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PLURISUBHARMONIC DEFINING FUNCTIONS

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Let Ω be a bounded pseudoconvex open set in n-dimensional complex Euclidean space C^n with a smooth ($\mathscr{C}^\infty$)-boundary. It has been known for some time that it is not always possible to choose a defining function ρ which is plurisubharmonic in a neighborhood of $\bar{\Omega}$. We study here the question whether for every point $p \in \partial\Omega$, there exists an open neighborhood on which ρ can be chosen to be plurisubharmonic. Our main conclusion is that this is not always the case.

1. Notation and results. In what follows, Ω will always be a bounded open set in C^n with $\mathscr{C}^\infty$-boundary. This means that there exists a real-valued $\mathscr{C}^\infty$-function $\rho: C^n \to R$ such that $\Omega = \{\rho < 0\}$ and $d\rho \neq 0$ on $\partial\Omega$. Let $z = (z_1, z_2, \cdots, z_n)$, $z_j = x_j + iy_j$, denote complex coordinates in C^n, and define

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

DEFINITION 1. The set Ω is pseudoconvex if for every $p \in \partial\Omega$, we have

$$\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(p) t_i \bar{t}_j \geqq 0 \tag{1}$$

whenever

$$t = (t_1, \cdots, t_n) \in C^n - (0) \quad \text{and} \quad \sum_{i=1}^{n} \frac{\partial \rho}{\partial z_i}(p) t_i = 0 .$$

If we have strict inequality in (1) for all $p \in \partial\Omega$, then Ω is said to be strongly pseudoconvex.

DEFINITION 2. A real-valued $\mathscr{C}^2$-function, u, defined on an open set V in C^n is plurisubharmonic if

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(p) t_i \bar{t}_j \geqq 0$$

whenever $p \in V$ and $t = (t_1, \cdots, t_n) \in C^n - (0)$.

If we have strict inequality for all $p \in V$, then u is strictly plurisubharmonic.

The following results are known:

THEOREM 3 [2]. *If Ω is strongly pseudoconvex, then ρ may be chosen to be strictly plurisubharmonic in some neighborhood of $\bar{\Omega}$.*

The next example shows that the theorem fails in general if we drop the hypothesis of *strong* pseudoconvexity.

EXAMPLE 4 [1]. There exists a bounded pseudoconvex domain Ω in C^2, with $\mathscr{C}^\infty$-boundary, such that no ($\mathscr{C}^2$) defining function ρ exists with

$$\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(p) t_i \bar{t}_j \geqq 0$$

whenever

$$p \in \partial\Omega \quad \text{and} \quad t = (t_1, \cdots, t_n) \in C^n .$$

There exists an example, similar to the one above, which has a real analytic boundary.

EXAMPLE 5. Let

$$\begin{aligned} \Omega = \Omega_K = \{(z_1, z_2) \in (C - (0)) \times C; \sigma \\ = |z_2 + e^{i \ln z_1 \bar{z}_1}|^2 - 1 + K(\ln z_1 \bar{z}_1)^4 < 0\} . \end{aligned}$$

Then, if, $K > 1$ is sufficiently large, Ω is a bounded pseudoconvex domain in C^2 with smooth real analytic boundary, such that no $\mathscr{C}^2$ defining function, ρ, exists such that

$$\sum_{i,j=1}^{2} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(p) t_i \bar{t}_j \geqq 0$$

whenever $p \in \partial\Omega$ and $(t_1, t_2) \in C^2$.

The details will be given in the next section.

EXAMPLE 6. There exists a bounded pseudoconvex domain Ω in C^3, with $\mathscr{C}^\infty$-boundary, and a point $p \in \partial\Omega$ such that whenever ρ is a $\mathscr{C}^2$ defining function for Ω,

$$\sum_{i,j=1}^{3} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(q) t_i \bar{t}_j < 0$$

for some $(t_1, \cdots, t_n)$ and $q \in \partial\Omega$ arbitrarily close to p.

