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EXISTENCE OF A STRONG LIFTING COMMUTING WITH A COMPACT GROUP OF TRANSFORMATIONS

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Let G be a locally compact group with left Haar measure γ. The well-known "Theorem LCG" ([10]) states that there is a strong lifting of $M^\infty(G, \gamma)$ commuting with left translations. We will prove partial generalizations of this theorem in case G is *compact*. Thus, let (G, X) be a *free* (left) transformation group with G, X compact such that (I) G is abelian, or (II) G is Lie, or (III) X is a product $G \times Y$. Let ν_0 be a Radon measure on $Y = X/G$, and let μ be the Haar lift of ν_0 We will show that, if ρ_0 is a strong lifting of $M^\infty(Y, \nu_0)$, then there is a strong lifting $M^\infty(X, \mu)$ which extends ρ_0 and commutes with the action of G.

The proof is modeled on the proof of LCG in ([10]), and follows it closely in several places. The main difference is in the present use of the fact that, if (H, X) is a free transformation group with H *Lie*, then (H, X) admits local sections.

DEFINITIONS 1.1. Let X be a compact Hausdorff space. Let $M_+(X)$ denote the set of positive Radon measures on X of norm 1 with the vague topology. For measure theory, we rely on [2], [3], [4]. If $\eta \in M_+(X)$, let $M^\infty(X, \eta)$ be the set of all bounded η-measurable complex functions on X. If $f \in M^\infty(X, \eta)$, let $N_\infty(f)$ denote its essential supremum. Let $L^\infty(X, \eta)$ be the usual set of equivalence classes modulo null functions.

Define $L^p(X, \eta)$ in the usual way; let N_p be its norm $(1 \leqq p < \infty)$. Since X is compact, we can and will assume that

$$L_p(X, \eta) \subset L^r(X, \eta) \quad (1 \leqq r \leqq p \leqq \infty) .$$

DEFINITIONS 1.2. Let W be a topological space, $f: X \rightarrow W$ a map. Say f is *η-Lusin-measurable* if there is a countable collection of pairwise disjoint compact sets K_i such that $X \backslash \bigcup_i K_i$ has η-measure zero and $f|_{K_i}$ is continuous $(i \geqq 1)$.

DEFINITIONS, NOTATION 1.3. Let G be a compact Hausdorff topological group. The pair (G, X) is a *free* (left) *transformation group* (t.g.) if there is a jointly continuous map $G \times X \rightarrow X$: $(g, x) \rightarrow g \cdot x$ such that, if $g \cdot x = x$ for any $g \in G$ and $x \in X$, then $g = \mathrm{idy}$, the identity in G. If $\eta \in M_+(X)$ and $f \in M^\infty(X, \eta)$, let $(f \cdot g)(x) = f(g \cdot x)$; also define $(g \cdot \eta)(f) = \eta(f \cdot g)$ if $f \in C(X)$. *Throughout the paper*, we will let (i) γ be normalized Haar measure on G; (ii) $Y = X/G$ (the quotient under identification of G-orbits) with canonical projection π_0; (iii) ν_0 be a fixed element of $M_+(Y)$ *whose support is all of* Y; (iv) μ be the *G-Haar life* of ν_0 $\left(\text{thus } \mu(f) = \int_Y \left(\int_G f(g \cdot x) d\gamma(g)\right) d\nu_0(y)\right.$ for $\left. f \in C(X)\right)$.

DEFINITION 1.4. Let $\eta \in M_+(X)$. A map ρ of $M^\infty(X, \eta)$ to itself is a *linear lifting of* $M^\infty(X, \eta)$ if (i) $\rho(f) = f$ η-a.e.; (ii) $f_1 = f_2$ η-a.e. $\Rightarrow \rho(f_1) = \rho(f_2)$ *everywhere*; (iii) $\rho(1) = 1$; (iv) $f \geqq 0 \Rightarrow \rho(f) \geqq 0$; (v) $\rho(af_1 + bf_2) = a\rho(f_1) + b\rho(f_2)$ if a, b are constants. If, in addition, $\rho(f \cdot g) = \rho(f) \cdot \rho(g)$, then ρ is a *lifting* of $M^\infty(X, \eta)$. If (i)–(iv) hold (if (i)–(v) hold), and in addition $\rho(f) = f$ all $f \in C(X)$, then ρ is a *strong linear lifting* (*strong lifting*). See ([11], p. 34).

Terminology 1.5. Let H be a closed subgroup of G, $\pi: X \to X/H \equiv Z$ the canonical projection, $\bar{\eta} = \pi(\eta)$. We can and will assume that $M^\infty(Z, \bar{\eta})$ is embedded in $M^\infty(X, \eta)$ via $f \to f \circ \pi$. Let $\bar{\rho}$ be a linear lifting of $M^\infty(X, \eta)$. A linear lifting ρ of $M^\infty(X, \mu)$ *extends* $\bar{\rho}$ if, for all $f \in M^\infty(Z, \bar{\eta})$, $\rho(f) = \bar{\rho}(f)$. Say ρ is *H-invariant if* $(f \cdot h) = \rho(f) \cdot h$ for all $h \in H$, $f \in M^\infty(X, \eta)$.

