PACIFIC JOURNAL OF MATHEMATICS
Vol. 79, No. 2, 1978

EXTENDING A BRANCHED COVERING OVER A HANDLE

ALLAN L. EDMONDS

It is shown that if $\varphi: M^n \to S^n$, $n \geqq 3$, is a branched covering of degree at least 3 and if W^{n+1} is $M^n \times [0, 1]$ with a 2-handle attached, then φ extends to a branched covering $W^{n+1} \to S^n \times [0, 1]$.

1. Introduction. Let $\varphi: M^n \to S^n$ be a branched covering, where M^n is a connected n-manifold, $f: \partial B^k \times D^{n-k+1} \to M^n$ be a flat embedding, and $W^{n+1} = M^n \times [0, 1] \cup_{(f,1)} B^k \times D^{n-k+1}$ be $M^n \times [0, 1]$ with a k-handle attached along $M^n \times 1$ via f. When can one extend φ to a branched covering $\theta: W^{n+1} \to S^n \times [0, 1]$?

If $k = 1$ and $\deg \varphi \geqq 2$, one always can extend φ [2; (6.1)]. But for $k = 2$ and $\deg \varphi = 2$ one meets obstructions indicated by the fact that the 3-torus T^3 is not a 2-fold branched covering of S^3[4].

In this paper we show (Theorem 4.4) that one can always extend φ if $k = 2$ provided that $\deg \varphi \geqq 3$ and $n \geqq 3$. (For $n = 2$ one would need to assume that $f(\partial B^2)$ does not separate M^2.) The prototype for a result of this sort was proved in a recent paper by J. Montesinos [8] forthe case $n = 3$, when φ is a particular standard 3-fold branched covering of a connected sum of $S^1 \times S^2$'s over S^3.

Again in the case when $k = 3$, $\deg \varphi = 3$, and $n \geqq 4$ one meets further obstructions indicated by the fact that T^4 is not a 3-fold branched covering of S^4 [1].

2. Preliminaries. We shall work in the PL category of piecewise linear manifolds and maps [6]. All embeddings of manifolds in manifolds will be required to be locally flat. The symbols M^n and N^n will denote compact orientable n-manifolds. The symbols B^n and D^n will be reserved for a standard model of a PL n-ball, say $\{x \in \boldsymbol{R}^n: |x_i| \leqq 1, i = 1, \cdots, n\}$, and $S^n = \partial B^{n+1}$ will denote the standard PL n-sphere.

A *branched covering* is a surjective, finite-to-one, open (PL) map $\varphi: M^n \to N^n$ between n-manifolds. The *singular set* of a branched covering $\varphi: M^n \to N^n$ is the set of $x \in M^n$ near which φ fails to be a local homeomorphism and is denoted by Σ_φ; the *branch set* of φ is $B_\varphi = \varphi\Sigma_\varphi \subset N^n$.

The *degree* of a branched covering $\varphi: M^n \to N^n$ is $\deg \varphi = \sup\{\#\varphi^{-1}(y): y \in N^n\}$. One easily verifies that deg φ is the absolute value of the ordinary homological degree of φ as a map.

A *branch homotopy* is a branched covering $\theta: M^n \times [0, 1] \to N^n \times$

$[0, 1]$ such that $\theta(M^n \times i) = N^n \times i$, $i = 0, 1$. Branched coverings $\varphi, \psi: M^n \to N^n$ are branch homotopic if there is a branch homotopy θ such that $\theta | M \times 0 = \varphi$ and $\theta | M \times 1 = \psi$. By the Alexander trick, two branched coverings $\varphi, \psi: D^n \to D^n$ which agree on ∂D^n are branch homotopic. In general the branch set of a branch homotopy is not assumed to have a locally flat manifold for its branch set.

3. The situation in degree two. If $\varphi: M^n \to N^n$ is a branched covering of degree 2, then φ may be identified with the orbit map $M^n \to M^n/T$ for the involution $T: M^n \to M^n$ which switches points in the fibers of φ. Then by Smith theory [3], $\Sigma_\varphi = \mathrm{Fix}\,(T) \cong B_\varphi$ is a $\boldsymbol{Z}_2$-homology $(n-2)$-manifold.

The *standard involution* $T: D^2 \times \boldsymbol{R}^n \to D^2 \times \boldsymbol{R}^n$ is given by $T(a, b, x_1, \cdots, x_n) = (a, -b, -x_1, x_2, \cdots, x_n)$. Then $\mathrm{Fix}\,(T)$ may be identified with $D^1 \times \boldsymbol{R}^{n-1}$. There are induced standard involutions on $D^2 \times D^n$ and on $S^1 \times D^n$. In particular

$$\mathrm{Fix}\,(T | S^1 \times D^n) \cong S^0 \times D^{n-1}$$

and the orbit space $D^2 \times D^n/T \cong D^{n+2}$ with $S^1 \times D^n/T \cong D^{n+1}$, a face of $D^2 \times D^n/T$. In $S^1 \times D^n/T$, $\mathrm{Fix}\,(T | S^1 \times D^n)$ is a pair of unknotted and unlinked properly embedded $(n-1)$-disks.

