PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No. 1, 1978

UNIQUENESS OF EXTENSIONS OF POSITIVE LINEAR FUNCTIONS

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In this paper necessary and sufficient conditions that every approximated function has a unique maximal approximated extension are given. When applied to the Choquet situation this gives a new approach to known uniqueness results for representing measures.

Extensions of positive linear functions preserving a certain approximation property were studied in a previous paper [4]. This led to a unified approach to integration theory and the Choquet-Bishop-de Leeuw theorem.

Notation will be that of [4]. In particular V and Y will designate ordered vector spaces; G will be a subspace of V; W will be a wedge in V such that $G \subset W \subset G + V^+$ and α will be a positive linear function from G to Y.

If $f \in V$ and $A \subset V$ we say that f *α-dominates* A if, for every g in A such that $g \leqq f$, the following holds; for every pair y, z in Y such that $y \geqq \alpha(f - V^+)$ and $z \leqq \alpha(g + V^+)$ we have $y \geqq z$. This is the condition of Theorem 2.1 in [4]. If $\underline{\alpha}(f) = \sup\{\alpha(h): f \geqq h \in G\}$ and $\bar{\alpha}(g) = \inf\{\alpha(h): g \leqq h \in G\}$ both exist in Y, this condition is equivalent to the requirement $\underline{\alpha}(f) \geqq \bar{\alpha}(g)$. Now the proof of Theorem 2.1 can be modified easily to yield the following: If $f \in W$ and f α-dominates $(-W)$ then f is in $\operatorname{dmn} \underline{\alpha}$ if and only if all maximal W-extensions are defined and give the same value on f. If Y is assumed Dedekind complete the converse holds; the equivalence implies that f α-dominates $(-W)$.

We define a "closure", G_1, of a subspace $G \subset V$ as $G_1 = \{f \in V: \exists g \in G^+ \text{ with } (f - \lambda g + V^+) \cap (f + \lambda g - V^+) \cap G \neq \varnothing \text{ for all } \lambda > 0\}$. It is easy to see that G_1 is a subspace containing G. If G contains an order unit u, then G_1 is just the closure of G in the order-unit normed space $(\boldsymbol{R}u + V^+) \cap (\boldsymbol{R}u - V^+)$.

If $f \in V$ and $A, B \subset V$ we say that *A separates f and B* if, for each $g \in B$, there exists $h \in A$ such that $f \geqq h \geqq g$.

THEOREM 1. *Let Y be an Archimedean space. If $f \in W$ is such that G_1 separates f and $(-W) \cap (f - V^+)$ then f α-dominates $(-W)$ for every α. Consequently if the separation holds for all $f \in W$ then every α has a unique maximal W-extension.*

Proof. Suppose $f \geqq g \in (-W)$. By hypothesis there exists $h \in G_1$

such that $f \geqq h \geqq g$. By definition of G_1 there is $p \in G^+$ such that, for any $\delta > 0$, there is $q_\delta \in G$ with $h - \delta p \leqq q_\delta \leqq h + \delta p$. Then $q_\delta + \delta p \in (g + V^+) \cap G$ so if $z \leqq \alpha(g + V^+)$ we have $z \leqq \alpha(q_\delta + \delta p)$. Similarly if $y \geqq \alpha(f - V^+)$ we find $y \geqq \alpha(q_\delta - \delta p)$. Then $z - y \leqq 2\delta\alpha(p)$ and the Archimedean property of Y gives $y \geqq z$.

Now we wish to investigate under what conditions the unique extension property implies the G_1 separation of Theorem 1.

LEMMA. *Assume G has an order unit u and suppose $Y^+ \neq \{0\}$. If every α has a unique maximal W-extension then, for every $f \in W \cap (G - V^+)$ and $\varepsilon > 0$, G separates $f + \varepsilon u$ and $(-W) \cap (f - V^+) \cap (G + V^+)$.*