DEFINITIONS, RESULTS 1.6. Let $f: X \to E$ where E is a Banach space. Say $f \in M^\infty(X, E, \eta)$ if (i) $f(X) \subset E$ is weakly compact, (ii) $x \to \langle f(x), e \rangle \in M^\infty(X, \eta)$ for each continuous linear functional e' on E. If $f \in M^\infty(X, E, \eta)$ and ρ is a linear lifting of $M^\infty(X, \eta)$, one can (abusing notation) define a map $\rho(f): X \to E$ which satisfies

$$\langle \rho(f)(x), e' \rangle = \rho \langle f(\bar{x}), e' \rangle (x)$$

for each $x \in X$ and $e' \in E' =$ topological dual of E (on the right-hand side, we apply ρ to the map $\bar{x} \to \langle f(\bar{x}), e' \rangle$, then valuate at x). If E is separable, then (iii) $\rho(f) = f$ η-a.e. For arbitrary E, (iv) $f_1 = f_2$ η-a.e. implies $\rho(f_1) = \rho(f_2)$ everywhere; (v) $\|f(x)\| \leqq M < \infty$ η-a.e. implies $\|\rho(f)(x)\| \leqq M$ for *all* x. For a more general discussion and proofs, see ([11], Chapter 6, §§4 and 5).

DEFINITIONS, RESULTS 1.7. A *D′-sequence in G* ([7]) is a sequence $(W_n)_{n=1}^\infty$ of γ-measurable subsets of G such that (i) $W_n \supset W_{n+1}$ $(n \geqq 1)$; (ii) $0 < \gamma(W_n \cdot W_n^{-1}) < C \cdot \gamma(W_n)$ for some $C > 0$ and all n; (iii) every neighborhood of idy contains some W_n. Every Lie group has a D' sequence consisting of compact neighborhoods of idy (for a stronger statement, see [7], Theorem 2.9). If (W_n) is a D'-sequence in G, then the Main Derivation Theorem ([7], Theorem 2.5) states that, if $f \in L^1(G, \gamma)$, then

$$\text{(version 1)} \qquad \lim_{n\to\infty} \frac{1}{\gamma(W_n)} \int_G f(g)\psi_{\bar{g}\cdot W_n}(g)d\gamma(g) = f(\bar{g}) \quad \text{for} \quad \gamma\text{-a.a. } \bar{g}\,;$$

$$\text{(version 2)} \qquad \lim_{n\to\infty} \frac{1}{\gamma(W_n)} \int_G f(g)\psi_{W_n\cdot\bar{g}}(g)d\gamma(g) = f(\bar{g}) \quad \text{for} \quad \gamma\text{-a.a. } \bar{g}\,;$$

here ψ denotes characteristic function. (Version 1 is Theorem 2.5; version 2 follows because γ is a *right* Haar measure as well as a *left* Haar measure.) If $f \in C(G)$, then it is easily seen that the equalities hold for *all* $\bar{g}$ in both versions.

2. A reduction.

NOTATION 2.1. Let X, G, μ, ν_0, etc. be as in 1.3; ρ_0 will henceforth denote a fixed strong lifting of $M^\infty(Y, \nu_0)$. Recall Support $(\nu_0) = Y$; hence Support $(\mu) = X$.

THEOREM 2.2. *Suppose (G, X) is a free left transformation group such that:* (I) *G is abelian, or* (II) *G is Lie, or* (III) *X is a product $G \times Y$. Then there is a strong lifting of $M^\infty(X, \mu)$ which extends ρ_0 and commutes with G.*

The goal in §2 is to show that 2.2 is a consequence of 2.7 below; 2.7 is then proved in §3. We begin with the following result; it is proved in ([10], p. 85, Remark 2).

LEMMA 2.3. *Let P be closed normal subgroup of G, $P \neq \{\text{idy}\}$. There exists a closed subgroup $K \subsetneqq P$ which is normal in G such that:* (i) *$P/K = H$ is a Lie group;* (ii) *$(G/K)/H \cong G/P$ (here H is assumed embedded in G/K).*

Discussion 2.4. Let P be as above; consider the free t.g. $(G/P, X/P)$. Note that H acts on X/K; it is easily seen that $(X/K)/H \cong X/P$. That is, X/K is a free *Lie* group extension of X/P.

We fix more terminology.

Terminology 2.5. Let H be a closed normal Lie subgroup of G. Let $Z = X/H$, $\pi: X \to Z$ the projection, $\nu = \pi(\mu)$. Then $(G/H, Z)$ is a free t.g. Let λ be normalized Haar measure on H.

Discussion 2.6. For $z \in Z$, let $\lambda_z \in M_+(X)$ be given by

$$\lambda_z(f) = \int_H f(h \cdot x) d\lambda(h)$$

for some (hence any) $x \in \pi^{-1}(z)(f \in C(X))$. The map $z \to \lambda_z$ is a disintegration of μ with respect to π ([4], p. 63); observe that the map $z \to \lambda_z$ is clearly vaguely continuous, hence ν-adequate. (See [3], Def. 1, p. 18; Prop. 2, p. 19.) Thus, if $f \in L^1(X, \mu)$ (in particular if f is the characteristic function ψ_A of a μ-measurable set A), then $z \to \lambda_z(f)$ is defined ν-a.e., is ν-measurable, and

$$\int_X f(x) d\mu(x) = \int_Z \lambda_z(f) d\nu(z)$$

(this follows from ν-adequacy; see [3], Thm. 1a, p. 26).