LEMMA 3.1. *Let $T': S^1 \times D^n \to S^1 \times D^n$ be an involution with $S^1 \times D^n/T' \cong D^{n+1}$ and $\mathrm{Fix}\,(T')$ consisting of two properly embedded unknotted and unlinked $(n-1)$-disks in $S^1 \times D^n/T$. Then T' is equivalent to the standard involution on $S^1 \times D^n$.*

The proof is an exercise in regular neighborhood theory and omitted.

Now consider the framing $\mathscr{F}: S^1 \times \boldsymbol{R}^n \to S^1 \times \boldsymbol{R}^n$ given by

$$\mathscr{F}(a, b; x_1, x_2, x_3, \cdots, x_n) = (a, b; ax_1 - bx_2, bx_1 + ax_2, x_3, \cdots, x_n)\,.$$

Notice that $\mathscr{F}T = T\mathscr{F}$, where T is the standard involution. The equivariant framings $\mathscr{F}^r$, $r \in \boldsymbol{Z}$, are called the *standard framings*. Note that any framing $\mathscr{G}: S^1 \times \boldsymbol{R}^n \to S^1 \times \boldsymbol{R}^n$ is isotopic through framings to a standard framing, since framings are classified by

$$\pi_1(\mathrm{PL}_n) \approx \begin{cases} \boldsymbol{Z} & (n = 2) \\ \boldsymbol{Z}_2 & (n \geqq 3) \end{cases}$$

and each class is represented by a standard framing.

Let $\varphi: M^n \to N^n$ be a branched covering of degree 2. A simple closed curve $C \subset M^n$ is said to be *invariant* if $\varphi^{-1}\varphi(C) = C$ and the map $C \to \varphi(C)$ is the orbit map for an involution with two fixed points (so that $\varphi(C)$ is an arc which meets B_φ precisely in its end points).

THEOREM 3.2. *Let $\varphi: M^n \to N^n$ be a branched covering of degree 2, $f: \partial B^2 \times D^{n-1} \to M^n \times 1$ an embedding, and $W^{n+1} = M^n \times [0, 1] \bigcup_f B^2 \times D^{n-1}$. Then φ extends to a branched covering $\theta: W^{n+1} \to N^n \times [0, 1]$ provided that $f(\partial B^2 \times 0)$ is isotopic to an invariant simple closed curve.*

Proof. It suffices to show that after perhaps changing f by an isotopy (which does not change W), f may be assumed to be equivariant with respect to the standard involution on $\partial B^2 \times \boldsymbol{R}^{n-1}$ and the involution of M corresponding to φ. For then W^{n+1} inherits an involution, standard on $B^2 \times D^{n-1}$, with orbit space $N^n \times [0, 1] \cup (B^2 \times D^{n-1}/T) \cong N^n \times [0, 1] \bigcup_{D^n} D^{n+1} \cong N^n \times [0, 1]$.

By hypothesis and the isotopy extension theorem, we may assume that $C = f(\partial B^2 \times 0)$ is invariant and that $f(\partial B^2 \times \boldsymbol{R}^{n-1}) = \operatorname{int} U$, where U is an invariant regular neighborhood of C in M^n. Let $A = \varphi(C)$, a simple arc in N^n such that $A \cap B_\varphi = \partial A$. Adjusting A, and hence C, slightly we may assume that A meets B_φ precisely in the interiors of $(n-2)$-simplices of B_φ when M^n and N^n are given triangulations with respect to which φ is simplicial. Then the involution on $U \cong S^1 \times D^{n-1}$ is equivalent to the standard involution by (3.1) and f may be assumed to be equivariant with respect to the standard involution by the remarks above concerning framings.

REMARK 3.3. The new branch set B_θ may be described as $B_\varphi \times [0, 1]$ plus a 1-handle attached in the manifold part of $B_\varphi \times 1$. Thus, if B_φ is a manifold, B_θ will also be a manifold.

REMARK 3.4. In general there are obstructions to making $f(\partial B^2 \times 0)$ invariant, as indicated in §1.

4. The situation in degree greater than two. A branched covering $\varphi: M^n \to N^n$ of degree d is said to be *simple* if $\# \varphi^{-1}(y) \geqq d - 1$ for all $y \in N^n$. A point $y \in B_\varphi$ is a *simple branch point* if $\# \varphi^{-1}(y) = d - 1$. One easily verifies that the nonsimple branch points constitute a subpolyhedron of B_φ.

