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ON THE INTERSECTION OF REGRESSIVE SETS

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Let A and B be regressive isols and let $\alpha \in A$ and $\beta \in B$. Let Y be the isol containing the set $\alpha \cap \beta$. We study some basic features of the isol Y, and features of Y in the special case the sum $A+B$ is regressive. We also show that there is a large variety of regressive isols A and B for which the values of Y that are associated, in the above way, are all finite.

1. **Introduction.** The results presented in this paper developed from an interest in sets that are the intersection of two regressive sets, and in the isols that contain such sets. (We shall agree that finite sets are both regressive and retraceable.) The following results are known: the intersection of two retraceable sets is retraceable (this is implicit in [7]), but the intersection of a cosimple retraceable set and a cosimple regressive set need not be regressive [1]. On the other hand ([2]) the intersection of any two regressive sets must, if infinite, have an infinite regressive subset. Several of the facts that are developed here were motivated by the following result from [5], which we state in terms of isols:

THEOREM (*Dekker*). *Let A be a regressive isol and let $\alpha, \beta \in A$. Then $\alpha \cap \beta$ is a regressive set.*

We shall prove one generalization of the above theorem. A part of our result gives the following: if α and β are regressive sets that belong to isols whose sum is regressive, then $\alpha \cap \beta$ is also a regressive set.

For regressive isols A and B, let $A \cap B$ denote the collection of all isols that contain a set which is the intersection of a set in A with a set in B. Various properties of the collection $A \cap B$ are derived in the paper. One, in particular, is the following: given any cosimple regressive isol A, there exists an infinite regressive isol B such that all isols in $A \cap B$ are finite.

We will assume that the reader is familiar with the basic notions in the theory of isols and in the theory of regressive isols. Certain special notions, like *T-retraceability*, are important in the paper, and these will be defined in the sequel. If α is an isolated set then $[\alpha]$ will denote the isol to which α belongs. We recall that sets α and β are *recursively equivalent* if there exists a one-to-one partial recursive function f defined on α and with $f(\alpha) = \beta$.

We write ω for the set $(0, 1, 2, \cdots)$. If f is a partial function from a subset of ω (or a subset of ω^2) into ω, then δf will denote the domain of f.

We let $j(x, y)$ denote the familiar recursive pairing function $x + 1/2(x + y)(x + y + 1)$ that maps ω^2 one-to-one and onto ω. The *degree of unsolvability* of a regressive isol is defined as the Turing degree of any retraceable set belonging to it. If A is a regressive isol then Δ_A will denote its degree of unsolvability. The notion of degree of unsolvability for a regressive isol is introduced in [6]. Among the results proved in [6] is the following: if two infinite regressive isols have a sum that is regressive, then their respective degrees are equal.

2. Intersections of regressive sets. Let A and B denote arbitrary regressive isols. We set

$$A \cap B = ([\alpha \cap \beta] \mid \alpha \in A \text{ and } \beta \in B) .$$

If either A or B is finite, then it is easy to see that $A \cap B$ is also finite. In this case, if $r = \text{minimum}\,(A, B)$, then $A \cap B = (0, \cdots, r)$. If Y is an isol with $Y \leqq A$ and $Y \leqq B$, then Y is regressive. For such a Y one can readily show $Y \in A \cap B$. Thus, if both A and B are infinite, then every finite r will belong to $A \cap B$, and in this case one therefore has:

$$\omega \subseteqq A \cap B .$$

It will follow from the second and third theorems given below that there exists a variety of pairs A, B of infinite regressive isols for which $A \cap B = \omega$. Moreover, it will be seen that one may choose such pairs of regressive isols to be cosimple and to have a regressive sum.

THEOREM 1. *Let A and B be regressive isols with $A + B$ regressive. Let $Y \in A \cap B$. Then*

1. *$Y = U + V$ with $U \leqq A$ and $V \leqq B$,*
2. *$Y \leqq A + B$, and*
3. *Y is regressive.*

Proof. It is clear that statement 1 implies statement 2. Also, since any predecessor of a regressive isol is also regressive, statement 2 implies statement 3. We therefore need only verify statement 1. This statement is clear if Y is a finite isol. Let us assume, therefore, that Y is an infinite isol. Let $Y = [\alpha \cap \beta]$ with $\alpha \in A$ and $\beta \in B$. Then both α and β are infinite regressive sets. Select separated sets $\alpha' \in A$ and $\beta' \in B$. Then $\alpha' \cup \beta'$ is a set in $A + B$ and is therefore regressive; we let c_n be a regressive function that ranges over $\alpha' \cup \beta'$. As both α and α' belong to A, they are recursively equivalent, and, similarly, β and β' are recursively equivalent. Let φ_0 and φ_1 be one-to-one partial recursive functions with $\alpha \subseteqq \delta\varphi_0$, $\varphi_0(\alpha) = \alpha'$, $\beta \subseteqq \delta\varphi_1$, and $\varphi_1(\beta) = \beta'$.

Define:

$$\delta = (a \in \alpha \cap \beta \,|\, \varphi_0(a) = c_s,\ \varphi_1(a) = c_t \text{ and } s > t)\ ,$$

$$\lambda = (a \in \alpha \cap \beta \,|\, \varphi_0(a) = c_s,\ \varphi_1(a) = c_t \text{ and } s < t)\ .$$

It is easy to see that the sets δ and λ are separated, and that their union is $\alpha \cap \beta$. In addition, we have,

(a) δ is a separated subset of α, and

(b) λ is a separated subset of β.