THEOREM 2.7. *Let H, Z, ν, π be as in* 2.5, *and suppose there is a strong lifting δ of $M^\infty(Z, \nu)$ which commutes with G/H. Then there is a strong lifting ρ of $M^\infty(X, \mu)$ which commutes with G and extends δ.*

Proof of 2.2, *using* 2.7. For each closed normal subgroup P of G, let $\pi_p: X \to X/P$ be the projection. Let J be the set of all pairs (P, β), where β is strong lifting of $M^\infty(X/P, \pi_p(\mu))$ which commutes with G/P and extends ρ_0. Note $J \neq \varnothing$, since $(G, \rho_0) \in J$. Order J as follows: $(P_1, \beta_1) \leqq (P_2, \beta_2)$ if and only if $P_2 \subset P_1$ and β_2 extends β_1. Then

(*) $\qquad J$ is inductive for $\leqq$

The proof of (*) is a straightforward modification of the (lengthy and sophisticated) proof of Theorem 4(i) in ([10]); therefore we omit it.

Let (P_∞, β_∞) be a maximal element of J, and suppose $P_\infty \neq \{\text{idy}\}$. By 2.3 and 2.4, we can find a free Lie group extension X/K of X/P_∞ with $K \subsetneqq P_\infty$. By 2.7, there is a strong lifting β_K of $M^\infty(X/K, \pi_K(\mu))$ which commutes with G/K. Hence (K, β_K) is a strict majorant of (P_∞, β_∞), contradicting maximality. Thus $P_\infty = \{\text{idy}\}$, and 2.2 is true if 2.7 is.

REMARK 2.8. In case II (G is Lie group), we can and will assume that $G = H$ in 2.5, 2.6, and 2.7. Hence $\nu_0 = \nu$, $\lambda = \gamma$, $\delta = \rho_0$, and $Z = Y$. In what follows, when case II is discussed, we will use the notation H, ν, λ, and Z, with the above identities taken for granted.

3. **Proof of 2.7.** Notation in §3 will be as in 1.3 and 2.5. In

addition, δ will always be a strong lifting of $M^\infty(Z, \nu)$ which commutes with G/H and extends ρ_0.

The idea of the proof is simple. Suppose X is the product $H \times Z$, and $f \in M(X, \mu)$ (observe $\mu = \lambda \times \nu$). "Define" $\tilde{F}$: $Z \to L^\infty(H, \lambda)$: $\tilde{F}(z) = [f|_{\pi^{-1}(z)}]$ ([ ] denotes equivalence class). Let $F(z) = \delta(\tilde{F})(z)$ (see 1.6). Then, if β is a strong lifting of $M^\infty(H, \lambda)$ commuting with left translations, let $\rho(f)(h, z) = \beta(F(z))(h)$. The difficulties are obvious: is $\tilde{F}$ ν-Lusin-measurable? If it is, is $\rho(f)$ measurable? These difficulties can be overcome. The local product structure of (H, X) will enable us to define an analogue of $\delta(\tilde{F})$ (3.5); we will then (basically) apply β to this analogue.

The following is an immediate consequence of ([12], Theorem 1, Sec. 5.4).

THEOREM 3.1. *For each $x \in X$, there is a compact neighborhood V of x and a compact $F \subset V$ and that* (i) $H \cdot F = V$; (ii) $\pi^{-1}(z) \cap F$ *is a single point whenever $z \in \pi(V)$.*

DEFINITION 3.2. A *proper triple* $(V, \mathcal{O}, \tau)$ *at* $z_0 \in Z$ is defined as follows. Pick $x \in \pi^{-1}(z_0)$, and let V, F be as in 3.1. Then $H \cdot V = V$. Let $\mathcal{O} \subset Z$ be an *open* set such that cls $\mathcal{O} = \pi(V)$. Let τ: $V \to H \times \pi(V)$ be "defined by F"; i.e., if $\pi(x) = z$ and $\pi^{-1}(z) \cap F = \{x_0\}$, then $\tau(x) = (h, z)$ where $h \cdot x_0 = x$.

Clearly τ is a homeomorphism, $\tau(h \cdot x) = h \cdot \tau(x)$ (define $h \cdot (\bar{h}, z) = (h\bar{h}, z)$), and $\tau(\mu|_V) = \lambda \otimes (\nu|_{\pi(V)})$.

In 3.3–3.7, fix $z_0 \in Z$.

3.3. Let $f \in M^\infty(X, \mu)$. Recall (1.1) that N_∞ refers to essential supremum. Let $(V, \mathcal{O}, \tau)$ be a proper triple at z_0. Let

$$f_z = f|_{\pi^{-1}(z)}(z \in Z) .$$

For each $z \in \pi(V) = K$ such that $f_z \in M^\infty(X, \lambda_z)$ and $N_\infty(f_z) \leqq N_\infty(f)$, define $b_p(z)$ to be the equivalence class in $L^p(H, \lambda)$ of the function

$$h \longrightarrow f_z \circ \tau^{-1}(h, z)(1 \leqq p < \infty) .$$

Let $b_p(z) = 0$ if f_z does not satisfy the above conditions or if $z \notin K$. By 2.6, $b_p(z)$ equals the equivalence class of $f_z \circ \tau^{-1}$ for ν-a.a.z. We will regard $L^\infty(H, \lambda) \subset L^p(H, \lambda) \subset L^r(H, \lambda)$ $(p \geqq r \geqq 1)$; one then has $b_p(z) = b_r(z)$ for all p, r, z.