This example shows that one does not have plurisubharmonic defining functions for pseudoconvex domains, even locally, in general.

2. Examples.

EXAMPLE 5. Clearly, Ω is bounded in $(\boldsymbol{C} - (0)) \times \boldsymbol{C}$. If $\partial\sigma/\partial z_2 = 0$, then $z_2 = -e^{i\ln z_1\bar{z}_1}$. Hence, if $d\sigma = 0$, then $0 = z_1\partial\sigma/\partial z_1 = 4K(\ln z_1\bar{z}_1)^3$. This implies that $|z_1| = 1$ and $z_2 = -1$. At such points, $\sigma(z_1, z_2) = -1$, so $d\sigma \neq 0$ on $\partial\Omega$.

To show that Ω is pseudoconvex, we compute the Leviform

$$\mathscr{L} = \frac{\partial^2\sigma}{\partial z_1\partial\bar{z}_1}\left|\frac{\partial\sigma}{\partial z_2}\right|^2 - \frac{\partial^2\sigma}{\partial z_1\partial\bar{z}_2}\frac{\partial\sigma}{\partial z_2}\frac{\partial\sigma}{\partial\bar{z}_1} - \frac{\partial^2\sigma}{\partial\bar{z}_1\partial z_2}\cdot\frac{\partial\sigma}{\partial z_1}\cdot\frac{\partial\sigma}{\partial\bar{z}_2} + \frac{\partial^2\sigma}{\partial z_2\partial\bar{z}_2}\cdot\left|\frac{\partial\sigma}{\partial z_1}\right|^2$$

to obtain

$$\mathscr{L} = \frac{z_2\bar{z}_2 + K(\ln z_1\bar{z}_1)^4 + 12K(\ln z_1\bar{z}_1)^2}{z_1\bar{z}_1}\cdot|z_2 + e^{i\ln z_1\bar{z}_1}|^2 + 4K\frac{(\ln z_1\bar{z}_1)^3}{z_1\bar{z}_1}(i\bar{z}_2e^{i\ln z_1\bar{z}_1} - iz_2e^{-i\ln z_1\bar{z}_1}) + 16K^2\frac{(\ln z_1\bar{z}_1)^6}{z_1\bar{z}_1}$$

on $\partial\Omega$.

If $|z_2 + e^{i\ln z_1\bar{z}_1}| \geqq 1/2$, we have

$$\mathscr{L} \geqq 3K(\ln z_1\bar{z}_1)^2/z_1\bar{z}_1 - 16K|\ln z_1\bar{z}_1|^3/z_1\bar{z}_1\,,$$

since $|z_2| \leqq 2$ on $\partial\Omega$. If K is sufficiently large, then $|\ln z_1\bar{z}_1| < 3/16$ on $\partial\Omega$ and hence $\mathscr{L} \geqq 0$.

Consider next a boundary point where $|z_2 + e^{i\ln z_1\bar{z}_1}| < 1/2$. Then $K(\ln z_1\bar{z}_1)^4 \geqq 3/4$, since $\sigma(z_1, z_2) = 0$. Hence

$$\begin{aligned}\mathscr{L} &\geqq -16K|\ln z_1\bar{z}_1|^3/z_1\bar{z}_1 + 16K^2(\ln z_1\bar{z}_1)^6/z_1\bar{z}_1\\ &= 16K|\ln z_1\bar{z}_1|^3/z_1\bar{z}_1(-1 + K(\ln z_1\bar{z}_1)^4/|\ln z_1\bar{z}_1|)\end{aligned}$$

which is nonnegative if K is sufficiently large.

