{-1}\cdot\sigma(g) = h^{-1}\cdot\sigma(h)$.*

Proof. For any subset X of G, denote by X_σ the σ-fixed point set of X; and for any element $x \in G$, set $\alpha(x) = x^{-1} \cdot \sigma(x)$. There is no non-identity element of the form $\alpha(x)$ in G_σ, because if $\alpha(x) \in G_\sigma$ for some $x \in G$, then

$$\begin{aligned}(\alpha(x))^n &= \alpha(x) \cdot \sigma(\alpha(x)) \cdot \sigma^2(\alpha(x)) \cdots \sigma^{n-1}(\alpha(x)) \\ &= x^{-1} \cdot \sigma(x) \cdot \sigma(x^{-1}) \cdot \sigma^2(x) \sigma^2(x^{-1}) \cdots \sigma^{n-1}(x^{-1}) \sigma^n(x) \\ &= x^{-1} \sigma^n(x) = e ;\end{aligned}$$

but only the identity element can be both unipotent and of order n.

Case I. H normal in G. H is unipotent, hence nilpotent, so one may use induction on the length l of the lower central series for H.

If $l = 1$, then H is commutative, so $\alpha|_H$ is an endomorphism of H with kernel H_σ and image $\alpha(H)$. Therefore $\dim H = \dim H_0 + \dim \alpha(H)$, and $H_\sigma \cap \alpha(H) = \{e\}$. So $H = H_\sigma \cdot \alpha(H)$ as a direct product.

Thus $\exists h_1 \in H_\sigma$, $h_2 \in H$ such that $\alpha(g) = h_1 \cdot \alpha(h_2)$. That is, $g^{-1} \cdot \sigma(g) = h_1 \cdot h_2^{-1} \cdot \sigma(h_2) = h_2^{-1} \cdot h_1 \cdot \sigma(h_2)$; and this implies that

$$h_2 \cdot g^{-1} \cdot \sigma(g) \sigma(h_2^{-1}) = (g h_2^{-1})^{-1} \cdot \sigma(g h_2^{-1}) = \alpha(g h_2^{-1}) = h_1 \in H_\sigma \subseteqq G_\sigma .$$

So $\alpha(g h_2^{-1}) = e = h_1$ and $\alpha(g) = \alpha(h_2)$.

Now suppose $l > 1$. If $\alpha(g) \in H^1$, then by induction $\exists h \in H^1$ with $\alpha(g) = \alpha(h)$. So suppose $\alpha(g) \notin H^1$. Then $\overline{\alpha(g)} \neq \bar{e}$ in $\bar{H} = \pi_{H^1}(H)$, where π_{H^1} is the projection of G with kernel H^1. $\bar{H}$ is commutative, and $\overline{\alpha(g)} = \overline{g^{-1} \cdot \sigma(g)} = \bar{g}^{-1} \cdot \bar{\sigma}(\bar{g}) = \bar{\alpha}(\bar{g})$, so as in the case for $l = 1$, $\exists \bar{h} \in \bar{H}$ such that $\overline{\alpha(g)} = \overline{\alpha(h)}$. That is,

$$\bar{g}^{-1} \cdot \bar{\sigma}(\bar{g}) = \bar{h}^{-1} \cdot \bar{\sigma}(\bar{h}) , \quad \text{and} \quad \overline{(gh^{-1})^{-1} \cdot \sigma(gh^{-1})} = \bar{e} .$$

In other words, $\alpha(gh^{-1}) \in H^1$, whence by induction $\exists h' \in H^1$ such that $\alpha(gh^{-1}) = \alpha(h')$. We now have $(gh^{-1})^{-1} \sigma(gh^{-1}) = hg^{-1} \sigma(g) \sigma(h)^{-1} = h'^{-1} \cdot \sigma(h')$, implying $g^{-1} \cdot \sigma(g) = h^{-1} h'^{-1} \cdot \sigma(h') \sigma(h) = (h'h)^{-1} \sigma(h'h)$. Hence $\alpha(g) = \alpha(h'h) \in \alpha(H)$.

Case II. If H is not normal in G, set $H = G_1$, and let G_i be the connected normalizer in G of G_{i-1}, for $i \geqq 2$. Since a proper subclag of a nilpotent clag is properly contained in its connected normalizer by ([1], 20.3), there is a chain of σ-stable subclags of G:

$$H = G_1 \underset{+}{\Delta} G_2 \underset{+}{\Delta} \cdots \underset{+}{\Delta} G_r = G ,$$

each of which is a normal and proper subclag of the following one.