A simple closed curve $C \subset M^n$ is *invariant* if $\varphi(C) = A$ is a simple arc which meets B_φ precisely in its boundary ∂A at two simple branch points. In this case $\varphi^{-1}(C)$ consists of C plus $(d-2)$ arcs. In particular, near C φ is an orbit map for an involution, and near any other component of $\varphi^{-1}(A)$, φ is a homeomorphism.

LEMMA 4.1. *Let M^2 be a closed, connected orientable 2-manifold and $\varphi: M^2 \to S^2$ be a simple branched covering of degree at least 3. Then any nonseparating simple closed curve $C \subset M^2$ is isotopic to an invariant simple closed curve.*

Proof. By [2; (3.4)] we have a standard picture for φ. By [7] there is a homeomorphism $h: M^2 \to M^2$ such that $h(C)$ is a standard invariant simple closed curve. By [5] and [1; (4.1)] h is isotopic to a homeomorphism $g: M^2 \to M^2$ which respects φ in the sense that g induces a homeomorphism of S^2. Then $g^{-1}h(C)$ is the desired simple closed curve.

LEMMA 4.2. *Let $\varphi: M^n \to N^n$ be any branched covering. Then φ is branch homotopic to a branched covering ψ such that the set of nonsimple branch points has dimension less than $n-2$.*

Proof. We may assume that M^n and N^n are triangulated so that φ is simplicial.

Suppose $\xi: D^2 \to D^2$ is any branched covering. Then by direct construction there is a simple branched covering $\zeta: D^2 \to D^2$ such that $\deg \zeta = \deg \xi$ and $\xi | \partial D^2 = \zeta | \partial D^2$. By the "Alexander trick" ξ and ζ are branch homotopic rel ∂D^2 (cf. [2; (3.3)]).

Now let $\sigma^{n-2} < B_\varphi$ and let $D^2 = D(\sigma^{n-2}, N^n)$ be the dual cell to σ^{n-2} (a subcomplex of the first barycentric subdivision of N^n). Then $\varphi^{-1}D(\sigma^{n-2}, N^n) = \bigcup D_i^2$, a disjoint union of 2-cells $D_i^2 = D(\tau_i^{n-2}, M^n)$ where $\varphi^{-1}(\sigma^{n-2}) = \bigcup \tau_i^{n-2}$. Replace $\varphi | D_i^2$ with a simple branched covering ψ_i such that $\psi_i | \partial D_i^2 = \varphi | \partial D_i^2$. We may assume that $B_{\psi_i} \cap B_{\psi_j} = \emptyset$, for $i \neq j$. Replace φ on the join $\partial\tau_i^{n-2} * D(\tau_i^{n-2}, M^n)$ by $\varphi | \partial\tau_i^{n-2} * \psi_i$, for each τ_i^{n-2}. Clearly $\varphi \ |\partial\tau_i^{n-2} * \psi_i$ is branch homotopic rel boundary to $\varphi | (\partial\tau_i^{n-2} * D(\tau_i^{n-2}, M^n))$. Doing this for each $\sigma^{n-2} < B_\varphi$ completes the proof.

REMARK 4.3. Using the techniques of [2] one can actually reduce the dimension of the nonsimple points of B_φ to $n-4$, but we shall not use this fact.

THEOREM 4.4. *Let $\varphi: M^n \to S^n$ be any branched covering with $n \geqq 3$ and $\deg \varphi \geqq 3$, let $f: \partial B^2 \times D^{n-1} \to M^n \times 1$ be a flat embedding, and let $W^{n+1} = M^n \times [0, 1] \bigcup_f B^2 \times D^{n-1}$. Then φ extends to a branched covering $\theta: W^{n+1} \to S^n \times [0, 1]$.*

Proof. Altering φ by a branch homotopy if necessary we may assume that the nonsimple part of B_φ has dimension less than $n-2$, by (4.2).

Let $C = f(\partial B^2 \times 0)$. By general position, we may assume that $\varphi | C$ is one-to-one. Let $K = \varphi(C)$.

We shall show that after an isotopy of C in M^n there is a 2-sphere $S^2 \subset S^n$ which meets B_φ transversely only in isolated points in the interior of $(n-2)$-simplices (over which φ is simple), such that $Q^2 = \varphi^{-1}(S^2)$ is a connected 2-manifold, and C lies on Q^2 as a non-separating simple closed curve.

