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SYMMETRIC SUBLATTICES OF A NOETHER LATTICE

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In this note we investigate questions about partitions of positive integers arising from multiplicative lattice theory and prove that the sublattice of $RL(A_i)$ ($A_1, \cdots, A_k$ is a prime sequence in a local Noether lattice) generated by the elementary symmetric elements in the A_i's is a π-lattice.

0. **Introduction.** If $A_1, A_2, \cdots, A_k$ is a prime sequence in L, a local Noether lattice, then the multiplicative sublattice it generates is isomorphic to RL_k, the distributive local Noether lattice with altitude k. We denote this sublattice of L by $RL(A_i)$. In $RL(A_i)$, every element is a finite join of products $A_1^{r_1}A_2^{r_2}\cdots A_k^{r_k}$ for $(r_1, \cdots, r_k)=(r_i)$ a k-tuple of nonnegative integers. Minimal bases for an element, T, in $RL(A_i)$ are unique and determined by the exponent k-tuples of the elements in the minimal base for T. We examine the sublattice of L generated by the elementary symmetric elements in the prime sequence $A_1, \cdots, A_k$. This multiplicative sublattice is a π-domain (Theorem 7.1).

Unless otherwise stated, all k-tuples will be nonnegative integers. A k-tuple (r_i) is *monotone* if and only if $r_i \geqq r_{i+1}$ for $1 \geqq i > k$. $(r_i) = (s_i)$ and $(r_i) + (s_i)$ refer to componentwise equality and addition respectively. $(r_i) \geqq_p (s_i)$ means $r_i \geqq s_i$ for $i = 1, \cdots, k$. We write $(r_i) \geqq_l (s_i)$ to mean the first nonzero entry in $(r_i - s_i)$ is strictly positive (lexicographic order). If (e_i) is a k-tuple we write e_i^* for $\sum_{j=i}^k e_j$ and e_i^{**} for $\sum_{j=i}^k e_j^*$. Throughout this note $A_1, \cdots, A_k$ is a prime sequence in L and $RL(A_i)$ is the multiplicative sublattice it generates.

1. **The symmetric sublattice.** If T is a principal element in $RL(A_i)$ and g is in S_k, the permutation group on $1, \cdots, k$, we define $T_g(T^g)$ to be the principal element in $RL(A_i)$ obtained by replacing $A_i^{t(i)}$ by the factor $A_{g(i)}^{t(i)}(A_i^{t(g(i))})$ in T for each i from 1 to k. If $C_1 \vee \cdots \vee C_p$ is a minimal base for C in $RL(A_i)$, then $C_g = (C_1)_g \vee \cdots \vee (C_p)_g$. C^g is defined similarly. Note that for each g in S_k and for C in $RL(A_i)$, $(C_g)^g = (C^g)_g = C$. Hence $C_g = C^{g^{-1}}$. An element C in $RL(A_i)$ is a *symmetric element* if and only if $C_g = C$ for each g in S_k.

THEOREM 1.1. *The set of all symmetric elements in $RL(A_i)$ forms a multiplicative sublattice of $RL(A_i)$ which is closed under residuation.*

Proof. We show that F_g, the set of elements fixed by the map ϕ from $RL(A_i)$ to $RL(A_i)$ defined $C \overset{\phi}{\mapsto} C^g$ for g in S_k is a residuated multiplicative lattice. For then the set of symmetric elements which is the intersection of all of the F_g's for g in S_k is also a multiplicative sublattice.

Let g be any permutation in S_k and ϕ be defined as above. ϕ is well defined and preserves join by definition. Since $(C_g)^g = (C^g)_g = C$ for each C in $RL(A_i)$, ϕ is a bijection.

Let $B = \Pi A_i^{b_i}$ and $C = \Pi A_i^{c_i}$ be principal elements in $RL(A_i)$. Then $(BC)^g = \Pi A_{g^{-1}(i)}^{b_i+c_i} = \Pi A_{g^{-1}(i)}^{b_i} \cdot \Pi A_{g^{-1}(i)}^{c_i} = B^g \cdot C^g$ and $(B \wedge C)^g = (\Pi \mathrm{A}_i^{\max(b_i, c_i)})^g = \Pi A_{g^{-1}(i)}^{\max(b_i, c_i)} = \Pi A_{g^{-1}(1)}^{b_i} \wedge \Pi A_{g^{-1}(i)}^{c_i} = B^g \wedge C^g$. Since elements in $RL(A_i)$ are joins of principal elements and multiplication and meet distribute over join, ϕ preserves products and meet.

Finally, the fact that ϕ preserves residuals and that F_g is a multiplicative sublattice of $RL(A_i)$ readily follows from the fact that ϕ is a multiplicative lattice isomorphism.

REMARK. If B is a principal element in $RL(A_i)$ such that $B^g = B$, then B is a principal element in F_g. However, F_g contains enough principal elements to make it a Noether lattice only if g is the identity in S_k (cf § 7) for $k > 1$.

2. Elementary symmetric elements. For $t = 1, \cdots, k$, a_t, the tth *elementary symmetric element* in $A_1, \cdots, A_k$ is the join of all products of $A_1, \cdots, A_k$ with t distinct factors. In this section we investigate the chain $0 < a_k < \cdots < a_1 < I$ of elementary symmetric elements in $RL(A_i)$.

We say the *weight* of a principal element in $RL(A_i)$ is the maximum of its exponents. If J is a t-tuple $(i_1, \cdots, i_t)$ with $i_j < i_{j+1}$ and $t \leqq k$ then we denote by (J) the set of all $(k-t)$-tuples $(j_1, \cdots, j_{k-t})$ such that $\{j_1, \cdots, j_{k-t}\} \cap \{i_1, \cdots, i_t\}$ is empty.

THEOREM 2.1. *The elementary symmetric elements together with* 0 *and* I *form a sublattice closed under residuation. In particular*

$$(a_t : a_p) = \begin{cases} I & \text{if } \ t \leqq p \\ a_t & \text{if } \ t > p \, . \end{cases}$$

Proof. From [8, p. 84] we have for $t > p$

$$(a_t : a_p) = \vee (J_1) \vee (J_2) \vee \cdots \vee (J_q)(A_{i_1} \cdot A_{i_2} \cdots A_{i_s} \wedge \cdots \wedge A_{q_1} \cdots A_{q_s})$$

where there are $C(k, p)$ (the binomial coefficient) join symbols each having indices in $(J_1), \cdots, (J_q)$ for J_i one of the $C(k, p)$ ordered p-

tuples which can be chosen from $\{1, \cdots, k\}$. Each intersection has weight one and by symmetry, $(a_t: a_p) = a_r$ for some r. Since $a_t \leqq (a_t: a_p)$ we only need show that $a_{t-1} \nleqq (a_t: a_p)$.