Proof. We will suppose that there is $f \in W \cap (G - V^+)$ and $\varepsilon > 0$ such that G does not separate $f + \varepsilon u$ and $(-W) \cap (f - V^+) \cap (G + V^+)$. Then there exists $g \in (-W) \cap (f - V^+) \cap (G + V^+)$ such that if $A = \{h \in G: h \geqq g + (\varepsilon/2)u\}$, $B = \{h \in G: h \leqq f + \varepsilon u\}$ and $U = \{h \in G: -(\varepsilon/2)u \leqq h \leqq (\varepsilon/2)u\}$ then $(A + U) \cap B = \varnothing$. Since U is radial at the origin as a subset of G a standard separation result [3; p. 23] shows that there exists a linear functional φ on G which strongly separates A and B. By taking $-\varphi$ if necessary we can assume that $r_0 = \sup\{\varphi(p): p \in B\} < \inf\{\varphi(q): q \in A\} = s_0$. Now let $p \in G^+$. Then if $f_1 \in B$ and $f_2 \in A$ we have $\varphi(f_2 + rp) = \varphi(f_2) + r\varphi(p) \geqq \varphi(f_1)$ for all $r \geqq 0$. This shows that $\varphi(p) \geqq 0$ and we see that φ is a positive linear functional on G. Then we have $\underline{\varphi}(f + \varepsilon u) = r_0 < s_0 = \bar{\varphi}(g + (\varepsilon/2)u)$. Since $g + (\varepsilon/2)u \in (-W) + G \subset (-W)$ we see that $f + \varepsilon u$ does not φ-dominate $(-W)$. Since $\boldsymbol{R}$ is Dedekind complete Theorem 2.1 of [4] shows that φ does not have a unique maximal W-extension. Now choose $y > 0$ in Y and define $\alpha: G \rightarrow Y$ by $\alpha(f) = \varphi(f)y$. Then it is easy to see that α does not have a unique maximal W-extension.

THEOREM 2. *In addition to the assumptions of the lemma we assume that V is Dedekind σ-complete and W is closed under finite infs. Then, if every α has a unique maximal W-extension, G_1 separates f and $(f - V^+) \cap (-W)$ for all $f \in W$.*

Proof. Given any $f \in W$ and $g \leqq f$ such that $g \in -W$ we wish to show that G_1 separates f and g. Now we can assume, without loss of generality, that $f \in G - V^+$ and $g \in G + V^+$. If this were not so we could choose $g' \in G \cap (f - V^+)$ and (using the assumptions that W is closed under finite infs and $W \subset G + V^+) f' \in G \cap (g \vee g' + V^+)$. Then we could replace f by $f \wedge f'$ and g by $g \vee g'$. Clearly any element which separates $f \wedge f'$ and $g \vee g'$ will separate f and g.

We adapt a technique of Edwards [2] to find $h \in G_1$ such that

$f \geqq h \geqq g$. Let $g_0 = g - u$ and $f_0 = f + u$ and use the lemma to choose $h_0 \in G$ such that $g_0 \leqq h_0 \leqq f_0$. Now assume that for $m = 1, 2, \cdots, n$ we have $f_m \in W$, $g_m \in -W$ and $h_m \in G$ such that $g - 2^{-m}u \leqq g_m \leqq h_m \leqq f_m \leqq f + 2^{-m}u$ and $-3 \cdot 2^{-m-1}u \leqq h_m - h_{m-1} \leqq 3 \cdot 2^{-m-1}u$.

Let

$$g_{n+1} = (g - 2^{-n-1}u) \vee (h_n - 3 \cdot 2^{-n-2}u) \in -W$$

while

$$\begin{aligned} f_{n+1} &= (f + 2^{-n-1}u) \wedge (h_n + 3 \cdot 2^{-n-2}u) \\ &= (f + 2^{-n-2}u) \wedge (h_n + 2^{-n-1}u) + 2^{-n-2}u \in W . \end{aligned}$$

Now $g_{n+1} + 2^{-n-2}u \leqq f_{n+1}$ results from the following inequalities:

$$g - 2^{-n-1}u \leqq f + 2^{-n-2}u \; ; \quad g - 2^{-n-1}u \leqq h_n + 2^{-n-1}u \; ;$$
$$h_n - 3 \cdot 2^{-n-2}u \leqq f + 2^{-n-2}u \; ; \quad \text{and} \quad h_n - 3 \cdot 2^{-n-2}u \leqq h_n + 2^{-n-1}u \; .$$

Hence we can use the lemma to choose $h_{n+1} \in G$ such that $g - 2^{-n-1}u \leqq g_{n+1} \leqq h_{n+1} \leqq f_{n+1} \leqq f + 2^{-n-1}u$ and $-3 \cdot 2^{-n-2}u \leqq g_{n+1} - h_n \leqq h_{n+1} - h_n \leqq f_{n+1} - h_n \leqq 3 \cdot 2^{-n-2}u$. This completes the inductive definition.