Now the element $g \in G$ with which we are concerned is contained in G_i for some (minimal) i, with $i \geqq 2$. Since $\alpha(g) \in H \subseteqq G_{i-1}$, and G_{i-1}

is normal in G_i, there is by Case I an element $g_{i-1} \in G_{i-1}$ for which $\alpha(g) = \alpha(g_{i-1})$.

If $(i-1) \geqq 2$, apply Case I again to obtain an element $g_{i-2} \in G_{i-2}$ for which $\alpha(g_{i-1}) = \alpha(g_{i-2})$, since $\alpha(g_{i-1}) \in H \subseteqq G_{i-2}$, and G_{i-2} is normal in G_{i-1}.

Similarly, by a total of $(i-1)$ application of Case I, one obtains an element $h \in H = G_1 = G_{i-(i-1)}$, for which $\alpha(h) = \alpha(g_2) = \alpha(g_3) = \cdots = \alpha(g_{i-1}) = \alpha(g)$.

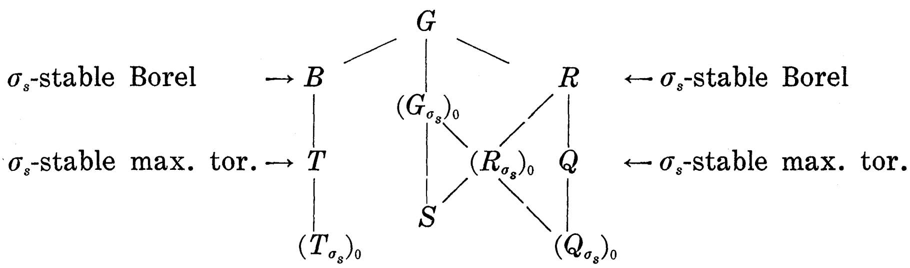
This completes the proof of Theorem 2.

THEOREM 4. *Let G have two σ-stable pairs, $T_i \subseteqq B_i$ $(i = 1, 2)$. If $p = 0$, or if σ is semisimple, then the T_i $(i = 1, 2)$ are conjugate by a σ-fixed point of G.*

Proof. Let $T \subseteqq B$ be any σ-stable pair of G.

First consider σ_s, the semisimple component of σ. (Any σ-stable clag is also σ_s-stable.)

Let S be a maximal torus of $(G_{\sigma_s})_0$. By ([3], 7.4), $S \subseteqq a$ σ_s-stable Borel subgroup R of G. S is also a maximal torus of $(R_{\sigma_s})_0$.



By ([3], 7.6), R has a σ_s-stable maximal torus Q. Now $R = Q \cdot V$ (semi-direct product), where V is the unipotent part of R. So any σ_s-fixed point $f \in R$ has Jordan decomposition $f = q \cdot v$ for some $q \in Q$, $v \in V$. Thus $f = \sigma_s(f) = \sigma_s(q)\sigma_s(v)$, with $\sigma_s(q) \in Q$, $\sigma_s(v) \in V$, whence $\sigma_s(q) = q$ and $\sigma_s(v) = v$. Hence $(R_{\sigma_s})_0 = (Q_{\sigma_s})_0 \cdot (V_{\sigma_s})_0$, and $(Q_{\sigma_s})_0$ is a maximal torus of $(R_{\sigma_s})_0$. Thus $\dim (Q_{\sigma_s})_0 = \dim S$, so $(Q_{\sigma_s})_0$ is also a maximal torus of $(G_{\sigma_s})_0$.

Now $\exists g \in G$ such that $gRg^{-1} = B$, $gQg^{-1} = T$, and (since $Q \subseteqq R$, $T \subseteqq B$ are all σ_s-stable), $g^{-1} \cdot \sigma_s(g) \in N_G(R) \cap N_G(Q) = R \cap N_G(Q) = C(Q)$, the Cartan subgroup of Q in G. This implies that $g(Q_{\sigma_s})_0 g^{-1} = (T_{\sigma_s})_0$, so that $\dim (T_{\sigma_s})_0 = \dim (Q_{\sigma_s})_0$, and $(T_{\sigma_s})_0$ is itself a maximal torus of $(G_{\sigma_s})_0$.

Moreover, $(T_{\sigma_s})_0$ is a torus of $(G_{\sigma_u})_0$, because $T \subseteqq (G_{\sigma_u})_0$. Therefore $(T_{\sigma_s})_0$ is a maximal torus of $(G_{\sigma_s})_0 \cap (G_{\sigma_u})_0 = (G_\sigma)_0$. Thus the $[(T_i)_{\sigma_s}]_0$ are both maximal tori of $(G_\sigma)_0$; so they are conjugate by a fixed point $y \in (G_\sigma)_0$, that is, $y(T_{1_{\sigma_s}})_0 y^{-1} = (T_{2_{\sigma_s}})_0$. Set $T_3 = yT_1y^{-1}$.