We shall verify only statement (a), for a similar argument may be given to verify (b). (Note that statement (a) implies $[\delta] \leqq A$, and statement (b) implies $[\lambda] \leqq B$.) To verify (a), assume $u \in \alpha$. We first compute the value of $\varphi_0(u) \in \alpha'$. Assume $\varphi_0(u) = c_s$. Since c_n is a regressive function, we can compute each of the values in the sequence $c_0, \cdots, c_s$. Some of these numbers may belong to the set β'. We let

$$\mu = (c_0, \cdots, c_s) \cap \beta'\ .$$

Since each of the numbers c_i belongs to the set $\alpha' \cup \beta'$, and since α' and β' are separated sets, it follows that we can effectively find all of the members of μ. It now remains only to observe that $u \in \delta$ iff $u \in \varphi_1^{-1}(\mu)$, and the desired separability of the sets δ and $\alpha - \delta$ follows. This completes the proof of statement (a).

Let $U = [\delta]$ and $V = [\lambda]$. Then $U \leqq A$ and $V \leqq B$, and, lastly,

$$\begin{aligned} Y &= [\alpha \cap \beta] \\ &= [\delta \cup \lambda] \\ &= [\delta] + [\lambda] \\ &= U + V\ . \end{aligned}$$

We mention one corollary to Theorem 1; it concerns regressive isols that are cosimple. We recall that a *cosimple* isol is one that contains a cosimple set; it is known that not all sets belonging to such an isol need be cosimple. We shall make use of the following basic fact about sets with r.e. (recursively enumerable) complements (The "Reduction Theorem"): if two co-r.e. sets are disjoint, then they are separated.

COROLLARY 1. *Let A and B be cosimple regressive isols and $A + B$ be regressive. Let $\alpha \in A$ and $\beta \in B$ with α and β cosimple sets. Then*

$$A + B = [\alpha \cup \beta] + [\alpha \cap \beta] .$$

Proof. Let the sets δ and λ be defined, as in the proof of Theorem 1, from the given sets α and β. Then δ is a separated subset of α, and λ is a separated subset of β. Since α and β are cosimple, it follows that $\alpha - \delta$ and $\beta - \lambda$ are cosimple also. From the proof of Theorem 1 it follows that $\alpha \cap \beta = \delta \cup \lambda$ and that δ and λ are separated sets. But $\alpha \cup \beta = (\alpha - \delta) \cup (\beta - \lambda)$. Since $\alpha - \delta$ and $\beta - \lambda$ are cosimple and disjoint, they are separated. Combining these facts gives the desired conclusion, as follows:

$$\begin{aligned} A + B &= [\alpha] + [\beta] \\ &= ([\alpha - \delta] + [\delta]) + ([\beta - \lambda] + [\lambda]) \\ &= ([\alpha - \delta] + [\beta - \lambda]) + ([\delta] + [\lambda]) \\ &= [(\alpha - \delta) \cup (\beta - \lambda)] + [\delta \cup \lambda] \\ &= [\alpha \cup \beta] + [\alpha \cap \beta] . \end{aligned}$$

A function t_n from ω into ω is called *T-retraceable* if it is retraceable and has the property: if φ is any partial recursive function then $\varphi(t_n) < t_{n+1}$ holds for all except at most finitely many values of n. A set is called *T-retraceable* if it is the range of a T-retraceable function, and an isol is called *T-retraceable* if it contains a T-retraceable set. The notion of T-retraceability has been used frequently in the literature on regressive isols. From [4] or [10], we know that there exist cosimple T-retraceable sets (and therefore also cosimple T-retraceable isols). The following result is Lemma 4 in [8].

LEMMA 1 (*Ellentuck*). *Let τ be a T-retraceable set. If $\alpha, \beta \subseteqq \tau$ are infinite recursively equivalent sets then $\alpha \cap \beta \neq \emptyset$.*

THEOREM 2. *Let Y be a T-retraceable isol. Let $Y = A + B$ with both A and B infinite. Then $A \cap B = \omega$.*

Proof. Note $\omega \subseteqq A \cap B$, since both A and B are infinite. To show that $A \cap B$ contains only finite isols, assume $D \in A \cap B$. Let $D = [\delta]$ with $\delta = \alpha \cap \beta, \alpha \in A$ and $\beta \in B$. Let τ be a T-retraceable set in Y. Since $Y = A + B$ it follows that there will exist separated sets τ_0 and τ_1 such that $\tau = \tau_0 \cup \tau_1, \tau_0 \in A$ and $\tau_1 \in B$. Thus, α and τ_0, and likewise β and τ_1, are recursively equivalent. Let φ_0 and φ_1 be one-to-one partial recursive functions such that $\varphi_0(\alpha) = \tau_0$

and $\varphi_1(\beta) = \tau_1$. Then $\varphi_0(\delta)$ and $\varphi_1(\delta)$ are disjoint subsets of τ, and, in addition, it is readily seen that they are recursively equivalent. It follows from Lemma 1 that each of these sets is finite. Hence δ is a finite set and D is a finite isol.

REMARK. We note that every infinite regressive isol can be written as the sum of two infinite summands. It follows, therefore, from Theorem 2 that there exist infinite regressive isols A and B that have a regressive sum and for which $A \cap B = \omega$. Moreover, we may choose such regressive isols A and B to be cosimple, since there exist cosimple T-retraceable isols.