Assume next that ρ is a $\mathscr{C}^2$ defining function for Ω such that

$$\sum_{i,j=1}^{2}\frac{\partial^2\rho}{\partial z_i\partial\bar{z}_j}(p)t_i\bar{t}_j \geqq 0$$

whenever $p \in \partial\Omega$ and $(t_1, t_2) \in \boldsymbol{C}^2$. In particular, $\rho = h\sigma$ for some $\mathscr{C}^1$-function $h > 0$. We observe that $\partial^2\rho/\partial z_1\partial\bar{z}_1(z_1, z_2) = 0$ whenever $|z_1| = 1$ and $z_2 = 0$. (All such points are in $\partial\Omega$.) Therefore, $\partial^2\rho/\partial\bar{z}_1\partial z_2(z_1, z_2) = 0$ at these points also. Hence

$$\left(\frac{\partial h}{\partial \bar{z}_1}\frac{\partial \sigma}{\partial z_2} + h\frac{\partial^2 \sigma}{\partial \bar{z}_1 \partial z_2}\right)(e^{i\theta}, 0) \equiv 0$$

and so

$$\frac{\partial}{\partial \bar{z}_1}(he^{i\ln z_1 \bar{z}_1})(e^{i\theta}, 0) \equiv 0 .$$

Multiplying with $e^{i\mathrm{Log} z_1}$ we get that

$$\frac{\partial}{\partial \bar{z}_1}(he^{-2\mathrm{Arg} z_1})(e^{i\theta}, 0) \equiv 0$$

which implies that $h(e^{i\theta}, 0) = ce^{2\theta}$ for some constant $c > 0$. This is of course impossible.

In the next example, we localize the above idea suitably.

EXAMPLE 6. Let us use coordinates (w, z_1, z_2) in $\boldsymbol{C}^3$ with $w = \eta + i\zeta$ and $z_j = x_j + iy_j$, $j = 1, 2$. We pick a $\mathscr{C}^\infty$, convex function $\chi_1(t)\colon \boldsymbol{R} \to \boldsymbol{R}$ such that $\chi_1(t) = 0$ when $t \leqq 1$ and $\chi_1(t) > 0$ when $t > 0$. Define $\sigma_1\colon \boldsymbol{C}^3 \to \boldsymbol{R}$ by

$$\sigma_1 = \eta + \eta^2 + K\zeta^2 + K(y_1^2 + y_2^2)^2 + (y_1^2 + y_2^2)\zeta^2 + \chi_1(x_1^2 + x_2^2) ,$$

and let $\Omega_1 = \{\sigma_1 < 0\}$. Here $K \gg 1$ is a constant which will be chosen later.

LEMMA 7. *The set Ω_1 is bounded and pseudoconvex with $\mathscr{C}^\infty$-boundary for all K sufficiently large.*

Proof. Computation shows that $d\sigma_1 = 0$ only at points $(-1/2, x_1, x_2)$ with $x_1^2 + x_2^2 \leqq 1$. Since $\sigma_1 = -1/4$ at these points, it follows that $d\sigma_1 \neq 0$ on $\partial\Omega_1$. Further computation shows that σ_1 is plurisubharmonic in a neighborhood of $\bar{\Omega}_1$ if K is sufficiently large.

In the following K, sufficiently large, is fixed.

The next step is to make an infinite number of perturbations of the boundary of Ω_1. Let $p_j = (0, 1/2^j, 0)$, $j = 1, 2, \cdots$ and let $\boldsymbol{B}(p_j, r) = \{(w, z_1, z_2); (|w|^2 + |z_1 - 1/2^j|^2 + |z_2|^2)^{1/2} < r\}$ be the ball centered at p_j of radius r. Choose functions $\chi^{(j)} \in \mathscr{C}_0^\infty(\boldsymbol{B}(p_j, 1/2^{j+2}))$ with $\chi^{(j)} \equiv 1$ on $\boldsymbol{B}(p_j, 1/2^{j+3})$ and $\chi^{(j)} \geqq 0$, $j = 1, 2, \cdots$. Observe that $\operatorname{supp}\chi^{(i)} \cap \operatorname{supp}\chi^{(j)} = \varnothing$ whenever $i \neq j$. We may arrange that $|d\chi^{(j)}|^2 \leqq C_j\chi^{(j)}$ and $|\partial\chi^{(j)}/\partial y_k| \leqq C_j|y_k|$ for suitable $C_1, C_2, \cdots$, and $k = 1, 2$. Let $\varepsilon = \{\varepsilon_j\}_{j=1}^\infty$ denote a rapidly decreasing sequence, $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ and define

$$\sigma_2 = \sigma_1 + \sum_{j=1}^\infty \varepsilon_j\chi^{(j)}\cdot(y_1^2 + y_2^2)\cdot x_2^2 .$$