Let $A_{i_1} \cdots A_{i_{t-1}}$ be any element in the minimal base for a_{t-1} and $A_{i_1} \cdots A_{i_p}$ be the product of the first p of these ($p \leqq t-1$). Then their product $A_{i_1}^2 \cdots A_{i_p}^2 \cdots A_{i_{t-1}}$ is an element which is not less than or equal to any element in the minimal base for a_t. Hence $a_{t-1} \nleqq (a_t: a_p)$.

REMARK. From the Reciprocity Theorem [9, Theorem 5.1] we can define a multiplication on the chain of elementary symmetric elements by $(a_t: a_p) \geqq a_s$ if and only if $a_t \geqq a_p \cdot a_s$, i.e., $a_p a_s = a_{\max\{p,s\}}$. This new multiplication makes every element in the chain idempotent and the order becomes $a \leqq b$ if and only if $a \cdot b = a$ for nonzero elements different from I.

3. The minimal base for $\pi a_i^{e_i}$: majorization.

In this section we determine the minimal base for a product of the elementary symmetric elements in $RL(A_i)$. We first dispense with the powers of the a_i.

LEMMA 3.1. *For $t < k$, a_t^e is the join of all powers of the A_i's whose exponents are bounded above by e and whose exponent sum is te. $a_k^e = A_1^e \cdots A_k^e$.*

Proof. For $k > 1$, let (k_i) be any k-tuple of nonnegative integers summing to te and bounded above by e. By symmetry we assume (k_i) is monotone. There are at least t nonzero k_i's no more than t of which are equal to e. Let

$$v_i = \begin{cases} k_i - 1 & 1 \leqq i \leqq t \\ k_i & t < i \leqq k \end{cases} \quad \text{and} \quad w_i = \begin{cases} 1 & 1 \leqq i \leqq t \\ 0 & t < i \leqq k \,. \end{cases}$$

Then $(v_i) + (w_i) = (k_i)$ and by induction $\Pi A_i^{v_i}$ and $\Pi A_j^{w_i}$ are elements in the minimal base for a_t^{e-1} and a_t respectively. Hence their product which has (k_i) as its exponent k-tuple is in the minimal base for a_t^e. The converse follows by writing down a product in a_t^e and observing the conditions hold.

LEMMA 3.2. *$\Pi A_j^{r_j}$ is in the minimal base for $\Pi a_i^{e_i}$ if and only if there is a nonnegative $k \times k$ matrix whose ith row sum is ie_i, whose ith row is bounded above by e_i, and whose* jth *column sum is r_j.*

Proof. If $\Pi A_j^{r_j} = C_1 \cdots C_k$ where C_i is in the minimal base for $a_i^{e_i}$, then $C_i = \Pi A_j^{r_{ij}}$ where $r_{ij} \leqq e_i$ and $\sum_j r_{ij} = ie_i$. Then $\Pi_i C_i = \Pi_j A_j^{r_j}$ where $r_j = \sum_i r_{ij}$ for $j = 1, \cdots, k$. (r_{ij}) is the desired matrix. The converse follows easily.

The existence of the matrix described in Lemma 3.2 is determined by the following generalization of the Gale-Ryser theorem on (0, 1)-matrices [7, p. 63].

DEFINITION 3.3. If $\mathfrak{M} = (e_1, e_2, \cdots, e_k)$ is a k-tuple of nonnegative integers, an $\mathfrak{M}$-*matrix* is a matrix of nonnegative integers with k rows whose ith row entries are bounded above by e_i. A $k \times t$ $\mathfrak{M}$-matrix is *maximal* with row sums (f_i) if each row is maximal in the lexicographic order of t-tuples.

In Lemma 3.4 (r'_j) is the monotone permutation of (r_j). If the condition of the lemma holds we say (r_j) is *majorized* by (s_j) and write $(r_j) \prec (s_j)$.

LEMMA 3.4. *If (t_{ij}) is the maximal $k \times t$ $\mathfrak{M}$-matrix with row sums (f_i) and column sums (s_j), then there exists an $\mathfrak{M}$-matrix (r_{ij}) with column sums (r_j) if and only if $\sum_1^\nu r'_j \leqq \sum_1^\nu s_j$ for $\nu = 1, \cdots, t-1$ with equality when $\nu = t$.*

Proof. The proof follows mutatis mutandus from [5, p. 1030].

Lemmas 3.2 and 3.4 allow us to characterize the elements in the minimal base for $\Pi a_i^{e_i}$.

THEOREM 3.5. *The minimal base for $\Pi a_i^{e_i}$ in $RL(A_i)$ is the join of all products of the A_i's whose exponent k-tuples are majorized by (e_i^*).*

Proof. The maximal $k \times k$ (e_i)-matrix with row sums (ie_i) has column sums e_i^*. Hence $(r_i) \prec (e_i^*)$ if and only if there exists an (e_i)-matrix with row sums (ie_i) and column sums (r_i). But this holds if and only if $\Pi A_i^{r_i}$ is an element in the minimal base for $\Pi a_i^{e_i}$.

REMARK. For $k \leqq 3$ we have determined that the product $\Pi a_i^{e_i}$ has as a minimal base the join of all products of the A_i's whose exponent k-tuples are bounded above by e_1^*, bounded below by e_k, sum to $\sum ie_i$ and whose breadth is less that or equal to $\sum_1^k (tk - t^2)e_t$. The breadth of $\Pi A_i^{r_i}$ is $\sum_{i<j} |r_i - r_j|$. However this characterization does not hold for $k > 3$.

4. $P(a_1, a_2, \cdots, a_k)$, A multiplicative sublattice. Let $P(a_1, \cdots, a_k) = P(a_i)$ be the set of all finite joins of products of the elementary symmetric elements in $A_1, \cdots, A_k$. We will show that this set is the multiplicative sublattice generated by $a_1, \cdots, a_k$.

If (u_i) and (v_i) are k-tuples we define the distance between them as $d((u_i), (v_i)) = \sum_i |u_i - v_i|$. The lemma which follows will aid us in identifying the minimal base for the meet of two products to the a_i's.

LEMMA 4.1. *Let (u_i) and (v_i) be k-tuples majorized by monotone k-tuples (r_i) and (s_i), respectively. Then if $w_i = \max(u_i, v_i)$ for $i = 1, \cdots, k$*

(1) $d((u_i), (v_i)) = |r_1^* - s_1^*|$ *if and only if* $w_1^* = \max(r_1^*, s_1^*)$.

(2) $d((u_i), (v_i)) \geqq |r_1^* - s_1^*|$.

(3) $d((u_i), (v_i)) > |r_1^* - s_1^*|$ *implies there exist k-tuples $(\bar{u}_i)$ and $(\bar{v}_i)$ such that $(w_i) \geqq_p (\max(\bar{u}_i, \bar{v}_i))$ and $d((\bar{u}_i), (\bar{v}_i)) = |r_1^* - s_1^*|$.*

Proof. (1) $2 \cdot w_1^* = \sum_i (u_i + v_i + |u_i - v_i|) = r_1^* - s_1^* + |r_1^* - s_1^*| = 2(\max(r_1^*, s_1^*))$ if and only if $\sum |u_i - v_i| = |r_1^* - s_1^*|$ since for any two integers a, b $2(\max(a, b)) = a + b + |a - b|$.