LEMMA 3.4. (a) *For* $1 \leqq p < \infty$, $b_p \in M^\infty(Z, L^p(H, \lambda))$ (1.6).

(b) *Let* $B_p(z) = \delta(b_p)(z)$ $(1 \leqq p < \infty)$. *If* $1 \leqq p \leqq r < \infty$, *then* $B_p(z) = B_r(z)$ *for all* z.

(c) *Let* $B(z) = B_p(z)$ *for one (hence all)* $p \in [1, \infty)$. *Then*

$$N_\infty(B(z)) \leqq N_\infty(f)$$

for all z.

Proof. (a) Note that f is a pointwise limit μ-a.e. of a sequence of bounded continuous functions f_n. Using 2.6 and the dominated convergence theorem, one shows that b_p is a pointwise limit ν-a.e. of maps $b^n \colon Z \to L^p(H, \lambda)$ which are (i) continuous on $K = \pi(V)$; (ii) zero outside K. The maps b^n are therefore ν-Lusin-measurable (1.2); hence ([2], Thm. 2, p. 175) b_p is ν-Lusin-measurable. Now the norm $N_p(b_p(z))$ (see 1.1) is $\leqq N_\infty(f)$ for all z. This implies that the range of b_p is bounded, hence weakly compact. We have shown that (i) and (ii) of 1.6 are satisfied, so $b_p \in M^\infty(Z, L^p(H, \lambda))$.

(b) and (c) We obtain (b) from 1.6 and the fact that, if $p < r$, then the dual space $L^p(H, \lambda)'$ may be identified with a subspace of $L^r(H, \lambda)'$. To prove (c), observe that $N_p(B(z)) = N_p(B_p(z)) \leqq N_\infty(f)$ (use v) of (1.6). But $N_\infty(B(z)) = \lim_{p\to\infty} N_p(B(z))$.

Recall $z_0 \in Z$ was fixed through 3.7. Let $pr \colon H \times Z \to H \colon (h, z) \to h$.

DEFINITION 3.5. Let u be an element of the equivalence class $B(z) \in L^\infty(H, \lambda)$. Let $v(x) = \begin{cases} u \circ pr \circ \tau(x) (x \in \pi^{-1}(z)) \\ 0 \quad \text{otherwise} \end{cases}$. Let $R^f(z_0)$ be the equivalence class in $L^\infty(X, \lambda_{z_0})$ of v.

One uses 1.6, 1.4, and the definition just made to prove the following; we omit details.

LEMMA 3.6. (a) $R^{af+bg}(z_0) = aR^f(z_0) + bR^g(z_0)$ $(a, b \in C)$.

(b) $R^f(z_0) \geqq 0$ *if* $f \geqq 0$.

(c) $R^1(z_0) = 1$.

In what follows, we will occasionally be sloppy, and think of $B(z_0)$, $R^f(z_0)$ as *functions*, not equivalence classes. We can write $R^f(z_0)(hx) = B(z_0)(h)$ if $\tau(x) = (\mathrm{idy}, z_0)$.

PROPOSITION 3.7. $R^f(z_0)$ *is independent of the proper triple used in its definition.*

Proof. We first make two observations.

(01) Let $\mathcal{O}^{\text{open}} \subset K^{\text{compact}} \subset Z$. Then $\mathcal{O} \subset \delta(\mathcal{O})(\equiv \delta(\psi_{\mathcal{O}})) \subset \delta(K) \subset K$ ([11], Thm 1, p. 105). Thus if $\varphi_1, \varphi_2 \in M^\infty(Z, \nu)$ and $\varphi_1 = \varphi_2$ for ν-a.a. $z \in K$, then $\delta(\varphi_1) = \delta(\varphi_2)$ on $\mathcal{O}$.

(02) Let u_{ij} $(1 \leqq i, j \leqq n)$ be *coordinate functions* on H defined by some irreducible unitary representation of H ([8], Sec. 27.5). Then $u_{ij}(h_1 \cdot h_2) = \sum_{r=1}^n u_{ir}(h_1) \cdot u_{rj}(h_2) (h_i \in H)$. From the Peter-Weyl theorem ([8], 27.40), the span of the set of all coordinate functions (defined by all irreducible unitary representations of H) is dense in $L^p(H, \lambda)(1 \leqq p < \infty)$.

Let $(V, \mathcal{O}, \tau)$, $(\tilde{V}, \tilde{\mathcal{O}}, \tilde{\tau})$ be proper triples at z_0. Define b_p, $\tilde{b}_p$, B, $\tilde{B}$ as in 3.3, 3.4. Let $K = \pi(V)$, $\tilde{K} = \pi(\tilde{V})$. On $\tilde{\tau}(V \cap \tilde{V})$, one has $\tau \circ \tilde{\tau}^{-1}(h, z) = (hh_z^{-1}, z)$, where $z \to h_z \colon K \cap \tilde{K} \to H$ is continuous. For fixed z, the map $h \to hh_z^{-1}$ induces a bounded linear operator A_z on $L^p(H, \lambda)$.