Clearly σ_2 is a $\mathscr{C}^\infty$-function, and if $\Omega_2 = \{\sigma_2 < 0\}$, then $d\sigma_2 \neq 0$ on $\partial\Omega_2$ and Ω_2 is a bounded domain which is pseudoconvex at every point in $\partial\Omega_2 - \bigcup_j B(p_j, 1/2^{j+2})$.

LEMMA 8. *The set Ω_2 is pseudoconvex if ε decreases sufficiently fast.*

Proof. Fix a $j \geqq 1$. It suffices to show that $\sigma_1 + \varepsilon_j \chi^{(j)} \cdot (y_1^2 + y_2^2)x_2^2$ is plurisubharmonic in $B(p_j, 1/2^{j+2})$ for all small enough $\varepsilon_j > 0$. This is checked by a direct computation.

We fix a sequence $\{\varepsilon_j\}$ decreasing sufficiently fast.

To complete the construction of the example, we will perturbe σ_2 inside each $B(p_j, 1/2^{j+3})$. More precisely, let $\chi_{(j)} \in \mathscr{C}_0^\infty(B(p_j, 1/2^{j+3}))$ with

$$\int_R \left(\frac{\partial \chi_{(j)}}{\partial x_1} + \chi_{(j)}\right)(0, x_1, 0)dx_1 \neq 0$$

for each j, $\chi_{(j)} \geqq 0$. We may assume that $|\partial\chi_{(j)}/\partial\eta|$, $|\partial\chi_{(j)}/\partial\zeta|$, $|\partial\chi_{(j)}/\partial y_k|$, $|\partial\chi_{(j)}/\partial x_2| \leqq C_j(|\eta| + |\zeta| + |x_2| + |y_1| + |y_2|)$, $k = 1, 2$, C_j some constant.

If $\delta = \{\delta_j\}_{j=j_0}^\infty$, $\delta_{j_0} > \delta_{j_0+1} > \cdots > 0$ is any sufficiently rapidly decreasing sequence,

$$\sigma = \sigma_2 + \sum_{j=j_0}^\infty \delta_j \chi_{(j)} \cdot (\eta + \zeta y_1)$$

is a $\mathscr{C}^\infty$-function and $d\sigma \neq 0$ on $\partial\Omega$, $\Omega = \{\sigma < 0\}$. Moreover, Ω is a bounded domain which is pseudoconvex on $\partial\Omega - \cup B(p_j, 1/2^{j+3})$.

LEMMA 9. *The set Ω is pseudoconvex if δ decreases sufficiently fast, and j_0 is sufficiently large.*

Proof. Fix a $j \gg 1$. It suffices to show that Ω is pseudoconvex at those boundary points which are in $B(p_j, 1/2^{j+3})$ for all δ_j sufficiently small. In $B(p_j, 1/2^{j+3})$, $\sigma = \eta + \eta^2 + K\zeta^2 + K(y_1^2 + y_2^2)^2 + (y_1^2 + y_2^2)\zeta^2 + \varepsilon_j(y_1^2 + y_2^2) \cdot x_2^2 + \delta_j \chi_{(j)} \cdot (\eta + \zeta y_1)$. Differentiating, we obtain:

$$\frac{\partial \sigma}{\partial w} = \frac{1}{2} + \eta - iK\zeta - i\zeta(y_1^2 + y_2^2) + \delta_j \frac{\delta \chi_{(j)}}{\delta w} \cdot (\eta + \zeta y_1) + \frac{1}{2}\delta_j \chi_{(j)} - \frac{i}{2}\delta_j \chi_{(j)} y_1 ,$$