(2) $|r_1^* - s_1^*| = |u_1^* - v_1^*| = |\sum_i (u_i - v_i)| \leqq \sum_i |u_i - v_i| = d((u_i), (v_i))$.

(3) $d((u_i), (v_i)) > |u_1^* - v_1^*|$ implies there exist indices i_1 and i_2 such that $u_{i_1} < v_{i_1}$ and $u_{i_2} > v_{i_2}$. Let $(u_i'), (v_i'')$ be the monotone representatives of $(u_i), (v_i)$ respectively. If $i_1' < i_2'$ then $v_{i_1}'' > u_{i_1}' \geqq u_{i_2}' > v_{i_2}''$ so that $v_{i_1}'' \geqq v_{i_2}'' + 2$. Let (t_i'') be the k-tuple equal to $(v_i'')$for $i \neq i_1''$, i_2'', $t_{i_1}'' = v_{i_1}'' - 1$ and $t_{i_2}'' = v_{i_2}'' + 1$. Then (t_i'') is majorized by (r_i). If (t_i) is obtained by reversing the permutation $(v_i) \to (v_i'')$ and applying it to (t_i'') then (t_i) is also majorized by (r_i). So

$$\max(u_i, t_i) = \begin{cases} \max(u_i, v_i)\,, & i \neq i_1 \\ v_{i_1} - 1\,, & i = i_1 \end{cases}$$

and $d((u_i), (t_i)) < d((u_i), (v_i))$. By induction on d, there exist $(\bar{u}_i), (\bar{v}_i)$ such that $d((\bar{u}_i), (\bar{v}_i)) = |r_1^* - s_1^*|$ and $\max(\bar{u}_i, \bar{v}_i) \leqq \max(u_i, t_i) \leqq \max(u_i, v_i)$ for $i = 1, \cdots, k$. The proof is complete if $i_1' < i_2'$.

Otherwise $i_1' > i_2'$ which implies that $i_1'' < i_2''$. The proof is similar if the latter holds.

Now suppose that (e_i) and (f_i) are k-tuples, then $\Pi a_i^{e_i}$ and $\Pi a_i^{f_i}$ are elements of $P(a_i)$. The next theorem characterizes the elements in the base for their meet in terms of the exponents of the A_i's.

THEOREM 4.2. *If $\Pi a_i^{e_i}$ and $\Pi a_i^{f_i}$ are elements of $P(a_i)$ with $f_1^{**} \geqq e_1^{**}$ then $\Pi a_i^{e_i} \wedge \Pi a_i^{f_i} = \{\Pi A_i^{v_i} | (v_i) \prec (f_i^*)$ and $(v_i) \geqq_p (u_i)$ for some $(u_i) \prec (e_i^*)\}$.*

Proof. Since $RL(A_i)$ is distributive, the meet described in the theorem is the join of all products of the A_i whose exponent k-tuples are $(\max(u_i, v_i))$ for $(u_i) \prec (e_i^*)$ and $(v_i) \prec (f_i^*)$. If $d((u_i), (v_i))$ is greater than $f_1^{**} - e_1^{**}$, then $(\max(u_i, v_i)) \geqq_p (\max(\bar{u}_i, \bar{v}_i))$ for some $(\bar{u}_i)$ and $(\bar{v}_i)$ majorized by (e_i^*) and (f_i^*) respectively. Hence the product of the A_i's with exponent k-tuple $(\max(u_i, v_i))$ can be left out of the minimal base for the meet. But $d((u_i), (v_i)) > f_1^{**} - e_1^{**}$ if and only if $(v_i) \ngeqq_p (u_i)$. Hence the elements left in the minimal base for the meet have the form desired.

To show that the meet of two products of the a_i's is again such a product, we need

LEMMA 4.3. *Let (e_i^*) and (f_i^*) be monotone k-tuples and $t_i^* = \max(e_i^{**}, f_i^{**}) - \max(e_{i+1}^{**}, f_{i+1}^{**})$ for $i = 1, \cdots, k$ where we agree that $e_{k+1}^* = f_{k+1}^* = 0$. Then (t_i^*) is also monotone.*

Proof.

$$\begin{aligned}
\max&(e_i^{**}, f_i^{**}) + \max(e_{i+2}^{**}, f_{i+2}^{**}) \\
&\geqq \max(e_i^{**} + e_{i+2}^{**}, f_i^{**} + f_{i+2}^{**}) \\
&\geqq \max(2e_{i+1}^{**}, 2f_{i+1}^{**}) \\
&= 2\max(e_{i+1}^{**}, f_{i+1}^{**}) .
\end{aligned}$$

So that $t_i^* \geqq t_{i+1}^*$ for $i = 1, \cdots, k-1$.

THEOREM 4.4. *Let (e_i) and (f_i) be k-tuples, then the meet of $\Pi a_i^{e_i}$ and $\Pi a_i^{f_i}$ is the product $\Pi a_i^{t_i}$ where t_i^* is given in Lemma 4.3.*

Proof. We may assume that $e_1^{**} \geqq f_1^{**}$. From above it suffices to show that the set $\mathfrak{B} = \{(u_i) | (u_i) \prec (e_i^*)$ and $(u_i) \geqq_p (v_i)$ for some $(v_i) \prec (f_i^*)\}$ is equal to the set $\mathfrak{C} = \{(u_i) | (u_i) \prec (t_i^*)\}$.

$\mathfrak{B} \subseteqq \mathfrak{C}$. If (u_i) is in $\mathfrak{B}$ then $(u_i) \prec (e_i^*)$ and $(u_i) \geqq_p (v_i)$ for $(v_i) \prec (f_i^*)$. Then $d((u_i), (v_i)) = e_1^{**} - f_1^{**}$ so that $w_1^* = e_1^{**}$ where $w_i = \max(u_i, v_i)$ for $i = 1, \cdots, k$. Moreover, for $j = 2, \cdots, k$, $u_j^* \geqq v_j^* \geqq f_j^{**}$ since if $v_j^* < f_j^{**}$, then $\sum_1^{j-1} v_i' \geqq \sum_1^{j-1} v_i > \sum_1^{j-1} f_i^*$ where (v_i') is the monotone representative of (v_i) which contradicts $(v_i) \prec (f_i^*)$. Therefore $\sum_1^{j-1} u_i = e_1^{**} - u_j^* \leqq e_1^{**} - f_1^{**}$. But

$$\begin{aligned}
\sum_{l=1}^{j-1} t_l^* &= \sum_{l=1}^{j-1} [\max(e_l^{**}, f_l^{**}) - \max(e_{l+1}^{**}, f_{l+1}^{**})] \\
&= \max(e_1^{**}, f_1^{**}) - \max(e_j^{**}, f_j^{**}) \\
&= \sum_1^{j-1} e_i^* - \begin{cases} 0 & \text{if} \quad e_j^{**} \geqq f_j^{**} \\ f_j^{**} - e_j^{**} & \text{if} \quad f_j^{**} > e_j^{**} \end{cases} \\
&= \begin{cases} \sum_1^{j-1} e_i^* & \text{if} \quad e_j^* \geqq f_j^* \\ e_1^{**} - f_j^{**} & \text{if} \quad f_j^{**} > e_j^{**} . \end{cases}
\end{aligned}$$

Hence $\sum_1^{j-1} u_i \leqq \sum_1^{j-1} t_i^*$ and $(u_i) \prec (t_i^*)$, i.e., (u_i) is in $\mathfrak{C}$.