To prove 3.7, it suffices to show that $\tilde{B}(z) = A_z(B(z))$ for all $z \in \mathcal{O} \cap \tilde{\mathcal{O}}$ (observe that, for ν-a.a. $z \in K \cap K'$, one has $\tilde{b}_p(z) = A_z(b_p(z))$). Thus we must show that, for some p,

$$\langle \tilde{B}(z), \sigma \rangle = \langle A_z(B(z)), \sigma \rangle$$

for all σ in the dual $L^p(H, \lambda)'$. By (02), we may assume σ is integration against some u_{ij} $\left(\text{thus } \langle w, \sigma \rangle = \int_H w(h) u_{ij}(h) d\lambda(h)\right)$. Extend each function $\eta_{rs} \colon z \to u_{rs}(h_z)$ continuously from $K \cap \tilde{K}$ to Z, calling the extensions η_{rs}, also.

For $z \in Z$, let $\varphi_1(z) = \langle \tilde{b}_p(z), \sigma \rangle$. Define a linear-functional-valued map $\hat{\sigma} \colon Z \to L^p(H, \lambda)'$ by $\hat{\sigma}(z) = \sum_r u_{ir} \cdot \eta_{rj}(z)$ (view u_{ir} as a linear functional). Let $\varphi_2(z) = \langle b_p(z), \hat{\sigma}(z) \rangle =$ (use 02) $\langle A_z(b_p(z)), \sigma \rangle = \varphi_1(z)$ for ν-a.a. $z \in K \cap \tilde{K}$. Now, $\delta(\varphi_1)(z) = \langle \tilde{B}(z), \sigma \rangle$ (3.4), while $\delta(\varphi_2)(z) =$ (since δ is a strong lifting)

$$\sum_r \eta_{rj}(z) \cdot (\delta \langle b_p, u_{ir} \rangle)(z) = \int_H [B_p(z)(h)][\sum_r u_{ir}(h) \eta_{rj}(z)] d\lambda(h)$$
$$= (\text{if } z \in K \cap K') \int_H [B(z)(h)] u_{ij}(hh_z) d\lambda(h) = \langle A_z(B(z)), \sigma \rangle .$$

By (01) and (02), $\tilde{B}(z) = A_z(B(z))$ for $z \in \mathcal{O} \cap \tilde{\mathcal{O}}$.

From now on, we assume $R^f(z)$ defined as in 3.5 for all $z \in Z$.

LEMMA 3.8. (a) *For ν-a.a. z, $R^f(z)$ is (the equivalence class of) $f_z \equiv f|_{\pi^{-1}(z)}$ in $L^\infty(X, \lambda_z)$.*

(b) *If f is continuous, the above holds for all $z \in Z$.*

(c) *If $f \in M^\infty(X/H, \nu)$, then $R^f(z)$ is (the equivalence class of) the constant $\delta(f)(z)$ in $L^\infty(X, \lambda_z)$.*

Proof. (a) and (b). Fix a proper triple $(V, \mathcal{O}, \tau)$ (the point z_0 doesn't matter), and fix p. As remarked in 3.3, $b_p(z) = f_z \circ \tau^{-1}$ for ν-a.a. $z \in K = \pi(V)$. Since $L^p(H, \lambda)$ is separable, 1.6 (iv) implies that $B(z) = f_z \circ \tau^{-1}$ for ν-a.a. $z \in K \supset \mathcal{O}$. Hence (3.5) $R^f(z) = f_z$ for ν-a.a. $z \in \mathcal{O}$. Since finitely many $\mathcal{O}$'s cover Z, (a) is proved. If f is continuous, then b_p is continuous on K. Use the method of ([1]) to extend $b_p|K$ to a continuous map $\tilde{b}_p\colon Z \to L^p(H, \lambda)$. Observe now that

(*) if $w \in M^\infty(Z, \nu)$ and $b \in M^\infty(Z, L^p(H, \lambda))$, then $\delta(w \cdot b)(z) = [\delta(w)(z)][\delta(b)(z)]$ (see [11], p. 76, equation (5)).

Using (*) and (01) in 3.7, we obtain, for $z \in \mathcal{O}$, $B(z) = \delta(\psi_K \cdot b_p)(z) = \delta(\psi_K \cdot \tilde{b}_p)(z) =$ (since δ is strong) $\tilde{b}_p(z) = f_z \circ \tau^{-1}$, and (b) follows.

(c) Pick z_0 and let $(V, \mathcal{O}, \tau)$ be a proper triple at z_0. For ν-a.a. $z \in K = \pi(V)$, one has $b_p(z) =$ the constant $f(z)$ in $L^p(H, \lambda)$. Let $\tilde{b}(z) = 1 \in L^p(H, \lambda)$ for all $z \in Z$; then $b_p(z) = f(z) \cdot \tilde{b}(z)$ ν-a.e. on K. Using (*) just above and (01) in 3.7, one obtains

$$B(z) = [\delta(f)(z)] \cdot \tilde{b}(z)(z \in \mathcal{O}) ,$$

which implies that $R^f(z_0) = \delta(f)(z_0) \in L^\infty(X, \lambda_z)$.

The next result will allow us to show that our still-to-be constructed lifting ρ is G-invariant. To motivate it, observe that $(f \cdot g)|_{\pi^{-1}(z)}(hx_0) = f|_{\pi^{-1}(gz)}(ghx_0) = f|_{\pi^{-1}(gz)}(ghg^{-1} \cdot gx_0)$ if $f \in M^\infty(X, \mu)$; here and below we write $g \cdot z$ for $(gH) \cdot z (g \in G, z \in Z)$.