Both T_2 and T_3 belong to the connected centralizer Z of $(T_{2_{\sigma_s}})_0$ in G. By ([4], Cor. 4), Z is solvable. Also, Z is σ-stable with maximal tori T_2 and T_3, so by (Thm. 2), T_2 and T_3 are conjugate under a σ-fixed point $z \in Z$; that is, $zT_2z^{-1} = T_3$. Then for $g = y^{-1}z$, g is a σ-fixed point of G for which $gT_2g^{-1} = T_1$.

[Note on the field of definition κ: If κ is algebraically closed, the point of conjugacy in Theorems 2 and 4 may be taken to be κ-rational; and theorems analogous to Theorems 2 and 4 hold for κ-groups. The proofs are mechanical glosses on those here and are found in the author's Ph. D. thesis.]

Counterexample A-1. σ is semisimple; G has two σ-stable maximal tori which are not both contained in σ-stable Borel subgroups, and are not conjugate by a σ-fixed point:

Take $G = SL(2, \Omega)$, $p \neq 2$. Let T_1 consist of matrices of the form

$$\begin{bmatrix} \alpha & 0 \\ 0 & \dfrac{1}{\alpha} \end{bmatrix}, \quad \alpha \neq 0\,;$$

and let T_2 be given by matrices of the form

$$\begin{bmatrix} \dfrac{1}{2}\left(\alpha + \dfrac{1}{\alpha}\right), & \dfrac{1}{2}\left(\alpha - \dfrac{1}{\alpha}\right) \\ \dfrac{1}{2}\left(\alpha - \dfrac{1}{\alpha}\right), & \dfrac{1}{2}\left(\alpha + \dfrac{1}{\alpha}\right) \end{bmatrix}, \quad \alpha \neq 0\,.$$

(T_1 is the maximal torus of G which has diagonal form; T_2 is the conjugate of T_1 by the element

$$\left.\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \in G\,.\right)$$

Take $\sigma = \mathrm{Inn}_G\, g$, where $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The effect of σ is to interchange diagonally the corner entries in each matrix of G. The σ-fixed point set G_σ of G is therefore

$$G_\sigma = \left\{\begin{bmatrix} a & b \\ b & a \end{bmatrix} : a^2 - b^2 = 1\right\}.$$

G_σ is infinite; and since $\sigma^2 = 1$ and $p \neq 2$, σ is semisimple.

Now T_2 is pointwise σ-fixed, and T_1 is not, although it is σ-stable. So T_1 and T_2 cannot be conjugate by a σ-fixed point of G.

(*Note*. The only Borel subgroups of G containing T_1 are the upper and lower triangular matrix groups in G, and σ leaves neither of these stable, but maps one onto the other.)

Counterexample A-2. σ (nonalgebraic) is the Frobenius map for $p=2$, having finite fixed-point set; G has two σ-stable maximal tori which are not both contained in a σ-stable Borel subgroup, and are not conjugate by a σ-fixed point.

Take $G = SL(2, \Omega)$, $p = 2$. Let

$$T_1 = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \dfrac{1}{\alpha} \end{bmatrix} : 0 \neq \alpha \in \Omega \right\} ;$$

and let

$$T_2 = \left\{ \begin{bmatrix} \alpha + a\left(\alpha + \dfrac{1}{\alpha}\right), & \left(\alpha + \dfrac{1}{\alpha}\right) \\ \left(\alpha + \dfrac{1}{\alpha}\right), & \dfrac{1}{\alpha} + a\left(\alpha + \dfrac{1}{\alpha}\right) \end{bmatrix} : 0 \neq \alpha \in \Omega ; \quad a \text{ fixed such that } a^2 + a + 1 = 0. \right\}$$

For σ take the Frobenius map $\sigma \colon (x_{ij}) \longrightarrow (x_{ij}^2)$. T_1 is clearly σ-stable. $T_2 = xT_1x^{-1}$, where

$$x = \begin{bmatrix} a & (a+1) \\ (a+1) & a \end{bmatrix}, \quad \text{and} \quad x^{-1}\cdot\sigma(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in N(T_1) ,$$

so T_2 is σ-stable too.

It can easily be seen that T_1 and T_2 are not conjugate by a σ-fixed point of G, since there are only 6 fixed points.

Counterexample B. σ is semisimple; G has two σ-stable pairs; but the σ-stable Borel subgroups are not conjugate by a σ-fixed point.