$\mathfrak{C} \subseteqq \mathfrak{B}$. Let (u_i) be a k-tuple majorized by (t_i^*). By symmetry, we may assume that (u_i) is monotone. Since, $(t_i^*) \prec (e_i^*)$, we have $(u_i) \prec (e_i^*)$. For $i = 1, \cdots, k$ let $v_i = \min(u_i, f_1^* + \cdots + f_i^* - \sum_1^{i-1} v_j)$ setting $v_0 = 0$. We claim

($\sharp$) $\sum_1^q v_i = \min_p \{\sum_0^p f_i^* + \sum_{p+1}^q u_i\}$ where the minimum is taken for p ranging from 0 to q and $f_0 = 0 = \sum_s^r u_i$ whenever $r < s$.

($\sharp$) is clear if $q = 1$. For $q > 1$,

$$\begin{aligned}\sum_1^q v_i &= \sum_1^{q-1} v_i + \min\left(u_q, f_1^* + \cdots + f_q^* - \sum_1^{q-1} v_i\right)\\ &= \min\left(u_q + \sum_1^{q-1} v_i, \sum_1^q f_i^*\right)\\ &= \min\left(u_q + \min_{p=0,\cdots,q-1}\left\{\sum_0^q f_i^* + \sum_{p+1}^q u_i\right\}, \sum_1^q f_i^*\right)\\ &= \min_{p=0,\cdots,q}\left\{\sum_0^p f_i^* + \sum_{p+1}^q u_i\right\}\end{aligned}$$

where the third equality follows by induction. Therefore ($\sharp$) holds.

Moreover, (v_i) is monotone: If q is any integer, $1 \leqq q < k - 1$, then

(1) $2(\sum_1^q u_i) \geqq 2(\sum_1^{q-1} u_i) + u_q + u_{q+1}$

(2) $2(\sum_1^p f_i^* + \sum_{p+1}^q u_i) \geqq 2(\sum_1^p f_i^* + \sum_{p+1}^{q-1} u_i) + u_q + u_{q+1}$

(3) $2(\sum_1^q f_i^*) \geqq 2(\sum_1^{q-1} f_i^*) + f_q^* + f_{q+1}^*$

since (u_i) and (f_i^*) are monotone. Hence each integer on the left of the inequalities of (1), (2), or (3) is greater than or equal to

$$\begin{aligned}\min\Big[&2\Big(\sum_1^{q-1} u_i\Big) + u_q + u_{q+1}, 2(f_1^* + u_2 + \cdots + u_{q-1})\\ &+ u_q + u_{q+1}, \cdots, 2(f_1^* + \cdots + f_{q-1}^*)\\ &+ f_q^* + u_{q+1}, 2\Big(\sum_1^q f_i^*\Big) + f_q^* + f_{q+1}^*\Big]\\ \geqq &\sum_1^{q+1} v_i + \sum_1^{q-1} v_i\,.\end{aligned}$$

So from ($\sharp$), $\sum_1^q v_i \geqq 1/2[\sum_1^{q+1} v_i + \sum_1^{q-1} v_i]$ and $v_q = \sum_1^q v_i - \sum_1^{q-1} v_i \geqq \sum_1^{q+1} v_i - \sum_1^q v_i = v_{q+1}$ for $q = 1, \cdots, k-1$. Hence (v_i) is monotone.

Finally, again from ($\sharp$) $v_1^* = \min_{j=1,\ldots,k}\{\sum_1^{j-1} f_i^* + u_j^*\}$ and since $u_j^* \geqq t_j^{**} = \max(e_j^{**}, f_j^{**}) \geqq f_j^{**}$ for $j = 1, \cdots, k$, we have $f_1^* + \cdots + f_{j-1}^* + u_j^* \geqq f_1^{**}$ for each j. Hence $v_1^* = f_1^{**}$. Therefore $(v_i) \prec (f_i^*)$ since by definition of the v_i's, $v_1 + \cdots + v_j \leqq f_1^* + \cdots + f_j^*$ for each j. Since $(v_i)\,{}_p{\leqq}\,(u_i)$ and $(u_i) \prec (e_i^*)$, we have (u_i) is in $\mathfrak{B}$.

It follows from the property in $RL(A_i)$ that multiplication in $P(a_i)$ distributes over joins. Consequently

THEOREM 4.5. *The set of all finite joins of products of the elementary symmetric elements in* $A_1, \cdots, A_k$ *is a (distributive) multiplicative sublattice of* $RL(A_i)$ *and is the sublattice generated by* $a_1, \cdots, a_k$.

In the next two sections we investigate the structure of the lattice $P(a_i)$. In § 5 we show that the factorization of products of the a_i is unique and in § 6 we investigate the principal elements and the residual division in $P(a_i)$.

5. Unique factorization of products of elementary symmetric elements. If $\Pi a_i^{e_i}$ and $\Pi a_i^{f_i}$ are products in $P(a_i)$ and $\Pi a_i^{e_i} \leqq \Pi a_i^{f_i}$, then every element in the minimal base for $\Pi a_i^{e_i}$ must be less than or equal to one of the elements in the minimal base for $\Pi a_i^{f_i}$. That is, whenever $(r_i) \prec (e_i^*)$ then $(r_i) \geqq_p (s_i)$ for some $(s_i) \prec (f_i^*)$. When this occurs we say that (e_i^*) is dominated by (f_i^*) and write (e_i^*) dom (f_i^*). Hence, $\Pi a_i^{e_i} \leqq \Pi a_i^{f_i}$ if and only if (e_i^*) dom (f_i^*). Hence,

LEMMA 5.1. *Dom is a partial order on the set of monotone k-tuples.*

Lemma 5.1 and the definition of dom establish the next theorem.

THEOREM 5.2. *The set of products of the* a_i*'s is order isomorphic to the poset of monotone k-tuples ordered by dom via the map* $\Pi a_i^{e_i} \mapsto (e_i^*)$. *In particular, since this mapping is well defined, factorization of a product of elementary symmetric elements is unique.*

Using the order dom, we show that in $P(a_i)$ any product of the elementary symmetric elements is join irreducible.