PROPOSITION 3.9. *Fix $z_0 \in Z$, $g \in G$, and $x_0 \in \pi^{-1}(z_0)$. Then*

$$R^{f \cdot g}(z_0)(hx_0) = R^f(gz_0)(ghg^{-1} \cdot gx) \quad \textit{for } \lambda\text{-a.a. } h \in H .$$

Proof. Let $(V, \mathcal{O}, \tau)$ be a proper triple at z_0. Then $(g \cdot V, g \cdot \mathcal{O}, \tilde{\tau})$ is a triple at $g \cdot z_0$, where $\tilde{\tau}(gx) = (ghg^{-1}, gz)$ if (and only if) $\tau(x) = (h, z)(x \in V)$. The map $h \to ghg^{-1}$ preserves λ ([8], 28.72e), hence induces a linear map $A_g\colon L^p(H, \lambda) \to L^p(H, \lambda)$. Define $b_p^{f \cdot g}$, $B^{f \cdot g}$ using the first triple, b_p^f, B^f using the second. We claim that 3.9 is implied by

$$(*) \qquad B^{f \cdot g}(z) = A_g(B^f(g \cdot z))(z \in \mathcal{O}) .$$

This is clear: if (*) holds, then (assuming $\tau(x_0) = (\mathrm{idy}, z_0)$) one has $R^{f \cdot g}(z_0)(hx_0) = B^{f \cdot g}(z_0)(h) = B^f(gz)(ghg^{-1}) =$ (definitions of R^f and $\tilde{\tau}$) $R^f(gz)(g \cdot hx_0) = R^f(gz)(ghg^{-1} \cdot gx_0)$ for λ-a.a. h.

We prove (*). Using the definitions of b_p^f and $b_p^{f \cdot g}$ together with the fact that the map $z \to g \cdot z$ preserves ν, one sees that $b_p^{f \cdot g}(z) = A_g(b_p^f(z))$ for ν-a.a. z. Let $\sigma \in L^p(H, \lambda)'$. Then $\langle B^{f \cdot g}(z_0), \sigma \rangle = \delta \langle b_p^{f \cdot g}, \sigma \rangle (z_0) = (\delta \langle A_g(b_p^f(gz)), \sigma \rangle)(z_0) = (\delta \langle b_p^f(gz), A_g^* \sigma \rangle)(z_0) =$ (since δ commutes with G/H) $\langle B^f(gz_0), A_g^* \sigma \rangle = \langle A_g(B^f(gz_0)), \sigma \rangle$; 3.9 is proved.

3.10. New let (W_n) be a D' sequence in H consisting of compact neighborhoods of idy (1.7). For $f \in M^\infty(X, \mu)$, we define functions T_n^f $(n \geqq 1)$ on X as follows.

Case I. If G is abelian, $x_0 \in X$, $z_0 = \pi(x)$, let

$$T_n^f(x_0) = \frac{1}{\lambda(W_n)} \int_X R^f(z)(\bar{x})\psi_{W_n \cdot x_0}(\bar{x}) = \frac{1}{\lambda(W_n)} \int_H R^f(z)(hx_0)\psi_{W_n}(h)d\lambda(h) .$$

Case II. Suppose $G = H$ is Lie (see 2.8); let $x_0 \in X$, $z_0 = \pi(x_0)$. Pick proper triples $(V_i, \mathcal{O}_i, \tau_i)_{i=1}^l$ such that $\bigcup_{i=1}^l \mathcal{O}_i = Z$. Pick any i such that $z_0 \in \mathcal{O}_i$. Letting $\tau_i(x_0) = (h_0, z_0)$, let

$$X \supset V_n = \tau_i^{-1}\{(h, z_0) \mid h \in h_0 \cdot W_n\} .$$

Define

$$Q_{i,n}^f(x_0) = \frac{1}{\lambda(W_n)} \int_X R^f(z_0)(\bar{x})\psi_{V_n}(\bar{x})d\lambda_{z_0}(\bar{x}) .$$

Letting $\tau_i(x_i) = (\mathrm{idy}, z_0)$, we also have

$$Q_{i,n}^f(x_0) = \frac{1}{\lambda(W_n)} \int_H R^f(z_0)(hx_i)\psi_{h_0 \cdot W_n}(h)d\lambda(h) .$$

Finally, let $(\alpha_i)_{i=1}^l$ be a partition of unity subordinate to $(\mathcal{O}_i)_{i=1}^l$, and $T_n^f(x_0) = \sum_{i=1}^l \alpha_i(z_0)Q_{i,n}^f(x_0)$.

Case III. If $X = G \times Y$ and $x_0 \in X$, $z_0 = \pi(x_0)$, write $x_0 = (g_0, y_0)$, let $V_n = \{(g, y_0) \mid g \in g_0 \cdot W_n\}$, and define

$$T_n^f(x_0) = \frac{1}{\lambda(W_n)} \int_X R^f(z_0)(\bar{x})\psi_{V_n}(\bar{x})d\lambda_{z_0}(\bar{x}) .$$

PROPOSITION 3.11. *In all three cases,* $T_n^{f\cdot g}(x_0) = T_n^f(g \cdot x_0)$ $(g \in G,$ $x_0 \in X)$.

Proof of Case I. *Let* $z_0 = \pi(x_0)$. One has

$$\int_H R^{f\cdot g}(z_0)(hx_0)\psi_{W_n}(h)d\lambda(h) = (\text{by } 3.9)$$

$$\int_H R^f(gz_0)(ghg^{-1} \cdot gx_0)\psi_{W_n}(h)d\lambda(h) = (\text{since } G \text{ is abelian})$$

$$\int_H R^f(gz_0)(h \cdot gx_0)\psi_{W_n}(h)d\lambda(h) .$$

Hence $T_n^{f\cdot g}(x_0) = T_n^f(g \cdot x_0)$.