PROPOSITION 1. *Let α and β be any two sets of numbers. Let $\alpha \cap \varphi(\beta)$ be a finite set, for every one-to-one partial recursive function φ. Then $\mu \cap \lambda$ is a finite set, for every pair (μ, λ) such that μ is recursively equivalent to α and λ is recursively equivalent to β.*

Proof. Assuming μ is recursively equivalent to α and λ is recursively equivalent to β, we have $\mu = \varphi_0(\alpha)$ and $\lambda = \varphi_1(\beta)$ for some one-to-one partial recursive functions φ_0 and φ_1. Assume $\mu \cap \lambda = \delta$. Then $\varphi_0(\alpha) \cap \varphi_1(\beta) = \delta$, with δ contained in the range of φ_0. This implies (since φ_0 is one-to-one) that $\alpha \cap \varphi_0^{-1}\varphi_1(\beta) = \varphi_0^{-1}(\delta)$. Since $\varphi_0^{-1}\varphi_1$ is one-to-one and partial recursive, it follows from our hypothesis that $\varphi_0^{-1}(\delta)$ is finite. Hence δ is also finite.

Since each isol A is the collection of all sets that are recursively equivalent to any given member of A, we obtain the following corollary from Proposition 1.

COROLLARY 2. *Let A and B be regressive isols. Let $\alpha \in A$ and $\beta \in B$ such that $\alpha \cap \varphi(\beta)$ is finite, for every one-to-one partial recursive function φ. Then $A \cap B \subseteqq \omega$.*

LEMMA 2. *Let α be any set which contains a simple set in its complement. Let $\varphi_0, \cdots, \varphi_s$ be any sequence of $s+1$ one-to-one partial recursive functions. Let λ be any infinite r.e. set. Then there exists an infinite r.e. set $\mu \subseteqq \lambda$ such that $\varphi_i(\mu) \cap \alpha$ is finite for each $i = 0, \cdots, s$.*

Proof. Let δ be a simple set contained in the complement of α.

Define a set λ_0 in the following way: consider the set $\varphi_0(\lambda) \cap \alpha$. If this set is finite, let $\lambda_0 = \lambda$. If this set is infinite, then the set $\varphi_0(\lambda) \cap \delta$ is also infinite. In this latter event, let $\lambda_0 = \lambda \cap \varphi_0^{-1}(\delta)$. Then λ_0 is an infinite r.e. subset of λ, and, $\varphi_0(\lambda_0) \cap \alpha$ is finite (in fact, empty).

We may iterate the previous procedure, to obtain sets $\lambda_s \subsetneqq \cdots \subsetneqq \lambda_0$ such that λ_i, for each $i = 0, \cdots, s$, is an infinite r.e. set and $\varphi_i(\lambda_i) \cap \alpha$ is finite. Then $\mu = \lambda_s$ has the desired properties.

We will introduce next some special terminology that is used in our subsequent proof of Theorem 3. Let $F(x)$ be a total function, η a set, and $m \in \eta$. We will interpret the expression "$F(m) = 1$ *stands alone in* η" as meaning $F(m) = 1$ and m is the only number in η whose F-value is 1. The expression "$F(m) = 1$ *does not stand alone in* η" we then interpret as meaning, either $F(m) \neq 1$ or else there exists a value $n \in \eta$ with $n \neq m$ and $F(n) = F(m) = 1$. Also, we set $\omega^+ = \omega - (0)$.

THEOREM 3. *Let A be an infinite regressive isol having the following property: there is a set $\alpha \in A$ such that α has a simple set contained in its complement. Then there exists a regressive isol B with the following properties: B is infinite, $A + B$ is not regressive, and $A \cap B = \omega$.*

Proof. Let δ be a retraceable set with $\delta \in A$, and let deg δ denote the Turing degree of δ. Then, $\Delta_A = \deg \delta$.

We shall define a retraceable function b_n, whose range β will have the following properties: (1) β is an isolated set, (2) β is not Turing reducible to δ, and (3) $\alpha \cap \varphi(\beta)$ is finite, for all one-to-one partial recursive functions φ. Let us assume such a set β can be constructed, and let $B = [\beta]$. Then B satisfies the desired requirements of the theorem. Clearly, B will be an infinite regressive isol. The sum $A + B$ is not regressive, since the degrees $\Delta_A = \deg \delta$ and $\Delta_B = \deg \beta$ are unequal. Lastly, from Corollary 2 it follows that $A \cap B = \omega$.

For use in the definition of b_n, let $\{\varphi_i\}$ be a listing of all one-to-one partial recursive functions, and let $\{F_i\}$ be a listing of all functions of one variable which are partial recursive in the characteristic function of δ.

Define b_0 in the following way: set $\lambda = j(0, \omega^+)$, apply Lemma 2, and let $\mu \subsetneqq \lambda$ be an infinite set with $\alpha \cap \varphi_0(\mu)$ finite. Set b_0 equal to the smallest number $m \in \mu$ such that $\varphi_0(m) \notin \alpha$ and $F_0(m) = 1$ does not stand alone in μ.

Assume the values $b_0, \cdots, b_s$ have already been defined. Define the value b_{s+1} in the following way: set $\lambda' = j(s + 1, j(b_s, \omega^+))$. Then λ' is an infinite r.e. set. Apply Lemma 2, and let $\mu' \subsetneqq \lambda'$ be an infinite set such that $\alpha \cap \varphi_i(\mu')$ is finite for each $i = 0, \cdots, s + 1$. Set b_{s+1} equal to the smallest number $n \in \mu'$ such that,

(a) $\varphi_i(n) \notin \alpha$ for each $i = 0, \cdots, s+1$, and

(b) $F_{s+1}(n) = 1$ does not stand alone in μ'.