Take G and σ as in Counterexample A-1 ($p \neq 2$). Let

$$T = T_2 = \left\{ \begin{bmatrix} \dfrac{1}{2}\left(\alpha + \dfrac{1}{\alpha}\right) & \dfrac{1}{2}\left(\alpha - \dfrac{1}{\alpha}\right) \\ \dfrac{1}{2}\left(\alpha - \dfrac{1}{\alpha}\right) & \dfrac{1}{2}\left(\alpha + \dfrac{1}{\alpha}\right) \end{bmatrix} : 0 \neq \alpha \in \Omega \right\} .$$

Set $\Delta = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{bmatrix} : \alpha, \beta \in \Omega, \alpha \neq 0 \right\} \subseteqq G$, a Borel subgroup of G. Set

$$x = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \in G\,, \quad \text{and} \quad y = \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \in G\,.$$

Take

$$B_1 = x\Delta x^{-1} = \left\{ \begin{bmatrix} \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) - \beta, & \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right) + \beta \\ \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right) - \beta, & \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) + \beta \end{bmatrix} : \alpha, \beta \in \Omega, \alpha \neq 0 \right\}$$

and

$$B_2 = y\Delta y^{-1} = \left\{ \begin{bmatrix} \frac{1}{2}\left(\frac{1}{\alpha} + \alpha\right) + \beta, & \frac{1}{2}\left(\frac{1}{\alpha} - \alpha\right) + \beta \\ \frac{1}{2}\left(\frac{1}{\alpha} - \alpha\right) - \beta, & \frac{1}{2}\left(\frac{1}{\alpha} + \alpha\right) - \beta \end{bmatrix} : \alpha, \beta \in \Omega, \alpha \neq 0 \right\}.$$

Recalling that σ diagonally interchanges the entries of a matrix, one sees that B_1 and B_2 are σ-stable, and T is pointwise σ-fixed. Moreover, T is clearly a maximal torus of both B_1 and B_2 (i.e., when $\beta = 0$). So $T \subsetneqq B_1$ and $T \subsetneqq B_2$ are σ-stable pairs.

Suppose now that B_1, B_2 are conjugate by a σ-fixed point $f \in G_\sigma$, i.e., that $B_1 = fB_2f^{-1}$. Then $B_1 = x\Delta x^{-1} = fB_2f^{-1} = fy\Delta y^{-1}f^{-1} \Rightarrow \Delta = x^{-1}fy\Delta y^{-1}f^{-1}x \Rightarrow x^{-1}fy \in N_G(\Delta) = \Delta$.

Say that $x^{-1}fy = b = \begin{bmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{bmatrix} \in \Delta$, and $f = \begin{bmatrix} \gamma & \delta \\ \delta & \gamma \end{bmatrix} \in G_\sigma$, for some α, β, γ, $\delta \in \Omega$ with $\gamma^2 - \delta^2 = 1$, and $\alpha \neq 0$. Then

$$\begin{aligned}
x^{-1}fy = b &\Longrightarrow fy = xb \\
&\Longrightarrow \begin{bmatrix} \gamma & \delta \\ \delta & \gamma \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 0 & \frac{1}{\alpha} \end{bmatrix} \\
&\Longrightarrow \begin{bmatrix} \gamma - \delta & \frac{1}{2}(\gamma + \delta) \\ \delta - \gamma & \frac{1}{2}(\gamma + \delta) \end{bmatrix} = \begin{bmatrix} \alpha & \beta - \frac{1}{2\alpha} \\ \alpha & \beta + \frac{1}{2\alpha} \end{bmatrix} \\
&\Longrightarrow \gamma - \delta = \delta - \gamma \Rightarrow \gamma = \delta \Rightarrow \gamma^2 - \delta^2 = 0\,,
\end{aligned}$$

a contradiction of the fact that $\gamma^2 - \delta^2 = 1$.

Thus B_1 and B_2 cannot be conjugate by a σ-fixed point of G.

Counterexample C. G solvable, σ unipotent, and $p > 0$. G has two σ-stable maximal tori which are not conjugate by a σ-fixed point.

Take $p = 2$.

Let T be the torus $\subsetneqq GL(6, \Omega)$ consisting of diagonal matrices t of the form

$$t = \left[\begin{array}{ccc|ccc} \tau_1 & & & & & \\ & \tau_2 & & & & \\ & & \tau_1 & & & \\ \hline & & & \tau_2 & & \\ & & & & \tau_1 & \\ & & & & & \tau_2 \end{array}\right], \quad \tau_1, \tau_2 \in \Omega, \quad \tau_1\tau_2 \neq 0 .$$

Let U be the unipotent clag consisting of upper triangular matrices u of the form

$$u = \left[\begin{array}{ccc|ccc} 1 & \alpha & x & & & \\ & 1 & \beta & & & \\ & & 1 & & & \\ \hline & & & 1 & \beta & y \\ & & & & 1 & \alpha \\ & & & & & 1 \end{array}\right], \quad \alpha, \beta, x, y \in \Omega ,$$

satisfying: $x + y - \alpha\beta = 0$.