REMARK. The proof just completed would work when G is nonabelian if one could replace $(W_n)_{n=1}^{\infty}$ by a D'-sequence $(V_n)_{n=1}^{\infty}$ satisfying $g^{-1}V_ng = V_n$ $(n \geqq 1, g \in G)$. If one defines $V_n = \bigcap_{g\in G} g^{-1}W_ng$, then V_n is a compact neighborhood of the identity. However, it is not clear that the inequalities $\lambda(V_nV_n^{-1}) < C\lambda(V_n)$ can be arranged.

Case II. Suppose $\pi(x_0) = z_0 \in \mathcal{O}_i$ for some i, $1 \leqq i \leqq l$. Observe that, since $G = H$, $g\cdot z_0 = z_0$. As in 3.10, let $\tau_i(x_i) = (\mathrm{idy}, z_0)$, and let $\tau_i(x_0) = (h_0, z_0)$. Then $\int_H R^{f\cdot g}(z_0)(hx_i)\psi_{h_0\cdot W_n}(h)d\lambda(h) =$ (by 3.9, noting that $ghg^{-1}\cdot g = gh$)

$$\int_H R^f(g\cdot z_0)(ghx_i)\psi_{h_0\cdot W_n}(h)d\lambda(h) = \int_H R^f(z_0)(hx_i)\psi_{h_0\cdot W_n}(g^{-1}h)d\lambda(h)$$
$$= \int_H R^f(z_0)(hx_i)\psi_{g\cdot h_0W_n}(h)d\lambda(h)\ .$$

Comparing the first and last terms, we obtain $Q_{i,n}^{f\cdot g}(x_0) = Q_{i,n}^f(gx_0)$. Hence

$$T_n^{f\cdot g}(x_0) = T_n^f(g\cdot x_0)\ . \tag{3.10}$$

Case III. A rehash of methods used in Cases I and II.

3.12. We now define functions S_n^f $(n \geqq 1)$ as follows.

Case I. If G is abelian, let

$$S_n^f(x) = \frac{1}{\lambda(W_n)}\int_X f(\bar{x})\psi_{W_n\cdot x}(\bar{x})d\lambda_z(\bar{x})\ \ (z = \pi(x))$$

for all x such that

$$f_z \in L^\infty(X, \lambda_z) \quad \text{and} \quad N_\infty(f_z) \leqq N_\infty(f)\ . \tag{**}$$

Let $S_n^f(x) = 0$ for all other x. By (3.8a), $S_n^f(x) = T_n^f(x)$ for μ-a.a. x.

Case II. If G is a Lie group, let

$$P_{i,n}(x) = \frac{1}{\lambda(W_n)}\int_X f(\bar{x})\psi_{V_n}(\bar{x})d\lambda_z(\bar{x})$$

($z = \pi(x)$; V_n is as in 3.10) for all $x \in \mathcal{O}_i$ satisfying (**). Then define $S_n^f(x) = \sum_{i=1}^{l}\alpha_i(z)P_{i,n}(x)$ for all such x. Let $S_n^f(x) = 0$ if x does not satisfy (**). By (3.8a), $S_n^f(x) = T_n^f(x)$ μ-a.e.

Case III. If $X = G \times Y$ and x satisfies (**), let

$$S_n^f(x) = \frac{1}{\lambda(W_n)}\int_X f(\bar{x})\psi_{V_n}(\bar{x})d\lambda_z(\bar{x})$$

(V_n is as in 3.10). Otherwise let $S_n^f(x) = 0$.

PROPOSITION 3.13. *For each n, S_n^f, and hence T_n^f, is μ-measurable.*

Proof. We prove this in Case I; the other cases are handled similarly. Let f_j be a bounded sequence of continuous functions such that $f_j \to f$ μ-a.e. Let

$$S_j(x) = \frac{1}{\lambda(W_n)} \int_X f_j(\bar{x})\psi_{W_n \cdot x}(\bar{x})d\lambda_z(\bar{x}) = \frac{1}{\lambda(W_n)} \int_H f_j(hx)\psi_{W_n}(h)d\lambda(h).$$

Then S_j is continuous (use uniform continuity of f_j and equicontinuity ([7]) of the transformation group (H, X)). Now, for z in a set $C \subset Z$ of ν-measure 1, $f_j|_{\pi^{-1}(z)} \to f_z$ λ_z-a.e. (2.6). Consider the set $C_1 = \{z \in C \mid (^{**}) \text{ holds for } f_z\}$. By dominated convergence, $S_j(z) \leftarrow S_n^f(x)$ for all $x \in \pi^{-1}(C_1)$. But $\mu(\pi^{-1}(C_1)) = 1$; hence 3.13 is proved.