This completes the definition of the function b_n. It is easy to see that b_n is a retraceable function. Let $\beta = (b_0, b_1, b_2, \cdots)$. It follows from our definition of β that if $\varphi = \varphi_i$ is any one-to-one partial recursive function, then $\alpha \cap \varphi(\beta) \subseteqq (b_0, \cdots, b_i)$, and, therefore, that $\alpha \cap \varphi(\beta)$ is finite. Since an infinite retraceable set is either recursive or isolated, it also follows, from the previous fact, that β is an isolated set. This verifies properties (1) and (3). For (2), assume $\deg \beta \leqq \deg \delta$. Then there would exist a total function F, partial recursive in the characteristic function of δ, such that $m \in \beta \Leftrightarrow F(m) = 1$. Assume $F = F_i$. Then, in particular, we know that b_i was defined so that $F(b_i) = 1$ does not stand alone in $j(i, \omega)$. Yet b_i is the only member of β that belongs to $j(i, \omega)$. The contradiction which follows establishes (3), and thus completes the proof of the theorem.

Since each infinite cosimple isol contains a cosimple set, the following corollary is obtained directly from the previous theorem.

COROLLARY 3. *Let A be an infinite cosimple regressive isol. Then there exists a regressive isol B with the properties: B is infinite, $A + B$ is not regressive, and $A \cap B = \omega$.*

3. **Invariantly T-retraceable isols.** A regressive isol is called *invariantly T-retraceable* if it is infinite and every set belonging to it is T-retraceable. Invariantly T-retraceable isols were introduced and studied in [10]. It follows from (for example) an observation of Dëgtev in [4] that all cosimple invariantly T-retraceable isols will have the same degree of unsolvability, namely $0'$. In this section we prove the existence of a pair of invariantly T-retraceable isols A and B, having the following properties: A and B are cosimple, $A + B$ is not regressive, and $A \cap B = \omega$.

We shall use some properties of semirecursive sets. This notion was introduced by Jockusch in [9], and it is defined in the following manner: a set α is called *semirecursive* if there is a recursive function $\psi(x, y)$, of two variables, such that for all numbers x and y we have $\psi(x, y) \in (x, y)$, and $(x \in \alpha$ or $y \in \alpha) \Rightarrow \psi(x, y) \in \alpha$. A recursive function $\psi(x, y)$ related to the set α by these properties is called a *selector function* for α. It follows from [9, Theorem 3.2] that if a set is both cosimple and regressive then it is semirecursive. We shall use this fact in our subsequent proof of Theorem 4, to establish the nonregressiveness of a particular set constructed in that proof. We also use the following notation:

if α is any set then $\bar{\alpha}$ will denote its complement.

THEOREM 4. *There exist cosimple regressive isols A and B with the following properties*:

(1) *A and B are invariantly T-retraceable,*

(2) *$A+B$ is nonregressive, and*

(3) *$A \cap B = \omega$.*

Proof. We use the technique of "*movable markers*". We shall construct sets α and β, and partial recursive functions $p(x)$ and $q(x)$, with the following properties:

(a) both α and β are infinite and cosimple, α is a subset of the even numbers, and β is a subset of the odd numbers,

(b) both α and β are retraceable sets, and p and q are partial recursive functions that retrace α and β respectively,

(c) the regressive isols $[\alpha]$ and $[\beta]$ are both invariantly T-retraceable,

(d) the set $\alpha \cup \beta$ is not semirecursive; and

(e) if φ is any one-to-one partial recursive function, then $\varphi(\alpha) \cap \beta$ is finite.

Let us assume that construction of sets α and β with these properties can be carried out, and let $A = [\alpha]$ and $B = [\beta]$. Then A and B so defined will satisfy the requirements of the theorem. From (a), (b), and (c), both A and B are cosimple and invariantly T-retraceable isols. Combining (d), the fact that α and β are separated sets, and [9, Theorem 3.2], we conclude that $A+B$ is a nonregressive isol. Lastly, combining (e) and Corollary 2, we obtain $A \cap B = \omega$.

For the construction, we use two kinds of movable markers on numbers: Λ_i for the construction of α and Σ_i for β. In addition, to obtain the property given in statement (d), we use one kind of tag on numbers, denoted by $*$. We let $\{\varphi_i\}$ be an effective enumeration of the collection of all partial recursive functions of one variable, and we let $\{\psi_i\}$ be an effective enumeration of the collection of all partial recursive functions of two variables. For each number s, φ_i^s will denote the portion of φ_i obtained at the end of the first s stages of computation; ψ_i^s is similarly related to ψ_i.

Stage 0. Attach markers Λ_0 to 0, Λ_1 to 2, Σ_0 to 1, and Σ_1 to 3. Define $p(2) = p(0) = 0$ and $q(3) = q(1) = 1$.

Let $s \geqq 0$ and assume the construction has been carried out up to and including stage s. Let $\tau(s)$ be the number characterized by the property that $\Lambda_0, \cdots, \Lambda_{\tau(s)}$ and $\Sigma_0, \cdots, \Sigma_{\tau(s)}$ are the only markers that are presently attached to numbers. For each number $j \leqq \tau(s)$, let a_j^s and b_j^s denote the numbers that Λ_j and Σ_j are attached to, respectively.

Stage $s+1$. We proceed in the construction in four separate Steps A, B, C and *D*. In Step A we arrange for the invariant T-retraceability of both $[\alpha]$ and $[\beta]$; in Step B, we work to insure that $\varphi(\alpha) \cap \beta$ is finite for each one-to-one partial recursive function φ; and in Step C, we seek to make each recursive function ψ fail in being a selector function for the set $\alpha \cup \beta$. Step D is an auxiliary step introduced to tie together certain parts of Step C. We will say that a number is *free* at a given point in the construction if it is larger than every number which bears at that point a marker or has borne a marker at any previous point in the construction.