$$\frac{\partial \sigma}{\partial z_1} = -2iK(y_1^3 + y_1 y_2^2) - iy_1\zeta^2 - i\varepsilon_j y_1 x_2^2 + \delta_j \frac{\partial \chi_{(j)}}{\partial z_1} \cdot (\eta + \zeta y_1) - \frac{i}{2}\delta_j \chi_{(j)} \cdot \zeta ,$$

$$\frac{\partial\sigma}{\partial z_2} = -2iK(y_1^2y_2 + y_2^3) - iy_2\zeta^2 - i\varepsilon_j y_2 x_2^2 + \varepsilon_j(y_1^2 + y_2^2)x_2$$
$$+ \delta_j \frac{\partial \chi_{(j)}}{\partial z_2}\cdot(\eta + \zeta y_1) ,$$
$$\frac{\partial^2\sigma}{\partial w\partial\bar{w}} = \frac{1}{2} + \frac{K}{2} + \frac{1}{2}(y_1^2 + y_2^2) + \delta_j\frac{\partial^2\chi_{(j)}}{\partial w\partial\bar{w}}\cdot(\eta + \zeta y_1)$$
$$+ \frac{1}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial w} + \frac{i}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial w}\cdot y_1 + \frac{1}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial\bar{w}}$$
$$- \frac{i}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial\bar{w}}\cdot y_1 ,$$
$$\frac{\partial^2\sigma}{\partial w\partial\bar{z}_1} = \zeta y_1 + \delta_j\frac{\partial^2\chi_{(j)}}{\partial w\partial\bar{z}_1}\cdot(\eta + \zeta y_1) + \frac{i}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial w}\cdot\zeta$$
$$+ \frac{1}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial\bar{z}_1} - \frac{i}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial\bar{z}_1}y_1 + \frac{1}{4}\delta_j\chi_{(j)} ,$$
$$\frac{\partial^2\sigma}{\partial w\partial\bar{z}_2} = \zeta y_2 + \delta_j\frac{\partial^2\chi_{(j)}}{\partial w\partial\bar{z}_2}\cdot(\eta + \zeta y_1) + \frac{1}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial\bar{z}_2} - \frac{i}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial\bar{z}_2}\cdot y_1 ,$$
$$\frac{\partial^2\sigma}{\partial z_1\partial\bar{z}_1} = 3Ky_1^2 + Ky_2^2 + \frac{1}{2}\zeta^2 + \frac{1}{2}\varepsilon_j x_2^2 + \delta_j\frac{\partial^2\chi_{(j)}}{\partial z_1\partial\bar{z}_1}\cdot(\eta + \zeta y_1)$$
$$+ \frac{i}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial z_1}\cdot\zeta - \frac{i}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial\bar{z}_1}\cdot\zeta ,$$
$$\frac{\partial^2\sigma}{\partial z_1\partial\bar{z}_2} = 2Ky_1y_2 - i\varepsilon_j y_1x_2 + \delta_j\frac{\partial^2\chi_{(j)}}{\partial z_1\partial\bar{z}_2}\cdot(\eta + \zeta y_1) - \frac{i}{2}\delta_j\frac{\partial\chi_{(j)}}{\partial\bar{z}_2}\cdot\zeta$$

and

$$\frac{\partial^2\sigma}{\partial z_2\partial\bar{z}_2} = Ky_1^2 + 3Ky_2^2 + \frac{1}{2}\zeta^2 + \frac{\varepsilon_j}{2}x_2^2 - i\varepsilon_j x_2y_2 + i\varepsilon_j y_2x_2$$
$$+ \frac{1}{2}\varepsilon_j(y_1^2 + y_2^2) + \delta_j\frac{\partial^2\chi_{(j)}}{\partial z_2\partial\bar{z}_2}\cdot(\eta + \zeta y_1) .$$