THEOREM 5.3. *Products of the elementary symmetric elements in* $P(a_i)$ *are join irreducible.*

Proof. Suppose that $\Pi a_i^{g_i} = \Pi a_i^{e_i} \vee \cdots \vee \Pi a_i^{f_i}$. Since minimal bases in $RL(A_i)$ are unique, the element $\Pi A_i^{g_i*}$ which is in the minimal base for $\Pi a_i^{g_i}$ must appear in the minimal base for one of the products of the a_i's on the right, say $\Pi a_i^{e_i}$. Then $(g_i^*) \prec (e_i^*)$. But since $\Pi a_i^{e_i} \leqq \Pi a_i^{g_i}$, (e_i^*) dom (g_i^*). So $(e_i^*) \geqq_p (v_i)$ where $(v_i) \prec (g_i^*)$. Therefore $(e_i^*) = (v_i)$ and $(e_i^*) \prec (g_i^*)$. Consequently $(e_i^*) = (g_i^*)$; and $\Pi a_i^{g_i}$ is join irreducible.

COROLLARY 5.4. *Elements in* $P(a_i)$ *have unique minimal bases as joins of products of the* a_i*'s.*

Proof. [2, p. 183].

6. Residuation and join principal elements in $P(a_i)$. In Lemma 4.1 we used the technique of subtracting one from a position in a k-tuple and adding one further to the right in such a way that monotonicity of the k-tuple was maintained. We call this process a *monotone* $(-1, 1)$*-change* and remark that these changes characterize majorization [cf. 4].

PROPOSITION 6.1. *Let (r_i) and (s_i) be manotone k-tuples such that $(r_i) \prec (s_i)$ and $(\bar{r}_i)$ be obtained from (r_i) by a monotone $(-1, 1)$-change. Then $(\bar{r}_i) \prec (s_i)$.*

PROPOSITION 6.2. *Every mototone k-tuple majorized by a monotone k-tuple (s_i) can be obtained from (s_i) by a sequence of monotone $(-1, 1)$-changes.*

Proof. Let (r_i) be a monotone k-tuple such that $(r_i) \prec (s_i)$. We show that (r_i) can be obtained by a sequence of $(-1, 1)$-changes by induction on $d((r_i), (s_i)) = \sum_1^k |r_i - s_i| = t$. If $t = 0$, $(r_i) = (s_i)$. For $t > 0$, let $\mathfrak{D} = \{i: s_i > r_i\}$. If $\mathfrak{D}$ is empty, then $(s_i) \,_p\!\leqq (r_i)$ and $(r_i) = (s_i)$ since $r_1^* = s_1^*$. Hence $\mathfrak{D}$ is nonempty. Set $i_0 = \max \mathfrak{D}$. Moreover, $i_0 < k$ since $i_0 = k$ implies $\sum_1^{k-1} r_i > \sum_1^{k-1} s_i$ contradicting $(r_i) \prec (s_i)$. Now let $j_0 = \max(\mathfrak{F}(i_0))$ where $\mathfrak{F}(i_0) = \{j: j > i_0 \text{ and } s_j < r_j\}$. If $\mathfrak{F}(i_0)$ is empty and $i_0 = 1$, then $j > 1$ implies $s_j \geqq r_j$ so that $s_j = r_j$ for $j > 1$. But then $s_1 = r_1$, a contradiction. If $\mathfrak{F}(i_0)$ is empty and $i_0 > 1$, then

$$\sum_1^{i_0} s_j + r_{i_0+1}^* \geqq \sum_1^{i_0} r_j + r_{i_0+1}^* = s_1^* \geqq \sum_1^{i_0} s_j + r_{i_0+1}^*$$

and $s_{i_0+1}^* = r_{i_0+1}^*$. But then $s_q = r_q$ for $i_0 + 1 \leqq q \leqq k$. Therefore $\sum_1^{i_0} r_j = \sum_1^{i_0} s_j$ with $s_{i_0} > r_{i_0}$. This implies $\sum_1^{i_0-1} r_j > \sum_1^{i_0-1} s_j$. Again this is a contradiction. Hence $\mathfrak{F}(i_0)$ is nonempty.

Let $(\bar{s}_i)$ be obtained from (s_i) by a monotone $(-1, 1)$-change at the i_0, j_0 places. Then $(\bar{s}_i)$ is monotone and we claim that $(r_i) \prec (\bar{s}_i)$. Since $(r_i) \prec (s_i)$ and $\bar{s}_{i_0} = s_{i_0} - 1 \geqq r_{i_0}$ the desired inequality holds for $1 \leqq q \leqq i_0$. If $i_0 < q < j_0$ and $\sum_1^q r_i > \sum_1^q \bar{s}_i$, then $\sum_1^q r_i = \sum_1^q s_i$. There is some $p > q$ such that $\sum_1^p r_i < \sum_1^p s_i$. Let p_0 be the least such p. Then $(r_{q+1}, \cdots, r_{p_0-1}) = (s_{q+1}, \cdots, s_{p_0-1})$ and $r_p < s_p$. This contradicts the choice of i_0 if $p_0 > q + 1$. If $p_0 = q + 1$, then $r_{q+1} < s_{q+1}$ again gives a contradiction to the choice of i_0. Hence for $1 \leqq q < j_0$, the sum of the first $q r_i$'s is less than or equal to the sum of the first $\bar{s}_i$'s. The inequalities are clear if $j_0 \leqq q \leqq k$ so that $(r_i) \prec (\bar{s}_i)$. Since $d((r_i), (\bar{s}_i)) < d((r_i), (s_i))$, the theorem follows by induction.

Note that if (r_i) can be obtained from (s_i) by a sequence of monotone $(-1, 1)$-changes, then we can obtain (s_i) from (r_i) by a sequence of $(1, -1)$-changes.

PROPOSITION 6.4. *If (r_i) is a monotone k-tuple, then each monotone k-tuple which majorizes (r_i) can be obtained from (r_i) by a finite sequence of monotone $(1, -1)$-changes.*

Our next objective is to show that $P(a_i)$ is closed under residuation. Since $P(a_i)$ is distributive and a product of the a_i's is join irreducible, the following lemma tells us that to check closure of residuation in $P(a_i)$ we only need check the residuation of a product of the a_i's by another such product.

LEMMA 6.5. *If every element in a distributive multiplicative lattice L is a join of join irreducibles and join irreducibles are closed under multiplication, then for Z join irreducible and X, Y in L,*

$$(X \vee Y: Z) = (X: Z) \vee (Y: Z) .$$

Proof. If W is join irreducible such that $WZ \leqq X \vee Y$, then $WZ = (WZ \wedge X) \vee (WZ \wedge Y)$. Hence $WZ \leqq X$ or $WZ \leqq Y$ and $W \leqq (X: Z) \vee (Y: Z)$. Therefore $(X \vee Y: Z) \leqq (X: Z) \vee (Y: Z)$. Since the opposite inequality holds, the lemma is proved.