Step A. Let $0 < j \leqq \tau(s)$. Consider the values of a_i^s and b_i^s, and the functions φ_i^s, for $i = 0, \cdots, j$. Define $\Delta_\alpha(j, e, f)$ to mean: $e, f \leqq j$, $(\forall k \leqq j)[\varphi_e^s(a_k^s)$ is defined], φ_e^s is one-to-one, and $(\exists k < j)[\varphi_f^s\varphi_e^s(a_k^s)$ is defined and has value $\geqq \varphi_e^s(a_j^s)]$.

Define $\Delta_\beta(j, e, f)$ exactly as $\Delta_\alpha(j, e, f)$ with the one exception that within $\Delta_\beta(j, e, f)$ each occurrence of the letter a is replaced by the letter b.

If $\Delta_\alpha(j, e, f)$ holds true, it implies the possible failure of α, as thus far witnessed by its "approximate initial segment" $a_0^s, \cdots, a_j^s$, to be invariantly T-retraceable. If $\Delta_\beta(j, e, f)$ holds true, then there is a similar meaning which follows, with respect to β and the values of $b_0^s, \cdots, b_j^s$. We consider separately two cases.

Case A1. For each number j with $0 < j \leqq \tau(s)$, and all numbers $e, f \leqq j$, neither $\Delta_\alpha(j, e, f)$ nor $\Delta_\beta(j, e, f)$ is true. In this event, proceed to Step B.

Case A2. There exist numbers j with $0 < j \leqq \tau(s)$ and numbers $e, f \leqq j$ for which either $\Delta_\alpha(j, e, f)$ is true or $\Delta_\beta(j, e, f)$ is true. Let j_0 be the smallest such number. Detach Λ_i and Σ_i for each i such that $j_0 \leqq i \leqq \tau(s)$. Detach $*$ from any number m with $j_0 \leqq m \leqq \tau(s)$ which happens to bear a $*$. Let $2r$ be the smallest even free number. Attach Λ_{j_0} to $2r$, and attach Σ_{j_0} to $2r+1$. Define $p_A^s = (\langle 2r, a_{j_0-1}^s \rangle)$ and $q_A^s = (\langle 2r+1, b_{j_0-1}^s \rangle)$, and then proceed to Step B.

Step B. Let $\tau^A(s)$ be the number characterized by the property that at the completion of Step A, Λ_i and Σ_i for $i = 0, \cdots, \tau^A(s)$, are the only attached markers. For each number j with $0 \leqq j \leqq \tau^A(s)$, let a_j^A and b_j^A denote the numbers to which the markers Λ_j and Σ_j, are, respectively, now attached.

Let w be a number with $0 < w \leqq \tau^A(s)$. Let $\square(w, e, z)$ express the statement: [$e \leqq w$ and $z \leqq \tau^A(s)$ and φ_e^s is one-to-one and $\varphi_e^s(a_w^A)$ is defined and $\varphi_e^s(a_w^A) = b_z^A$]. Then, if $\square(w, e, z)$ holds, we want to try to secure some changes in the eventual values of a_w or b_z, or both, to deal with the possibility that φ_e is actually one-to-one. We consider seperately two cases.

Case B1. For no number w with $0 < w \leqq \tau^A(s)$ do there exist numbers e and z with $\square(w, e, z)$ holding true. In this event, proceed to Step C.

Case B2. There exist numbers $w, e,$ and z with $0 < w \leqq \tau^A(s)$ and with $\square(w, e, z)$ holding true. Let $\langle w_0, e_0, z_0 \rangle$ denote the least such triple of numbers, relative to the lexicographic ordering of all triples of numbers. We consider separately three subcases.

Subcase B2.1. $w_0 = z_0$ and w_0 currently does not carry a $*$ on account of $a_{w_0}^A$. Detach all markers Λ_i with $w_0 \leqq i \leqq \tau^A(s)$, and detach all markers Σ_j with $w_0 < j \leqq \tau^A(s)$. Remove a $*$ from all numbers k with $w_0 < k \leqq \tau^A(s)$ that are carrying a $*$. Let $2r$ be the smallest even free number. Attach Λ_{w_0} to $2r$, and let $p_B^s = (\langle 2r, a_{w_0-1}^A \rangle)$ and $q_B^s = \varnothing$. Now proceed to Step C.

Subcase B2.2. $w_0 = z_0$ and w_0 currently carries a $*$ on account of $a_{w_0}^A$. Detach all markers Λ_i with $w_0 < i \leqq \tau^A(s)$ and detach all markers Σ_j with $w_0 \leqq j \leqq \tau^A(s)$. Remove any $*$ that may be attached to a number k with $w_0 < k \leqq \tau^A(s)$. Let $2r + 1$ be the smallest odd free number. Attach Σ_{w_0} to $2r + 1$, and let $p_B^s = \varnothing$, $q_B^s = (\langle 2r + 1, b_{w_0-1}^A \rangle)$. Now proceed to Step C.

Subcase B2.3. $w_0 \neq z_0$. Set $y_0 =$ maximum (w_0, z_0). Detach all markers Λ_i and Σ_i with $y_0 \leqq i \leqq \tau^A(s)$. Remove the $*$ from any number k with $y_0 \leqq k \leqq \tau^A(s)$ that may carry a $*$. Let $2r$ be the smallest even free number. Attach Λ_{y_0} to $2r$ and Σ_{y_0} to $2r+1$. Set $p_B^s = (\langle 2r, a_{y_0-1}^A \rangle)$ and $q_B^s = (\langle 2r + 1, b_{y_0-1}^A \rangle)$. Then proceed to Step C.

