PACIFIC JOURNAL OF MATHEMATICS
Vol. 77, No. 2, 1978

DUALS OF LORENTZ SPACES

G. D. ALLEN

An explicit representation of the duals of Lorentz sequence spaces having "regular" weights is provided.

1. Introduction. Lorentz spaces are rearrangement-invariant weighted l_p spaces, and as such it is necessary to consider only the positive cone of positive sequences in their study. Indeed it is necessary to consider only the sub-cone of positive decreasing sequences.

If X is a sequence space, denote by X^+ the subset of X of non-negative sequences, and by X^{++} the subset of X^+ of decreasing sequences (e.g., l_p, l_p^+, l_p^{++}). For any infinite sequence $\{\mu_n\}$ converging to zero, define $\{\hat{\mu}_n\}$ to be the decreasing rearrangement of $\{|\mu_n|\}$. As is common, c_0 denotes the set of infinite sequences with limit 0.

Let $\{\pi_n\} \in c_0^{++} \backslash l_1^+$. For any $1 \leqq p < \infty$ define $d(\pi, p)$ to be the set of all sequences $\{\mu_n\}$ such that

$$\sum \pi_n (\hat{\mu}_n)^p < \infty .$$

The norm on $d(\pi, p)$ is $(\sum \pi_n (\hat{\mu}_n)^p)^{1/p}$, and $d(\pi, p)$ is called a Lorentz space. The duals of the Lorentz spaces $d(\pi, p)$ are denoted by $d^*(\pi, p)$. These are rearrangement-invariant Banach spaces and hence if one can characterize $d^*(\pi, p)^{++}$ all of $d^*(\pi, p)$ is characterized.

For $1 < p < \infty$, Garling [3] has characterized the duals $d^*(\pi, p)$ as follows: $\{\alpha_n\} \in d^*(\pi, p)^{++}$ *if and only if there is a sequence* $\{\eta_n\} \in l_q^{++}((1/p + 1/q) = 1)$ *such that*

$$\sup_n \frac{\sum_{j=1}^n \alpha_j}{\sum_{j=1}^n \eta_j \pi_j^{1/p}} < \infty . \tag{1}$$

In the case $p = 1$, $\{\alpha_n\} \in d^*(\pi, 1)^{++}$ *if and only if*

$$\sup_n \frac{\sum_{j=1}^n \alpha_j}{\sum_{j=1}^n \pi_j} < \infty . \tag{2}$$

We intend to show that these duals have a particularly simple structure for a broad class of sequences $\{\pi_n\}$. Namely; the sequence $\{\pi_n\} \in c_0^{++}$ is said to be *regular* if

$$\sum_{j=1}^n \pi_j = 0(n\pi_n) . \tag{3}$$

For example, the sequences $\{n^{-p}\}$, $0 < p < 1$, and $\{(\log n)^{-p}\}$, $p > 0$, are regular but the sequence $\{n^{-1}\}$ is not. The concept of regular sequences was first used by Gohberg and Krein [4], and was also used by Altshuler [2] to give necessary and sufficient conditions that the Lorent spaces $d(\pi, p)$, $p > 1$, are uniformly convexifiable.

A necessary and sufficient condition that a sequence $\{\pi_n\} \in c_0^{++}$ be regular is that

$$\inf_n \frac{\sum_{j=1}^{kn} \pi_j}{\sum_{j=1}^{n} \pi_j} \geqq c > 1 \tag{4}$$

for some (and hence all) integers $k \geqq 2$ (cf. Allen and Shen [1]).

2. Main Result. Our main result is stated as follows.

THEOREM 1. *If $\{\pi_n\}$ is regular, then for $p > 1$, $p^{-1} + q^{-1} = 1$,*

$$\{\alpha_n\} \in d^*(\pi, p) \quad \textit{if and only if} \quad \{\hat{\alpha}_n/\pi_n^{1/p}\} \in l^q \,. \tag{5}$$

Note that Theorem 1 is the complete generalization of the result (Allen and Shen [1]):

$$d^*(\pi, 1) = \{\{\xi_n\} \mid \hat{\xi}_n = 0(\pi_n)\} \ \textit{if and only if} \ \{\pi_n\} \ \textit{is regular} \,.$$

The proof of Theorem 1 proceeds in a series of lemmas.