COROLLARY 6.6. *$P(a_i)$ is closed under residuation if and only if $(X: Y)$ is in $P(a_i)$ for any join irreducibles X, Y in $P(a_i)$.*

Proof. If $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ are products of the a_i's in $P(a_i)$, then

$$(X_1 \vee \cdots \vee X_m: Y_1 \vee \cdots \vee Y_n) = \bigwedge_{j=1}^{n} \left(\bigvee_{i=1}^{m} (X_i: Y_j) \right)$$

by Lemma 6.5.

Technical Lemmas 6.7 and 6.8 allow us to prove $P(a_i)$ is closed under residuation.

LEMMA 6.7. *If $(q_i) \prec (g_i)$ and $(g_i) \geqq_p (b_i)$ for some $(b_i) \prec (e_i^*)$, then $(q_i) \geqq_p (a_i)$ for some $(a_i) \prec (e_i^*)$.*

Proof. First we assume (q_i) is monotone and we may assume that (b_i) is monotone. Let $(\bar{q}_i)$ be obtained from (q_i) by a monotone $(-1, 1)$-change at the l, m places where $l < m$. If $(\bar{q}_i) \geqq_p (b_i)$, let $(a_i) = (b_i)$. If not, then $\bar{q}_i \geqq b_i$ for $i \neq l$ implies that $\bar{q}_l < b_l$. Since $q_l \geqq b_l$, we have $q_l = b_l$ and $b_{l+1} < b_l$. (If $b_{l+1} = b_l$ then $b_l = b_{l+1} \leqq q_{l+1} < q_l = b_l$, a contradiction.) Let $\bar{b}_l = b_l - 1$ and $\bar{b}_i = b_i$ for $i \neq l$. If $\bar{b}_{m-j} < \bar{b}_{m-(j+1)}$ and $q_{m-j} > \bar{b}_{m-j}$ for some $0 \leqq j \leqq m - l + 1$ then (a_i) defined by

$$a_i = \begin{cases} \bar{b}_i & \text{for} \quad i \neq m - j \\ \bar{b}_i + 1 & \text{for} \quad i + m - j \end{cases}$$

satisfies the conclusion of the lemma. Otherwise $\bar{b}_{m-1} = \bar{b}_m$ so that $q_{m-1} \geqq \bar{q}_m > \bar{b}_m = \bar{b}_{m-1}$. Then we can construct (a_i) as desired unless $\bar{b}_{m-1} = \bar{b}_{m-2}$ in which case $q_{m-2} \geqq \bar{q}_{m-1} > \bar{b}_{m-1} = \bar{b}_{m-2}$. Again we can construct the desired (a_i) unless $\bar{b}_{m-2} = \bar{b}_{m-3}$. Continuing, we conclude all of the $\bar{b}_i$'s for i from l to m are equal if (a_i) cannot be constructed. But we know that $\bar{b}_m < \bar{q}_m = q_m + 1 \leqq \bar{q}_l - q_l - 1 = b_l - 1 = \bar{b}_l$; that is, $\bar{b}_m < \bar{b}_l$, a contradiction. Hence (a_i) exists such that $(a_i) \prec (e_i^*)$ and $(\bar{q}_i) \geqq_p (a_i)$. Since any monotone k-tuple majorized by (g_i) can be obtained by a finite sequence of monotone $(-1, 1)$-changes, the lemma is proved for (q_i) monotone.

If (q_i) is not monotone, let (q_i') be its monotone representative. Then for some $(a_i') \prec (e_i^*)$, $(q_i') \geqq_p (a_i')$. But then $(q_i) \geqq_p (a_i)$ and $(a_i) \prec (e_i^*)$.

LEMMA 6.8. *Let (u_i), (f_i^*), (b_i), and (e_i^*) be monotone k-tuples with $(u_i) + (f_i^*) \geqq_p (b_i)$ for some $(b_i) \prec (e_i^*)$ and suppose $(q_i) \prec (f_i^*)$, then $(u_i) + (q_i) \geqq_p (c_i)$ for some $(c_i) \prec (e_i^*)$.*

Proof. Since $(q_i) \prec (f_i^*)$, $(u_i + q_i) \prec (u_i + f_i^*)$. Moreover, since $(u_i) + (f_i^*) = (u_i + f_i^*) \geqq_p (b_i)$ for some $(b_i) \prec (e_i^*)$, by Lemma 6.7 $(u_i + q_i) = (u_i) + (q_i) \geqq_p (c_i)$ for some $(c_i) \prec (e_i^*)$.

COROLLARY 6.9. *If (u_i) is a monotone k-tuple then $\Pi A_i^{u_i} \leqq \Pi a_i^{e_i} : \Pi a_i^{f_i}$ if and only if $(u_i + f_i^*) \geqq_p (b_i)$ for some $(b_i) \prec (e_i^*)$.*

Proof. If $(\bar{q}_i)$ is the monotone representative for (q_i) and $(\overline{u_i + q_i})$ is the monotone representative for $(u_i + q_i)$ for some $(q_i) \prec (f_i^*)$, then

$$\sum \overline{u_i + q_i} \leqq \sum u_i + \sum \bar{q}_i \leqq \sum u_i + \sum f_i^* = \sum (u_i + f_i^*)$$

where the indices run from 1 to j for $1 \leqq j \leqq k - 1$ and $(u_1 + q_1)^* = u_1^* + q_1^* = u_1^* + f_1^{**} = (u_1 + f_1^*)^*$. Hence the condition is sufficient.

Necessity is clear.

Note that a symmetric element E in $RL(A_i)$ is the join of products of the a_i's if and only if whenever $\Pi A_i^{r_i} \leqq E$ with (r_i) monotone and (s_i) is obtained from (r_i) by a sequence monotone $(-1, 1)$-changes, then $\Pi A_i^{s_i} \leqq E$; for then $E = \bigvee \{\Pi a_i^{t_i - t_{i+1}}: (t_i)$ is monotome and $\Pi A_i^{t_i}$ is in the minimal base for $E\}$. As before we set $t_{k+1} = 0$.

THEOREM 6.10. *$P(a_i)$ is closed under residuation.*

Proof. Suppose that (u_i) is monotone and that $\Pi A_i^{u_i} \leqq (\Pi a_i^{e_i}: \Pi a_i^{f_i})$. Let (v_i) be obtained from (u_i) by a monotone $(-1, 1)$-change. Then $\Pi a_i^{u_i} \cdot \Pi A_i^{f_i^*} \leqq \Pi a_i^{e_i}$ so that $(u_i) + (f_i^*) \geqq_p (b_i)$ for some $(b_i) \prec (e_i^*)$. So by Lemma 6.8 $(v_i) + (f_i^*) \geqq_p (c_i)$ for some $(c_i) \prec (e_i^*)$ since $(v_i) + (f_i^*)$ is obtained from $(u_i) + (f_i^*)$ by a monotone $(-1, 1)$-change. Hence $\Pi A_i^{v_i} \leqq (\Pi a_i^{e_i}: \Pi a_i^{f_i})$ by Corollary 6.9. Therefore $\Pi a_i^{u_i - u_{i+1}} \leqq (\Pi a_i^{e_i}: \Pi a_i^{f_i})$ so the residual is the join of all such products $\Pi a_i^{u_i - u_{i+1}}$ where (u_i) is monotone and $\Pi A_i^{u_i} \cdot \Pi a_i^{f_i} \leqq a_i^{e_i}$. (We set $u_{k+1} = 0$.) Since this is an element in $P(a_i)$ our proof is complete.