Step C. There are two aims in this step. The first is to insure that in the eventual determination of the sets α and β, the set $\alpha \cup \beta$ is not semirecursive. This we do by arranging that each $\psi_i(x, y)$ fail to be a selector function for $\alpha \cup \beta$. The second aim is to tie together the construction of values in the previous and present steps so that they may be continued further at the next stage; and this we do here and in an auxiliary Step D.

Let $\tau^B(s)$ be the number characterized by the property that Λ_i and Σ_i for $0 \leqq i \leqq \tau^B(s)$, are the only attached markers, at the conclusion of Step B. For each number i with $0 \leqq i \leqq \tau^B(s)$, let a_i^B and b_i^B be the respective numbers to which the markers Λ_i and Σ_i are now attached.

Let j be a number with $0 < j \leqq \tau^B(s)$, and let ∇_j denote the statement: $[\psi_j^s(a_j^B, b_j^B)$ is defined and $\psi_j^s(a_j^s, b_j^s) \in (a_j^B, b_j^B)]$. The number j may or may not bear a $*$. The significance of a $*$ on j is that it indicates the current impossibility of $\psi_j(x, y)$ being a selector function for $\alpha \cup \beta$. Thus, we wish to "hang stars". The criteria for application of a $*$ to a number are introduced in this step. We consider separately two cases.

Case C1. There is a number j such that $0 < j \leqq \tau^B(s)$, j does not bear a $*$, and ∇_j holds true. Let j_0 be the smallest such number j. Let $2r$ denote the smallest even free number.

Subcase C1.1. $\psi_{j_0}^s(a_{j_0}^B, b_{j_0}^B) = a_{j_0}^B$. Detach all markers Λ_j and Σ_k with $j_0 \leqq j \leqq \tau^B(s)$ and $j_0 < k \leqq \tau^B(s)$. Remove $*$ from each number i for which $j_0 \leqq i \leqq \tau^B(s)$ and i currently bears a $*$. Attach Λ_{j_0} to $2r$, and set $p_C^s = (\langle 2r, a_{j_0-1}^B\rangle)$ and $q_C^s = \emptyset$. Tag j_0 with a $*$ on account of $b_{j_0}^B$. Proceed now to Step D.

Subcase C1.2. $\psi_{j_0}^s(a_{j_0}^B, b_{j_0}^B) = b_{j_0}^B$. Detach all markers Λ_j and Σ_k with $j_0 < j \leqq \tau^B(s)$ and $j_0 \leqq k \leqq \tau^B(s)$. Remove $*$ from each number i such that $j_0 \leqq i \leqq \tau^B(s)$ and i currently bears a $*$. Attach Σ_{j_0} to the number $2r+1$, and set $p_C^s = \emptyset$ and $q_C^s = (\langle 2r+1, b_{j_0-1}^B\rangle)$. Tag j_0 with a $*$ on account of $a_{j_0}^B$. Proceed now to Step D.

Case C2. Either no number j exists with $0 < j \leqq \tau^B(s)$ and with ∇_j holding true, or if such a number j does exist, then it carries a $*$. In either of these events, we proceed as follows. Let $2u$ be the smallest even free number. Attach $\Lambda_{\tau^B(s)+1}$ to $2u$ and attach $\Sigma_{\tau^B(s)+1}$ to $2u+1$. Set $p(x) = y$ in case $\langle x, y\rangle \in p_A^s \cup p_B^s \cup (\langle 2u, a_{\tau^B(s)}^B\rangle)$, and set $q(x) = y$ in case $\langle x, y\rangle \in q_A^s \cup q_B^s \cup (\langle 2u+1, b_{\tau^B(s)}^B\rangle)$. Then proceed to stage $s+2$.

Step D. Let the number j_0 be defined as in Case C1. Let $2u$ be the smallest even free number. Attach Λ_{j_0+1} to $2u$ and attach Σ_{j_0+1} to $2u+1$. Set $p(x) = y$ in case $\langle x, y\rangle \in p_A^s \cup p_B^s \cup p_C^s$ or if $x = 2u$ and $y =$ the number to which Λ_{j_0} is currently attached. Set $q(x) =$ y in case $\langle x, y\rangle \in q_A^s \cup q_B^s \cup q_C^s$ or if $x = 2u+1$ and $y =$ the number to which Σ_{j_0} is currently attached. We now proceed to stage $s + 2$.

This completes our construction. We verify first the following two facts:

I. Each of the functions $p(x)$ and $q(x)$, as defined by the construction, is partial recursive; and

II. Each of the markers Λ_m and Σ_m, for $m = 0, 1, 2, \cdots$, eventually attains a permanently-held position on some particular number.

Re. I. Since numbers that enter into the domains of p and q are free just prior to their entry, it follows that both p and q are well-defined functions. Also, since the construction, as a whole, is effective, it follows that each of the sets of ordered pairs $\{\langle x, p(x)\rangle\}$ and $\{\langle x, q(x)\rangle\}$ is recursively enumerable. Hence both of the functions p and q are partial recursive.

Re. II. It clearly follows from the construction that the marker Λ_0 is permanently attached to the number 0, and Σ_0 is permanently attached to the number 1.