PROPOSITION 3.14. *In Case* I, II, *and* III:
(a) $\lim_{n\to\infty} T_n^f(x) = f(x)$ μ-a.e. $(f \in M^\infty(X, \mu))$;
(b) *if f is continuous, then* $\lim_{n\to\infty} T_n^f(x) = f(x)$ *everywhere*;
(c) *if $f \in M^\infty(X/H, \nu)$, then* $\lim_{n\to\infty} T_n^f(x) = \delta(f)(\pi(x))$ *for all x.*

Proof. (a) *Case* I. It is sufficient to show that $S_n^f(x) \to f(x)$ μ-a.e. By version 2 of the Main Derivation Theorem (1.7), one has, for $g \in L^1(H, \lambda)$, $1/\lambda(W_n)\int_H g(\tilde{h})\psi_{W_n \cdot h}(\tilde{h})d\lambda(\tilde{h}) \to g(h)$ λ-a.e. Consider the set $C = \{z \in Z \mid (^{**}) \text{ of 3.12 is satisfied}\}$. Note $\nu(C) = 1$. Fix $z \in C$ and $x_0 \in \pi^{-1}(z)$. Then if $x = hx_0$, one has

$$\begin{aligned}(S_n^f x) &= \frac{1}{\lambda(W_n)} \int_H f(\tilde{h}x_0)\psi_{W_n \cdot hx_0}(\tilde{h}) \\ &= \frac{1}{\lambda(W_n)} \int_H f(\tilde{h}x_0)\psi_{W_n \cdot h}(\tilde{h})d\lambda(\tilde{h}) \longrightarrow f(hx_0) = f(x)\end{aligned}$$

for λ-a.a. h; i.e., for λ_z-a.a. x.

Now if $A = \{x \in X \mid \lim_{n\to\infty} S_n^f(x) \text{ exists and equals } f(x)\}$, then A is μ-measurable. We have just shown that, for ν-a.a. z, A intersects $\pi^{-1}(z)$ in a set of λ_z-measure 1. Hence (2.6) A has μ-measure 1. So $S_n^f(x)$, and therefore $T_n^f(x)$, converges to $f(x)$ μ-a.e.

Case II. We use the notation of 3.12. Observe that, if $x \in \pi^{-1}(\mathcal{O}_i)$, $\pi(x)$ satisfies $(^{**})$, $\tau_i(x) = (h, z)$, and $\tau_i(x_i) = (\text{idy}, z)$, then

$$P_{i,n}(x) = \frac{1}{\lambda(W_n)} \int_H f(\tilde{h}x_i)\psi_{h \cdot W_n}(\tilde{h})d\lambda(\tilde{h}) .$$

By version 1 of 1.7, the right-hand side tends to $f(hx_i) = f(x)$ for

λ-a.a. h; i.e., for λ_z-a.a. x. Let $A_i = \{x \in \pi^{-1}(\mathcal{O}_i) \mid P_{i,n}(x) \to f(x)\}$. Arguing as in Case I, we find that $\mu(A_i) = \mu(\pi^{-1}(\mathcal{O}_i))$. Let $A = \{x \mid S_n^f(x) \to f(x)\}$. Let z satisfy (**). Then $A \cap \pi^{-1}(z)$ has λ_z-measure 1. For, let $i_i, \cdots, i_k$ $(1 \leqq k \leqq l)$ be those indices i such that $z \in \mathcal{O}_i$. Then $\pi^{-1}(z) \cap A_{i_j}$ $(1 \leqq j \leqq k)$ has λ_z-measure 1, since $P_{i,n}(x) \to f(x)$ λ_z-a.e. The definition of S_n^f now implies that $\lambda_z(A \cap \pi^{-1}(z)) = 1$. Again argue as in Case I to obtain $\mu(A) = 1$.

Case III. The proof contains nothing new, hence we omit it.

(b) *Case I, II, III.* By 3.8b, $R^f(z) = f_z$ for all z. The Main Derivation Theorem for *continuous* functions gives convergence *everywhere* (as noted in 1.7, this is a simple observation). Combining these two facts with the definition(s) of T_n^f yields the result.

(c) *Case* I, II, III. Use 3.8c and the definition(*s*) of T_n^f.

We are ready prove 2.7.

3.15. *Proof of* 2.7. Let U be an ultrafilter on $N = \{1, 2, 3, \cdots\}$ finer than the Fréchet filter (see [5], and [10], p. 83). Since $|T_n^f(x)| \leqq N_\infty(f)$ for all x (3.4c and 3.5), we may define $T^f(x) = \lim_U T_n^f$. Let $\rho(f)(x) = T^f(x)(x \in X, f \in M^\infty(X, \mu))$. By choice of U and 3.14a, $\rho(f) = f$ μ-a.e. Hence (i) of 1.4 is satisfied. By 3.6, (iii), (iv), and (v) are also satisfied. If $f = 0$ μ-a.e., then $|T_n^f(x)| = 0$ for all n, x, and this together with linearity shows that 1.4 (ii) holds. Combining these facts with 3.14b, c shows that ρ is a strong linear lifting which extends δ.

By 3.12, ρ commutes with G. Now, the group G of self-mappings of X satisfies the condition of Theorem 1 of ([9]). Hence we may apply the method of Remark 2 following ([9], Theorem 1) to obtain a lifting $\bar{\rho}$ commuting with G. By the proof of $(j) \Rightarrow (jj)$ in ([11], Theorem 2, p. 105), $\bar{\rho}$ is strong. By the proof of ([11], Theorem 2, p. 39), $\bar{\rho}$ extends δ. So $\bar{\rho}$ has all the necessary properties.

REMARK 3.16. It should be emphasized that the only point in the proof which requires special assumptions on G occurs in the proof of 3.11. If one could assume $g^{-1}W_n g = W_n$ $(g \in G)$, Theorem 2.2 would hold for any compact G.

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