LEMMA 1. *Let $\{\alpha_n\} \in c_0^{++}$. If, for some positive integer $k \geqq 2$ the $\sup_j \alpha_j/\alpha_{kj} = c < k$, then $\{\alpha_j\}$ is regular and*

$$\inf_n \sum_{j=1}^{kn} \alpha_j \Big/ \sum_{j=1}^{n} \alpha_j \geqq k/c \,.$$

Proof. This result follows from (4) since

$$\sum_{j=1}^{kn} \alpha_j \geqq \sum_{j=1}^{n} k\alpha_{kj} \geqq k/c \sum_{j=1}^{n} \alpha_j \,. \tag{6}$$

LEMMA 2. *Let $\{\eta_j\} \in l_q^{++}$, $q > 1$, and $\{\eta_j\} \notin l_r$ for all $r < q$. Then there is a regular sequence $\{\xi_j\} \in l_q^{++}$ for which $\eta_j \leqq \xi_j$, $j = 1, 2, \cdots$. Moreover for any $\varepsilon > 0$ the sequence $\{\xi_j\}$ may be chosen so that*

$$\inf_n \sum_{j=1}^{2n} \xi_j \Big/ \sum_{j=1}^{n} \xi_j > 2^{(q-1)/q} - \varepsilon \,.$$

Proof. For $1 < q_1 < q$, $\{\eta_j^{q_1}\} \in l_{q/q_1}$. Define $\{\xi_n\}$ by

$$\xi_n = \left(\frac{1}{n}\sum_{j=1}^{n}\eta_j^{q_1}\right)^{1/q_1}.$$

Then

$$\xi_n^{q_1}/\xi_{2n}^{q_1} = 2\sum_{j=1}^{n}\eta_j^{q_1}\Big/\sum_{j=1}^{2n}\eta_j^{q_1} < 2\,.$$

By Hardy's inequality $\{\xi_j^{q_1}\} \in l_{q/q_1}^{++}$; so $\{\xi_j\} \in l_q^{++}$. By Lemma 1 $\{\xi_n\}$ is regular, from above $\xi_j/\xi_{2j} < 2^{1/q_1}$, and from (6)

$$\sum_{j=1}^{2n}\xi_j\Big/\sum_{j=1}^{n}\xi_j > 2^{(q_1-1)/q_1} > 2^{(q-1)/q} - \varepsilon\,,$$

for any $\varepsilon > 0$ if q_1 is sufficiently close to q.

LEMMA 3. *Let $\{\pi_n\}$ be a regular sequence and let $p > 1$. Define $\{\pi_n^*\}$ by $\pi_n^* = (1/n)\sum_{j=1}^{n}\pi_j$. Then*

$$\inf_n \sum_{j=1}^{2n}(\pi_j^*)^{1/p}\Big/\sum_{j=1}^{n}(\pi_j^*)^{1/p} > 2^{(p-1)/p}\,.$$

Proof. From the regularity of $\{\pi_j\}$ it follows that

$$\sum_{j=1}^{2n}\pi_j\Big/\sum_{j=1}^{n}\pi_j \geqq k_1 > 1\,.$$

Thus $\pi_{2n}^*/\pi_n^* > k_1/2$, and $(\pi_n^*/\pi_{2n}^*)^{1/p} < (2/k_1)^{1/p}$. By Lemma 1

$$\inf_n \sum_{j=1}^{2n}(\pi_j^*)^{1/p}\Big/\sum_{j=1}^{n}(\pi_j^*)^{1/p} > 2/(2/k_1)^{1/p} = 2^{(p-1)/p}k_1^{1/p}\,.$$

Call two sequences $\{\alpha_n\}$ and $\{\beta_n\} \in c_0^{++}$ *equivalent* if

$$0 < \inf \alpha_n/\beta_n \leqq \sup \alpha_n/\beta_n < \infty\,,$$

and denote equivalence by $\{\alpha_n\} \sim \{\beta_n\}$.

LEMMA 4. *If $\{\alpha_n\}$ and $\{\beta_n\} \in c_0^{++}$ are equivalent and if $\{\alpha_n\}$ is regular, then $\{\beta_n\}$ is regular.*

The proof is a simple application of the definitions. A useful application of Lemma 4 is the following

COROLLARY. *Let $\{\pi_n\}$ and $\{\lambda_n\}$ be regular sequences, for which*

$$\inf_n \sum_{j=1}^{2n}\pi_j\Big/\sum_{j=1}^{n}\pi_j \geqq k_1 > 1 \quad and \quad \inf_n \sum_{j=1}^{2n}\lambda_j\Big/\sum_{j=1}^{n}\lambda_j \geqq k_2 > 1\,.$$

If $k_1k_2 > 2$, then $\{\pi_n\lambda_n\}$ is regular.