Let us assume that $n \geqq 0$ and that each of the markers Λ_i and Σ_i, for $i = 0, \cdots, n$, attains a permanent attachment to a corresponding number. We wish to show that each of the markers Λ_{n+1} and Σ_{n+1} will also become permanently attached. Let s_0 be a stage such that each of the markers Λ_n and Σ_n is found in its permanent position at the conclusion of stage s_0. In view of the construction, it follows that both markers Λ_{n+1} and Σ_{n+1} will be attached at the end of every stage $t \geqq s_0 + 1$. If these markers are never detached during a stage $t > s_0 + 1$ then they will be permanently attached at the conclusion of stage $s_0 + 1$, and the desired result follows. Let us, therefore, assume there does occur some detachment of Λ_{n+1} or of Σ_{n+1} during a stage $t > s_0 + 1$. This can occur in only three different ways, during each individual stage: either from an application of Case A2 (with $j_0 = n + 1$), or an application of Case B2 (with $n+1 \in (w_0, z_0)$), or an application of Case C1 (with $j_0 = n + 1$). In addition, once a detachment of one of the markers Λ_{n+1} or Σ_{n+1} is made during a particular stage $t > s_0+1$ of the construction, then, within that stage, the particular marker is immediately reattached to a new number.

Let us consider first the number of times one of the markers Λ_{n+1} or Σ_{n+1} can be detached, in the course of the construction past stage s_0, via an application of Case A2. By the procedure defined in Case A2, in particular, because of the one-to-one feature placed on the function φ_e^s, it follows that there can occur at most a finite number of detachments of either Λ_{n+1} or of Σ_{n+1} by an application of this case. Let t_0 denote a number such that $t_0 \geqq s_0 + 2$, and neither Λ_{n+1} nor Σ_{n+1} moves because of an application of Case A2 during any stage u with $u \geqq t_0$.

Let us consider next the number of times that a detachment of either of the markers Λ_{n+1} or Σ_{n+1} can occur at a stage $u \geqq t_0$ through an application of Case C1. Note that the procedure in Step B serves to preserve the attachment of a $*$ to $n+1$, at stage t_0 and at each following stage, except in Subcase B2.3 when $y_0 = w_0 = n+1$. In view of this fact, and the procedure defined in Case C1, it is easy to check that (subsequent to stage t_0) each of the markers Λ_{n+1} and Σ_{n+1} can be detached, by an application of Case C1, at most a finite number of times, provided they suffer at most finitely many detachments via Subcase B2.3.

With respect to applications of Case B2, let us first, then, consider the number of times either of Λ_{n+1} or Σ_{n+1} could be detached via an application of Subcase B2.3, in a stage $u \geqq t_0$. By the one-to-one condition placed on φ_e^s throughout Case B2, we see that there can occur at most finitely many such applications of Subcase B2.3. Let u_0 be the smallest number u with the properties: $u \geqq t_0 + 1$ and neither Λ_{n+1} nor Σ_{n+1} moves by an application of either Subcase B2.3 or Case C1 at any stage subsequent to stage u. Then movements of Λ_{n+1} or Σ_{n+1}, in a stage $t \geqq u_0 + 1$, will occur only within Subcase B2.1 or Subcase B2.2. We consider seperately two possibilities, based on the $*$-bearing status of the number $n+1$ at the conclusion of stage u_0.

Case 1. The number $n+1$ bears a $*$ at the conclusion of stage u_0. Then $n+1$ will retain a $*$ throughout all stages $t \geqq u_0 + 1$. Hence, by the construction, either (a) all the movements being considered are movements of Λ_{n+1} under Subcase B2.1, or else (b) all of the movements are of Σ_{n+1} under Subcase B2.2. In either of these events, because of the one-to-one condition placed on φ_e^s throughout Case B2, it follows that there can be at most $n+2$ movements of either of the markers Λ_{n+1} or Σ_{n+1}.

Case 2. The number $n+1$ does not bear a $*$ at the conclusion of stage u_0. Since we are in a situation where Case C1 will henceforth no longer apply toward moving the markers Λ_{n+1} and Σ_{n+1}, it follows that throughout the construction beyond stage u_0, the number $n+1$ will always be without a $*$. Based upon the construction, this fact implies that all of the movements which we are now considering are movements only of the marker Λ_{n+1}, and moreover, that each such movement can occur only by an application of Case B2.1. It follows that the constraints placed upon the application of Case B2.1 make possible at most $n+2$ such detachments of the marker Λ_{n+1}.

In view of these remarks it follows that each of the markers Λ_{n+1} and Σ_{n+1} will achieve a permanent position of attachment after a finite number of stages in the construction. Hence, by induction, each of the markers Λ_j and Σ_j, for $j = 0, 1, 2, \cdots$, attains a permanently-held position on some particular number.

DEFINITION. For each number n, we let a_n and b_n be, respectively, the terminal positions of the marker Λ_n and of Σ_n. In view of the construction, it follows that $a_n = \lim_{s\to\infty} a_n^s$ and $b_n = \lim_{s\to\infty} b_n^s$. We also define: $\alpha = (a_0, a_1, a_2, \cdots)$ and $\beta = (b_0, b_1, b_2, \cdots)$.

It is easy to see, from the construction, that α is a subset of the even numbers and β is a subset of the odd numbers. Hence α and β are separated sets. To complete the proof, in view of our earlier remarks, it suffices to verify the following properties:

III. $\bar{\alpha}$ and $\bar{\beta}$ are recursively enumerable sets,

IV. $\alpha \cup \beta$ is not semirecursive, and therefore $\alpha \cup \beta$ is not regressive,

V. Each of the sets α and β is retraceable and immune, and each of the isols $[\alpha]$ and $[\beta]$ is invariantly T-retraceable, and

VI. $\varphi_t(\alpha) \cap \beta$ is a finite set, for every one-to-one partial recursive function φ_t.

Re. III. Markers move, in the construction, only from previously-held positions up to free numbers. Therefore the set $\bar{\alpha}$ can be characterized in the following way:

$\bar{\alpha} = (x \mid (\exists n)(\exists s)(\exists u)(\exists w)[u < x < w$ and Λ_n is attached to u at the end of stage s, and Λ_{n+1} is attached to w at the end of stage $s])$.