Observe that $\eta = 0(\zeta^2 + y_1^2 + y_2^2)$ on $\partial\Omega \cap \boldsymbol{B}(p_j, 1/2^{j+3})$. Hence there is a $D_j \gg 1$ such that for all sufficiently small $\delta_j > 0$, $\partial^2\sigma/\partial w\partial\bar{w} \geqq K/2$,

$$\left|\frac{\partial^2\sigma}{\partial w\partial\bar{z}_1} - \zeta y_1 - \frac{1}{4}\delta_j\frac{\partial\chi_{(j)}}{\partial x_1} - \frac{1}{4}\delta_j\chi_{(j)}\right| \leqq D_j\delta_j\|(w, iy_1, z_2)\| ,$$
$$\left|\frac{\partial^2\sigma}{\partial w\partial\bar{z}_2} - \zeta y_2\right| \leqq D_j\delta_j\|(w, iy_1, z_2)\| ,$$
$$\frac{\partial^2\sigma}{\partial z_1\partial\bar{z}_1} \geqq (3K - 1)y_1^2 + (K - 1)y_2^2 + \frac{1}{4}\zeta^2 + \frac{1}{4}\varepsilon_j x_2^2 ,$$
$$\left|\frac{\partial^2\sigma}{\partial z_1\partial\bar{z}_2} - 2Ky_1y_2 + i\varepsilon_j y_1x_2\right| \leqq D_j\delta_j\|(w, iy_1, z_2)\|^2$$

and

$$\frac{\partial^2\sigma}{\partial z_2\partial\bar{z}_2} \geqq Ky_1^2 + 3Ky_2^2 + \frac{1}{4}\zeta^2 + \frac{\varepsilon_j}{4}x_2^2 .$$

We compute the Leviform,

$$\begin{aligned}\mathscr{L}_\sigma = \sigma_{w\bar{w}}t_0\bar{t}_0 + 2\,\mathrm{Re}\,\sigma_{w\bar{z}_1}t_0\bar{t}_1 + 2\,\mathrm{Re}\,\sigma_{w\bar{z}_2}t_0\bar{t}_2 \\ + \sigma_{z_1\bar{z}_1}t_1\bar{t}_1 + 2\,\mathrm{Re}\,\sigma_{z_1\bar{z}_2}t_1\bar{t}_2 + \sigma_{z_2\bar{z}_2}t_2\bar{t}_2\end{aligned}$$

for vectors (t_0, t_1, t_2) such that

$$t_0 = (-1/\sigma_w)\cdot(\sigma_{z_1}t_1 + \sigma_{z_2}t_2) .$$

Using the above estimates, we obtain

$$\begin{aligned}\mathscr{L}_\sigma \geqq &\Big((3K-2)y_1^2 + (K-2)y_2^2 + \frac{1}{8}\zeta^2 + \frac{1}{8}\varepsilon_j x_2^2\Big)t_1\bar{t}_1 \\ &+ \Big((K-2)y_1^2 + (3K-2)y_2^2 + \frac{1}{8}\zeta^2 + \frac{\varepsilon_j}{8}x_2^2\Big)t_2\bar{t}_2 \\ &+ 2\,\mathrm{Re}\,(2Ky_1y_2 - i\varepsilon_j y_1 x_2)t_1\bar{t}_2 \\ &+ 2\,\mathrm{Re}\Big(\frac{1}{4}\delta_j\frac{\partial\chi_{(j)}}{\partial x_1} + \frac{1}{4}\delta_j\chi_{(j)}\Big)\cdot\Bigg[\Bigg(\frac{-1}{\frac{1}{2}+\frac{1}{2}\delta_j\chi_{(j)}}\Bigg)\cdot\frac{-i}{2} \\ &\qquad\times\delta_j\chi_{(j)}\zeta t_1\Bigg]\bar{t}_1\end{aligned}$$

which clearly is nonnegative.

Assume that there exists a $\mathscr{C}^2$-function $\rho\colon \boldsymbol{C}^3 \to \boldsymbol{R}$, such that $\Omega = \{\rho < 0\}$ and $d\rho \neq 0$ on $\partial\Omega$, with a nonnegative complex Hessian on some neighborhood U of 0 in $\partial\Omega$.