Given this, the proof is completed as follows. By (4.1) and the isotopy extension theorem we may assume that $C \subset Q^2$ is invariant. We may now appeal to the degree 2 case in the following way. Let $A = \varphi(C)$ (an arc such that $A \cap B_\varphi = \partial A$). Let V a regular neighborhood of A in the second barycentric subdivision of N, let $\varphi^{-1}(A) = C \cup A_1 \cup \cdots \cup A_{d-2}$ and $\varphi^{-1}(V) = U \cup U_1 \cup \cdots \cup U_{d-2}$, where $\varphi | U : U \to V$ is a 2-fold branched covering and $\varphi | U_i : U_i \to V$ is a homeomorphism. By (3.1) we may equivariantly add a handle $B^2 \times D^{n-1}$ to $M^n \times I$ along $C \subset U \times 1$ using the given framing. We simply add copies of $B^2 \times D^{n-1}/T$ at each $U_i \times 1$, to extend to a d-fold branched covering.

It remains to construct the 2-sphere S^2 as needed. First consider the case $n = 3$.

Using the notion of a regular projection we may isotope the standard S^2 in S^3 until S^2 meets B_φ transversely in the interiors of (simple) 1-simplices and so that K lies on S^2 except for isolated standard overcrossings away from B_φ. See Figure 4.1.

We may assume that S^2 meets B_φ in enough different points so that the 2-manifold $Q^2 = \varphi^{-1}(S^2)$ is connected. Then C lies on Q^2 except for a finite number of standard small overcrossings which may be assumed to take place in one side of a bicollar neighborhood of Q^2. The local picture in M^3 is the same as that in S^3 (Fig. 4.1).

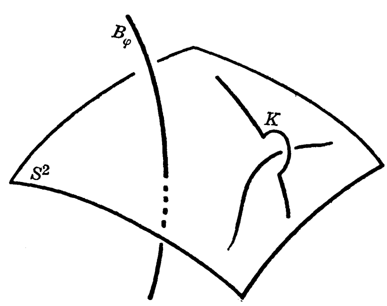

FIGURE 4.1

By perturbing S^2 in S^3 slightly as follows we may add some trivially embedded handles to Q^2 within a given regular neighborhood of Q^2. Push a small 2-disc in S^2 up until it meets B_φ transversely in two new simple branch points. See Fig. 4.2. This adds a small handle to Q^2. See Fig. 4.3.

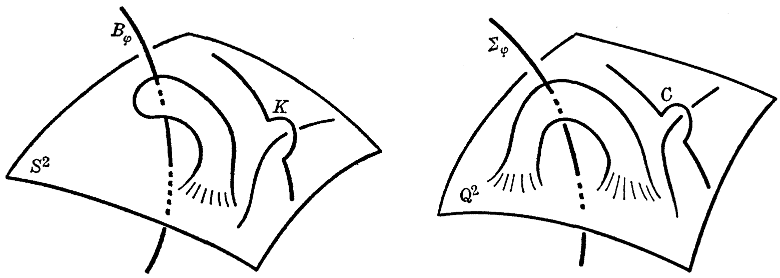

FIGURE 4.2 and 4.3

Do this once for each overcrossing. Then in M^3 we can isotope C onto the new surface Q^2, by making the overcrossings lie on the new handles. See Fig. 4.4. Finally $Q^2 - C$ might not be connected; but this can be rectified by adding another trivial handle to Q^2 and isotoping C in M^3 so that the new handle connects the two sides of C.

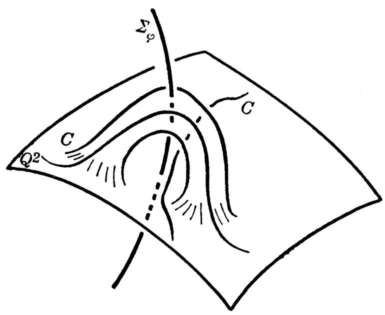

FIGURE 4.4

Now consider the case $n \geqq 4$.

Since $n \geqq 4$, $K = \varphi(C)$ is unknotted, and so we may isotope the standard S^2 in S^n until $K \subset S^2$ and S^2 meets B_φ transversely in enough simple branch points so that $Q^2 = \varphi^{-1}(S^2)$ is connected. Then $C \subset Q^2$. It may happen that $Q^2 - C$ is not connected. But as in the case $n = 3$, we may perturb S^2 slightly and move C so that this does not happen. This completes the proof.

REMARK 4.5. Clearly a similar result holds when $n = 2$ if $f(\partial B^2 \times 0)$ does not separate M^2.

REMARK 4.6. If $n \geqq 4$ one only needs the target manifold for φ to be simply connected.

REMARK 4.7. The overriding difficulty which arises when trying to extend a branched covering over a k-handle, $k > 2$, is that the attaching sphere often most intersect the branch set.

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Received March 13, 1978 and in revised form May 10, 1978. Supported in part by an NSF grant.

INDIANA UNIVERSITY
BLOOMINGTON, IN 47401