PROPOSITION 6.11. *Each product of the elementary symmetric elements is a weak join principal element in $P(a_i)$.*

Proof. Let $k > 1$. It suffices to show that $(\Pi a_i^{e_i}: a_t) = \Pi_{i \neq t} a_i^{e_i} \cdot a_t^{e_t - 1}$ whenever $e_t \geqq 1$. And since the product on the right is clearly less than or equal to the residual, we only need demonstrate the opposite inequality. So suppose that $\Pi A_i^{t_i} \leqq (a_i^{e_i}: a_t)$ where $e_t \geqq 1$. By symmetry we assume (t_i) is monotone. Let $(f_i^*) = (1, 1, \cdots, 1, 0, \cdots, 0)$ with 1's in the first t positions. Then

$$(\nabla) \qquad (t_i) + (f_i^*) \geqq_p (b_i) \text{ for some } (b_i) \prec (e_i^*) .$$

Let (u_i) be the lexicographic maximum of the p-minimal k-tuples which are ${}_p\!\leqq (t_i)$ and satisfy (∇) with (u_i) in place of (t_i). Note that (u_i) is monotone since if $(\bar{u}_i)$ is the monotone representative of (u_i) then $(\bar{u}_i)\, {}_p\!\leqq (t_i)$ and by symmetry $\Pi A_i^{\bar{u}_i} \leqq (\Pi a_i^{e_i}: a_t)$. But $(\bar{u}_i) \geqq_l (u_i)$ and since $(\bar{u}_i)$ is p-minimal $(u_i) = (\bar{u}_i)$. Moreover, $(u_i) + (f_i^*) = (u_i + f_i^*)$ is monotone so we can choose (b_i) monotone and l-maximum satisfying (∇) with (t_i) replaced by (u_i).

Claim. $(u_i) \prec ((e_i - f_i)^*)$. For then $\Pi A_i^{t_i} \leqq \Pi A_i^{u_i} \leqq \Pi_{i \neq t} a_i^{e_i} \cdot a_t^{e_t - 1}$.

First suppose that $\sum_1^r b_i = \sum_1^r e_i^*$ for some $r < k$. Set $(g_1, \cdots, g_r) = (f_1, \cdots, f_{r-1}, f_r^*)$ and $(h_1, \cdots, h_r) = (e_1, \cdots, e_{r-1}, e_r^*)$. Then $(h_i) \geqq_p (g_i)$. Also $g_i^* = f_i^*$ and $h_i^* = e_i^*$ for $i = 1, \cdots, r$. So $(u_1 + g_1^*, \cdots, u_r + g_r^*) \geqq_p (b_1, \cdots, b_r)$ with $(b_1, \cdots, b_r) \prec (h_i^*)$. By induction on k $(u_1, \cdots, u_r) \geqq_p (c_1, \cdots, c_r)$ for some $(c_1, \cdots c_r) \prec (h_1^* - g_1^*, \cdots, h_r^* - g_r^*)$. Also by induction on k, since $(u_{r+1}, \cdots, u_k) + (f_{r+1}^*, \cdots, f_k^*) \geqq_p (b_{r+1}, \cdots, b_k)$ for

$(b_{r+1}, \cdots, b_k) \prec (e^*_{r+1}, \cdots, e^*_k)$ there is a $k-r$-tuple $(c_{r+1}, \cdots, c_k)$ such that $(c_{r+1}, \cdots, c_k) \prec ((e_{r+1} - f_{r+1})^*, \cdots, (e_k - f_k)^*)$ and $(u_{r+1}, \cdots, u_k) \geqq_p (c_{r+1}, \cdots, c_k)$. But then $(u_i) \geqq_p (c_i)$ with $(c_i) \prec ((e_i - f_i)^*)$. Hence we may assume that $\sum_1^r b_i < \sum_1^r e_i^*$ for any $r < k$.

If $(b_i) = (u_i + f_i^*)$, then $(u_i) = (b_i - f_i^*)$ and $(u_i) \prec ((e_i - f_i)^*)$. So suppose there exists some i such that $b_i < u_i + f_i^*$. Let i_0 be the first such i. Then for any j, $1 \leqq j \leqq i_0 - 1$, $b_j = u_j + f_j^*$ and by the l-maximality of (b_i), either $b_{i_0-1} = b_{i_0}$, $i_0 = 1$, or if $b_{i_0-1} > b_{i_0}$, then for all $q > i_0$, $b_q = 0$ since otherwise we could perform a mototone $(1, -1)$-change on (b_i). Moreover, by the p-minimality of (u_i), u_{i_0} cannot be reduced in any coordinate so that $u_{i_0} + f^*_{i_0} > b_{i_0}$ implies that $u_{i_0} = 0$. Since f_i^* is either 0 or 1 for each i, we conclude that $1 = f_{i_0} > b_{i_0} = 0$. Hence $i_0 \neq 1$ (for if $i_0 = 1$ then $(b_i) = (0, \cdots, 0)$) and $b_{i_0} \neq b_{i_0-1}$ (for if $b_{i_0-1} = b_{i_0}$, then $b_{i_0-1} = 0 < 1 + u_{i_0-1} = f^*_{i_0-1} + u_{i_0-1}$ contradicting the choice of i_0). So $b_{i_0-1} > b_{i_0}$ and $q > i_0$ implies that $b_q = 0$. Since $e^*_{i_0} > f^*_{i_0}$, $e^*_{i_0} > 0$. Therefore $e_1^* + \cdots + e^*_{i_0-1} < e_1^{**} = b_1^* = b_1 + \cdots + b_{i_0-1} \leqq e_1^* + \cdots + e^*_{i_0-1}$, a contradiction. Therefore the i_0 does not exist and the theorem is proved.

COROLLARY 6.12. *Each product of the elementary symmetric elements is join principal in* $P(a_i)$.

Proof. If A, B, and C are in $P(a_i)$ with A a product of the a_i's, then $(AB \vee C\colon A) = (AB\colon A) \vee (C\colon A) = B \vee (C\colon A)$ since B and C are joins of join irreducibles in $P(a_i)$.

REMARK. In general if A and B are join irreducible in $P(a_i)$, $A\colon B$ is not join irreducible; for example, $a_2^2\colon a_1 = a_2^2 \vee a_3$ in $P(a_1, a_2, a_3)$. Of course the residual $A\colon B$ is join irreducible if $A = CB$ for some C in $P(a_i)$.