Let γ_i, $i = 1, 2, 3, 4$, be straight lines in the (x_1, x_2)-plane,

γ_1 goes from $\left(\frac{1}{2^j} - \frac{1}{2^{j+2}}, 0\right)$ to $\left(\frac{1}{2^j} + \frac{1}{2^{j+2}}, 0\right)$,

γ_2 goes from $\left(\frac{1}{2^j} + \frac{1}{2^{j+2}}, 0\right)$ to $\left(\frac{1}{2^j} + \frac{1}{2^{j+2}}, \frac{1}{2^{j+2}}\right)$,

γ_3 goes from $\left(\frac{1}{2^j} + \frac{1}{2^{j+2}}, \frac{1}{2^{j+2}}\right)$ to $\left(\frac{1}{2^j} - \frac{1}{2^{j+2}}, \frac{1}{2^{j+2}}\right)$ and

γ_4 goes from $\left(\frac{1}{2^j} - \frac{1}{2^{j+2}}, \frac{1}{2^{j+2}}\right)$ to $\left(\frac{1}{2^j} - \frac{1}{2^{j+2}}, 0\right)$.

We fix j so large that each $\gamma_i \subset U$. The function $\rho = \sigma h$ for some $\mathscr{C}^1$-function $h > 0$.

We will show that $\int_{\gamma_1} d(\ln h) \neq 0$ for all small enough $\delta_j > 0$, while

$$\int_{\gamma_i} d(\ln h) = 0,\ i = 2, 3, 4\ .$$

First consider the curves γ_2 and γ_4. There $\rho = (\eta + \eta^2 + K\zeta^2 + K(y_1^2 + y_2^2)^2 + (y_1^2 + y_2^2)\zeta^2)h$ from which it follows that $\partial^2\rho/\partial z_2\partial\bar{z}_2 \equiv 0$ on $\gamma_2 \cup \gamma_4$. Hence $\partial^2\rho/\partial w\partial\bar{z}_2 \equiv 0$ on $\gamma_2 \cup \gamma_4$ as well. This reduces to the equation $\partial h/\partial\bar{z}_2 = 0$ from which it follows that $\int_{\gamma_i} d(\ln h) = 0$, $i = 2, 4$. Similarly $\int_{\gamma_3} d(\ln h) = 0$.

Finally, consider the curve γ_1. Here $\sigma = \eta + \eta^2 + K\zeta^2 + K(y_1^2 + y_2^2)^2 + (y_1^2 + y_2^2)\zeta^2 + \varepsilon_j \chi^{(j)}\cdot(y_1^2 + y_2^2)\cdot x_2^2 + \delta_j \chi_{(j)}\cdot(\eta + \zeta y_1)$. Clearly $\partial^2\rho/\partial z_1\partial\bar{z}_1 \equiv 0$ on γ_1 and hence $\partial^2\rho/\partial w\partial\bar{z}_1 \equiv 0$ there also. This reduces to the equation

$$\partial^2\sigma/\partial w\partial\bar{z}_1\cdot h + \partial\sigma/\partial w\cdot\partial h/\partial\bar{z}_1 \equiv 0 \quad \text{on} \quad \gamma_1\ .$$

Hence

$$\frac{\partial}{\partial x_1}(\ln h) = (-\delta_j)(\partial\chi_{(j)}/\partial x_1 + \chi_{(j)})/(1 + \delta_j\chi_{(j)})\ .$$

Since we choose $\chi_{(j)}$ such that

$$\int_R \left(\frac{\partial\chi_{(j)}}{\partial x_1} + \chi_{(j)}\right)(0, x_1, 0)dx_1 \neq 0\ ,$$

it follows that $\int_{\gamma_1} d(\ln h) \neq 0$ for all small enough $\delta_j > 0$.

So $\int_{\gamma_1+\cdots+\gamma_4} d(\ln h) \neq 0$, which contradicts the assumption that h was well defined.

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