Proof. Define $\lambda_n^* = (1/n)\sum_{j=1}^n \lambda_j$. Summing by parts,

$$\begin{aligned}\sum_{j=1}^{2n} \pi_j\lambda_j^* &= \sum_{j=1}^{2n-1}\Big(\sum_{i=1}^{j}\pi_i\Big)(\lambda_j^* - \lambda_{j+1}^*) + \Big(\sum_{i=1}^{2n}\pi_i\Big)\lambda_{2n}^* \\ &\geqq \sum_{j=1}^{n-1}\Big\{\Big(\sum_{i=1}^{2j}\pi_i\Big)(\lambda_{2j}^* - \lambda_{2j+1}^*) + \Big(\sum_{i=1}^{2j+1}\pi_i\Big)(\lambda_{2j+1}^* - \lambda_{2j+2}^*)\Big\} \\ &\quad + \Big(\sum_{i=1}^{2n}\pi_i\Big)\lambda_{2n}^* \\ &\geqq k_1\Big(\sum_{j=1}^{n-1}\Big(\sum_{i=1}^{j}\pi_i\Big)(\lambda_{2j}^* - \lambda_{2j+2}^*) + \Big(\sum_{i=1}^{n}\pi_i\Big)\lambda_{2n}^*\Big) \\ &\geqq k_1k_2/2\Big(\sum_{j=1}^{n}\pi_j\lambda_j^*\Big)\,.\end{aligned}$$

Thus $\{\pi_j\lambda_j^*\}$ is regular, and so, by Lemma 4 is $\{\pi_j\lambda_j\}$.

Proof of Theorem 1. Suppose that $\{\alpha_j\} \in d^*(\pi, p)^{++}$. Select $\{\eta_j\}$ satisfying (1) to additionally satisfy $\{\eta_j\} \notin l_r$ for all $r < q$. By Lemma 2 we can assume by taking $\{\eta_j\}$ larger if necessary that $\{\eta_j\}$ is regular and

$$\inf_n \sum_{j=1}^{2n} \eta_j \quad \sum_{j=1}^{n} \eta_j > 2^{(q-1)/q} - \varepsilon \tag{7}$$

for any fixed $\varepsilon > 0$. It is apparent that $d(\pi, p)$ is isomorphic to $d(\pi^*, p)$ where $\{\pi_n^*\}$ is the sequence defined by $\pi_n^* = \sum_{j=1}^n \pi_j/n$. Hence $d^*(\pi, p)$ and $d^*(\pi^*, p)$ are isomorphic. By Lemma 3 and (7) we can assume that $\{\pi_j^{1/p}\}$ and $\{\eta_j\}$ satisfy the hypotheses of the corollary. Thus by (6) $\alpha_j = 0(\eta_j\pi_j^{1/p})$. Writing $\alpha_j = \xi_j\pi_j^{1/p}$ it follows that $\xi_j = 0(\eta_j)$, and thus $\{\xi_j\} \in l_q$. This proves, together with our earlier remarks, that $d^*(\pi, p) \subset l_q \cdot d^*(\pi^{1/p}, 1)$. The reverse inclusion is immediate. The converse of the theorem follows by taking for $\{\eta_j\}$ the decreasing rearrangement of $\{|\alpha_j/\pi_j^{1/p}|\}$.

3. REMARKS. It could be said that the most important aspect of Theorem 1 is that it provides an explicit representation of the duals of Lorentz spaces with regular weights. We also remark that the function analogue of regularity includes all "weights" currently used in classical interpolation theory. Whether or not Theorem 1 remains true for Lorentz function spaces remains unknown.

REFERENCES

1. G. D. Allen and L. C. Shen, *On the structure of principal ideals of operators*, Trans. Amer. Math. Soc., to appear.

2. Z. Altshuler, *Uniform convexity in Lorentz sequence spaces*, Israel J. Math., **20** (1975), 260–274.

3. D. J. H. Garling, *A class of reflexive symmetric BK-spaces*, Canad. J. Math., **21** (1969), 602–608.
4. I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Nonselfadjoint Operators in Hilbert Space*, English transl. Monographs, Vol. 18, Amer. Math. Soc., Providence, R. I., 1969.

Received July 27, 1977 and in revised form November 10, 1977.

TEXAS A & M UNIVERSITY
COLLEGE STATION, TX 77843