The reader may verify that U is closed under multiplication, and since $u^4 = e$, $\forall u \in U$, U is also closed under inverses. Hence U is well-defined.

Moreover, U is normalized by T, as the reader again may verify.

One may therefore form the solvable clag $G = T\cdot U$ (semi-direct product).

Let the automorphism σ on $t\cdot u \in G$ be given by the following action on the entries of t and u

$$\begin{aligned} \sigma\colon \tau_1 &\longleftrightarrow \tau_2 \\ \alpha &\longleftrightarrow \beta \\ x &\longleftrightarrow y \end{aligned}$$

σ is thus conjugation by the permutation matrix:

$$s = \left[\begin{array}{ccc|ccc} & & & 1 & & 0 \\ & 0 & & & 1 & \\ & & & 0 & & 1 \\ \hline 1 & & 0 & & & \\ & 1 & & & 0 & \\ 0 & & 1 & & & \end{array}\right].$$

So s and σ are unipotent of order 2.

T is a σ-stable maximal torus of G, whose Cartan subgroup is $C(T) = T \times C(T)_u$, where

$$C(T)_u = \left\{ \left[\begin{array}{ccc|ccc} 1 & 0 & x & & & \\ & 1 & 0 & & 0 & \\ 0 & & 1 & & & \\ \hline & & & 1 & 0 & x \\ & 0 & & & 1 & 0 \\ & & & 0 & & 1 \end{array}\right] : x \in \Omega \right\}.$$

Now if $u \in U$, then uTu^{-1} is σ-stable if and only if $u^{-1}\cdot\sigma(u) \in C(T)_u$. Moreover, $\exists$ a σ-fixed element $f \in U_\sigma$ such that $uTu^{-1} = fTf^{-1}$ if and only if $f^{-1}u \in C(T)_u$; i.e., if and only if $\exists c \in C(T)_u$ such that $uc^{-1} = f$ is σ-fixed.

However, all $c \in C(T)_u$ are σ-fixed; So a σ-stable maximal torus uTu^{-1} of G is conjugate to T by a σ-fixed point if and only if u itself is σ-fixed.

However, for the unipotent matrix

$$u = \left[\begin{array}{ccc|ccc} 1 & \alpha & x & & & \\ & 1 & \alpha & & 0 & \\ 0 & & 1 & & & \\ \hline & & & 1 & \alpha & y \\ & 0 & & & 1 & \alpha \\ & & & 0 & & 1 \end{array}\right],$$

satisfying $x + y - \alpha^2 = 0$, $\alpha \neq 0$, one gets

$$u^{-1}\cdot\sigma(u) = \left[\begin{array}{ccc|ccc} 1 & 0 & -x+y & & & \\ & 1 & 0 & & 0 & \\ 0 & & 1 & & & \\ \hline & & & 1 & 0 & -x+y \\ & 0 & & & 1 & 0 \\ & & & 0 & & 1 \end{array}\right].$$

That is, $u^{-1}\cdot\sigma(u) \in C(T)_u$, so uTu^{-1} is σ-stable. But u is not σ-fixed, so T and uTu^{-1} are not conjugate by a σ-fixed element of G.

(*Note*. This counterexample in $p = 2$ is due to D. Winter. The present author has generalized it in a separate paper for all $p > 0$. The resulting group may be of some interest in itself.)

References

1. Armand Borel, *Groupes lineares algebriques*, Annals of Math., **64** (1956), 20-82.
2. ———, *Linear Algebraic Groups*, W. A. Benjamin, New York, 1969.
3. Robert Steinberg, *Endomorphisms of linear algebraic groups*, A. M. S. Memoirs, **80** (1968).
4. David Winter, *On automorphisms of algebraic groups*, Bull. A. M. S., **72** (1966), 706-708.
5. ———, *Algebraic group automorphisms having finite-fixed point sets*, preprint, 1966.

Received January 12, 1977 and in revised form March 23, 1977. This work represents part of the author's doctoral dissertation Rutgers University, May, 1973. The author wishes to express her gratitude to her advisor, Professor Earl Taft, for his guidance and encouragement.

PURDUE UNIVERSITY
WEST LAFAYETTE, IN 47907