Now $-3 \cdot 2^{-n-2}u \leqq h_{n+1} - h_n \leqq 3 \cdot 2^{-n-2}u$ implies $-3 \cdot 2^{-m-1}u \leqq h_p - h_m \leqq 3 \cdot 2^{-m-1}u$ for all $p \geqq m$.

Now let $h = \inf_n (\sup_{k \geqq n} h_k)$ which exists by the inequality for $h_p - h_m$ and the fact that V is Dedekind σ-complete. From the inequalities $g - 2^{-n-1}u \leqq h_{n+1} \leqq f + 2^{-n-1}u$ we conclude, since a Dedekind σ-complete space is Archimedean, that $g \leqq h \leqq f$. Since we can replace h_p by h in the inequality for $h_p - h_m$ we see that $h \in G_1$ as desired.

Now in the approach to Choquet boundary theory given in [4] we assume that V is the space of continuous functions on a compact Hausdorff space X, G is a closed subspace and W is a wedge of bounded continuous functions on X closed under finite infs. Then $G = G_1$ and, since W-approximated linear functionals are maximal measures, we see that uniqueness of representing "Choquet" measures implies the separation of Theorem 2. This gives the "geometric simplex" result of Boboc and Cornea [1, Th. 4]. If we let X be a convex compact subset of a locally convex space, G the continuous affine functions and W the wedge of finite infs from G then we find that the separation property reduces in this case to the interpolation version of the Riesz decomposition property. This gives the "Choquet simplex" result of Edwards [2].

We now investigate an alternate characterization of the space G_1. We define G_Y to be the largest subspace of V such that every positive linear $\alpha: G \to Y$ has a unique positive linear extension to G_Y. In the notation of [4] we can write G_Y as $\bigcap$ {dmn α_G: α positive and linear from G to $Y\}$.

THEOREM 3. *If Y is Dedekind σ-complete $G_1 \subset G_Y$.*

Proof. If $f \in G_1$ then there exists $g \in G^+$ and a sequence $\{h_n\} \subset G$ such that $h_n - (1/n)g \leqq f \leqq h_n + (1/n)g$ for all n. This gives us

$$-\left(\frac{1}{n} + \frac{1}{m}\right)g \leqq h_n - h_m \leqq \left(\frac{1}{n} + \frac{1}{m}\right)g$$

for all n and m. Now let α be any positive linear function from G to Y. Then

$$-\left(\frac{1}{n} + \frac{1}{m}\right)\alpha(g) \leqq \alpha(h_n) - \alpha(h_m) \leqq \left(\frac{1}{n} + \frac{1}{m}\right)\alpha(g)$$

gives $-(1/m)\alpha(g) \leqq y - \alpha(h_m) \leqq (1/m)\alpha(g)$ for $y = \inf_n (\sup_{k \geqq n} \alpha(h_k))$. Then $h_n - (1/n)g \leqq f \leqq h_n + (1/n)g$ for all n implies $\bar{\alpha}(f) \leqq y \leqq \underline{\alpha}(f)$. From this it is not hard to see that every maximal positive extension of α assumes the value y. Since α was arbitrary we conclude that $f \in G_Y$.

THEOREM 4. *If G has an order-unit and $Y^+ \neq \{0\}$ then $G_Y \subset G_1$.*

Proof. Note first that if $f \in G_Y$ we must have $f \in (G - V^+) \cap (G + V^+)$. For if f is not in $G - V^+$ let $\hat{\alpha}$ be a maximal positive extension of a positive linear α from G to Y. Choose $y > 0$ in Y and define α_1 by $\alpha_1(g + rf) = \alpha(g) + r(\hat{\alpha}(f) + y)$. Then since $\hat{\alpha}$ is positive and $f \notin G - V^+$ it is easy to see that α_1 is positive. Then any maximal extension of α_1 contradicts $f \in G_Y$. A symmetric argument gives $f \in G + V^+$.

Now if $f \in G_Y$ let $W = G + \boldsymbol{R}f$. Then the lemma applies and we can assume that G separates $f + \varepsilon u$ and $(-W) \cap (f - V^+) \cap (G + V^+)$ for all $\varepsilon > 0$. Since $f \in (W) \cap (-W) \cap (f - V^+) \cap (G + V^+)$ we see there exists $h_\varepsilon \in G$ such that $f \leqq h_\varepsilon \leqq f + \varepsilon u$ for all $\varepsilon > 0$. Hence $f \in G_1$ as desired.

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Received June 24, 1977.

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