7. **Principal elements in $P(a_i)$.** In general a product of elementary symmetric elements in $P(a_i)$ is not a principal element in $P(a_i)$. In particular a_1 is not weak meet principal if $k > 1$ since from § 2 $(a_k\colon a_1) = a_k$ so $(a_k\colon a_1)a_1 = a_1a_k$ while $a_k \wedge a_1 = a_k \neq a_1a_k$. However, there is a nontrivial principal element, a_k, in $P(a_i)$ since a_k is a principal element in $RL(A_i)$. We show that a_k and its powers are the only nontrivial principal elements in $P(a_i)$.

A *Π-domain* is a multiplicative lattice, L', which contains a subset, S, of elements of L' which generates L' under joins such that every element of S is a product of prime elements and in which 0 is a prime element [1, § 4].

THEOREM 7.1. *$P(a_i)$ is a Π-domain in which the only principal* *elements are* 0, a_k^t *for* $t \geqq 1$, *and* I.

Proof. 0 is a prime element in $P(a_i)$ since 0 is a prime element in $RL(A_i)$. Moreover, $P(a_i)$ is a multiplicative lattice which is generated under joins by products of the elementary symmetric elements.

If A and B are joins of products of the a_i's such that $A \not\leqq a_j$ and $B \not\leqq a_j$ for a fixed $j, 1 \leqq j \leqq k$, then there are products $\Pi a_i^{e_i}$ and $\Pi a_i^{f_i}$ in the minimal bases in $P(a_i)$ respectively such that $\Pi a_i^{e_i} \not\leqq a_j$ and $\Pi a_i^{f_i} \not\leqq a_j$. Then there exist $(r_i) \prec (e_i^*)$ and $(s_i) \prec (f_i^*)$ such that both (r_i) and (s_i) have fewer than j nonzero integers. By symmetry (r_i') and (s_j'), the monotone representatives of (r_i) and (s_i) are in the minimal bases for $\Pi a_i^{e_i}$ and $\Pi a_i^{f_i}$ respectively and $(r_i')+(s_i')$ has fewer than j nonzero entries. Therefore $\Pi A_i^{r_i'} \cdot \Pi A_i^{s_i'} \not\leqq a_j$ and hence $AB \not\leqq a_j$. Hence a_j is a prime element in $P(a_i)$.

0 and I are principal elements in $P(a_i)$. The fact that any weak meet principal element in $P(a_i)$ is join irreducible follows from [1, Theorem 1.2]. So in $P(a_i)$ the only nontrivial candidates for principal elements are products of the a_i's. Moreover, since AB principal implies that A is principal and $a_1 \cdots, a_{k-1}$ are not principal elements in $P(a_i)$, the only principal elements in $P(a_i)$ are powers of a_k, 0, and I.

8. **Remarks (multiplicative lattices).** Elements in $RL(A_i)$ and $P(a_i)$ are joins of unique products of their generators. Moreover, both of these multiplicative lattices have a partial order which naturally induces an order on k-tuples associated with their exponent k-tuples. If we define $\phi\colon RL(A_i) \to P(a_i)$ by sending A_i to a_i for each i and extending ϕ via products and joins, we see that ϕ is a join-morphism which preserves products, primes, and join principalness. However $RL(A_i)$ is the lattice of ideals of a semigroup while $P(a_i)$ is not [1]. The problem in $P(a_i)$ is the absence of weak meet principal generators.

In $P(a_i)$ $(k > 1)$ every prime contains the only principal prime element, a_k.

9. **Remarks (partitions of integers).** Brylawski [4] has studied certain sublattices of $P(a_i)$. He defined L_k to be the lattice of monotone partitions of k of length k. Extending Brylawski's notation, we write L_n^k for the lattice of monotone partitions of n with the understanding that the last $n-k$ entries are zero if $n \geqq k$ and the last $k-n$ entries are zero if $n < k$.

For $\mathfrak{B}, \mathfrak{C} \subseteq P(a_i)$, we write $\mathfrak{B} \cdot \mathfrak{C}$ for $\{AB \mid A \in \mathfrak{B} \text{ and } B \in \mathfrak{C}\}$.

PROPOSITION 9.1. *$P(a_i)$ is the disjoint union of isomorphic* *images of* L_n^k, $\bigcup_{n\geqq 0 \text{ or } n=\infty} \psi(L_n^k)$ *where we set* $L_0^k = \{(0, \cdots, 0)\}$ *and* $L_\infty^k = \{(\infty, \cdots, \infty)\}$ *with* $\psi(s_1, \cdots, s_k) = \Pi a_i^{s_i - s_{i+1}}$ *and* $s_{k+1} = 0$. *Moreover* $\psi(L_{n_1}^k) \cdot \psi(L_{n_2}^k) = \psi(L_{n_1+n_2}^k)$ *if* $n_1, n_2 \geqq k$.

Proof. That L_n^k and $\psi(L_n^k)$ are isomorphic as lattices follows from Theorem 5.2 and the fact that dom restricted to L_n^k is simply majorization. Clearly $\psi(L_{n_1}^k) \cap \psi(L_{n_2}^k) = \phi$ for $n_1 \neq n_2$ and $\bigcup_n \psi(L_n^k) = P(a_i)$ if we agree $\psi(L_0^k) = I$ and $\psi(L_\infty^k) = 0$. That $\psi(L_{n_1}^k) \cdot \psi(L_{n_2}^k) = \psi(L_{n_1+n_2}^k)$ if $n_1, n_2 \geqq k$ follows from the addition of exponents of the a_i's in $P(a_i)$ under multiplication.

10. Remarks (symmetric elements). We asked whether the multiplicative sublattice of symmetric elements, $\mathfrak{N}$ (§1) can be generated naturally by a proper subset of generators. We note here that a large subset of $\mathfrak{N}$ does not generate $\mathfrak{N}$ under products and joins.

If (s_i) is a k-tuple of nonzero integers then in $RL(A_i)$, $A_1^{s_1}$, $A_2^{s_2}, \cdots, A_k^{s_k}$ is a prime sequence [6]. So $P(a_1^{(s_1)}, \cdots, a_k^{(s_k)})$ is a Π-domain isomorphic with $P(a_i)$ where $a_i^{(s_i)}$ is the ith elementary symmetric element in $A_1^{s_1}, \cdots, A_k^{s_k}$. Moreover, in terms of the A_i's, $\Pi_{i=1}^k (a_i^{(s_i)})^{e_i} = \{\Pi A_i^{t_i} \mid t_i = s_i r_i \text{ for some } (r_i) \prec (e_i^*)\}$. Elements in $P(a_i^{(s_i)})$ are all symmetric. However, $\bigcup_{(s_i)} P(a_i^{(s_i)})$ generates a proper subset of $\mathfrak{N}$. For example, if $C = A_1^5 A_2^3 A_3$ in $RL(A_1, A_2, A_3)$, then $\bigvee_{g \in S_3} C^g$ is a symmetric element which is not the join of products of any of the $a_i^{(s_i)}$'s.

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