A similar expression exists for $\bar{\beta}$ in terms of the markers Σ_i. Hence both $\bar{\alpha}$ and $\bar{\beta}$ will be recursively enumerable sets.

Re. IV. Let us assume otherwise, and let ψ_n be a selector function for $\alpha \cup \beta$. Then, by definition, $\psi_n(x, y) \in (x, y)$ for all x and y, and if $x \in \alpha \cup \beta$ or $y \in \alpha \cup \beta$ then $\psi_n(x, y) \in \alpha \cup \beta$. Consider the number n and the pair of numbers $\langle a_n, b_n \rangle$. Assume we are at the beginning of a stage s in the construction where Λ_n is already permanently attached to a_n and Σ_n is already permanently attached to b_n. If the number n carries a $*$, then this $*$ could only have been received by an application of Case C1 prior to stage s. From the description of Case C1, it then follows that either $(\exists y)[\psi_n(a_n, y) \in \bar{\beta}]$ the odd numbers] or $(\exists x)[\psi_n(x, b_n) \in \bar{\alpha}]$ the even numbers]. Hence, in this case, ψ_n could not be a selector function for $\alpha \cup \beta$. If the number n does not carry a $*$, then n will not carry a $*$ at any stage beyond stage s. (Since awarding it one would cause either Λ_n or Σ_n to move.) It follows therefore, in view of Step C, that $\psi_n(a_n, b_n) \notin (a_n, b_n)$, and hence, in this case also, ψ_n could not be a selector function for $\alpha \cup \beta$.

Combining these facts we may conclude that $\alpha \cup \beta$ is not semirecursive. Then, by our earlier remarks about semirecursive sets, it also follows that $\alpha \cup \beta$ is not a regressive set.

Re. V. It follows directly from statement I and from the definitions of the functions p and q and of the sets α and β that the functions p and q are partial recursive and, respectively, retrace the sets α and β. (Thus, note that $p(a_{n+1}) = a_n$ and $q(b_{n+1}) = b_n$ follow from the construction.) So, α and β are both retraceable sets. We know, from [7], that a retraceable set is either recursive or immune. If either α or β is recursive, it would follow at once that $\alpha \cup \beta$ is semirecursive. By statement IV, therefore, both α and β are immune sets. We prove next that the isols $[\alpha]$ and $[\beta]$ are each invariantly T-retraceable; and we shall do this only for $[\alpha]$, since a similar proof can be given for $[\beta]$. Let φ_e be any one-to-one partial recursive function such that $\alpha \subseteqq \delta\varphi_e$. We wish to verify that $\varphi_e(\alpha)$ is a T-retraceable set. First note that $\varphi_e(a_n)$ is a regressive function, since it is recursively equivalent to the retraceable function a_n. Let us consider any partial recursive function φ_f. Let j = maximum (e, f). It follows from the construction that Step A is applied as often as is needed in order to obtain the property: $(\forall k)[(k > j \text{ and } \varphi_e(a_k) \in \delta\varphi_f) \Rightarrow \varphi_f(\varphi_e(a_k)) < \varphi_e(a_{k+1})]$. Thus if $\varphi_e(\alpha)$ is a retraceable set, then it is a T-retraceable set. To see that $\varphi_e(\alpha)$ is actually a retraceable set, first note that $\varphi_e(a_n)$ is an eventually increasing function; for we may choose, in the previous discussion, the function φ_f to be the identity function. Since $\varphi_e(a_n)$ is a regressive function, this fact gives the retraceability of the function $\varphi_e(a_n)$, and, therefore, of the set $\varphi_e(\alpha)$.

Re. VI. We wish to verify that if φ_t is any one-to-one partial recursive function then $\varphi_t(\alpha) \cap \beta$ is a finite set. But, Step B of the construction is so designed that the only elements of α that φ_t can map into β are among the values $a_0, \cdots, a_t$. Hence $\varphi_t(\alpha) \cap \beta$ has cardinality $\leqq t + 1$, and therefore is a finite set. This verifies statement VI, and completes the proof of the theorem.

4. Concluding remarks. (1) The results of §§ 2-3 leave open the question whether there is an isol theoretic characterization of the regressive intersection property $A \cap B \subseteqq \Lambda_R$ (where Λ_R is the set of regressive isols). Some of the intersection and separation properties considered in § 2 of this paper are further studied in [12], yet no such characterization is yet at hand.

(2) One way of proving Lemma 1 is to note that if φ_e is a one-to-one partial recursive function, τ a T-retraceable set, and $\lambda = (x \in \tau \mid \varphi_e(x)$ is defined and $\varphi_e(x) \in \tau)$, then φ_e restricted to λ is the identity function except on a finite set. This observation suggests the possibility of proving Theorem 4 without an ad hoc construction, by choosing suitable disjoint subsets α and β of a suitable T-retraceable set, with α and β being representable as differences of r.e. sets. This approach, however, cannot be applied to prove Theorem 4 in view of the footnote on p. 574 of [11].

(3) Let us assume that A and B are infinite regressive isols, and $A + B$ is also regressive. Let $Y \in A \cap B$. From Theorem 1 it follows that $Y \leqq A + B$. It is natural to feel that perhaps in this case one also has either $Y \leqq A$ or $Y \leqq B$. This feature need not hold, and we simply state, without proof, the following result which implies this fact.

THEOREM. *There exist cosimple, invariantly T-retraceable isols A and B with the following two properties*: (1) $A + B$ *is regressive, and* (2) *there is an isol* $Y \in A \cap B$ *such that neither* $Y \leqq A$ *nor* $Y \leqq B$.

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