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SIMPLICIAL SUBDIVISION OF INFINITE DIMENSIONAL COMPACT CUBES

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Finite dimensional polyhedra may be characterized as those finite dimensional convex sets S admitting a finite a simplicial subdivision with vertex set equal to the extreme point set of S. Alfsen has shown the existence of α-polytope which doesn't admit even an infinite simplicial subdivision of this type. It is shown here that any infinite dimensional compact cube does admit subdivisions of this type.

1. **Introduction.** In [5] cubes (i.e., affine isomorphs of unit balls of $\mathscr{C}(X)$ spaces with X compact and Hausdorff) were examined. In particular the compact cubes (i.e., unit balls of dual $\mathscr{C}(X)$ spaces and their affine isomorphs) were examined. The unit ball of $\mathscr{C}(X)$ is a compact cube iff $\mathscr{C}(X)$ is Banach lattice isomorphic to $L^\infty(S, \Sigma, \mu)$ for some positive localizable measure space (S, Σ, μ). The predual of such a $\mathscr{C}(X)$ space is a unique subspace of $\mathscr{M}(X)$ which consists of the signed normal measures $\mathscr{N}(X)$ on X, [7], [23], [5]. $\mathscr{N}(X)$ is Banach lattice isomorphic to $L^1(S, \Sigma, \mu)$, [7], [23], [5]. X is said to be hyperstonian, iff it is extremally disconnected and possesses a collection $\{\mu_\alpha\}$ of normal probability measures such that $X_\alpha = \mathrm{supp}\,(\mu_\alpha)$ forms a disjoint collection of clopen subsets of X with dense union. The measure space (S, Σ, μ) can be taken to be $S = \bigcup_\alpha X$ with $\mu = \mu_\alpha$ on each X_α. $L^1(S, \Sigma, \mu)$ or $\mathscr{N}(X)$ is Banach lattice isomorphic with the: l^1-direct sum $[\sum_\alpha L^1(\mu_\alpha)]_1$, [5]. There is a unique affine toplogy on any compact cube rendering it compact. This may be considered as the weak* topology $\sigma(\mathscr{C}(X), \mathscr{N}(X))$ or $\sigma(L^\infty, L^1)$.

Examples of compact cubes include the Hilbert cube $\{(\chi_n) \in l^2 : |\chi_n| \leqq n^{-1}$ when $n \in N\}$, which is norm compact in the Hilbert space l^2, and its affine homeomorph the unit ball of l^∞ with topology $\sigma(l^\infty, l^1)$. The Tychonoff cube over a set T is a more general example and is defined as the unit ball of $l^\infty(T)$ with the topology $\sigma(l^\infty(T), l^1(T))$ and is usually thought of as $\prod_{t \in T}[-1, 1]$. Another, more esoteric, example is the unit ball of $\mathscr{H}^\infty(\theta)$ of all bounded harmonic functions on an open subset θ of a harmonic space of Constantinescu and Cornea on which constants are superharmonic, [4]. This compact cube has as its compact affine topology the topology of locally uniform convergence on θ. $\mathscr{H}^\infty(\theta)$ is isomorphic to $\mathscr{C}(X)$ for a hyperstonian compact Hausdorff space X which is the harmonic part of the Feller boundary of θ. $\mathscr{N}(X)$ is generated by the harmonic measures on X, [4], [24].

In [5] the infinite dimensional notions of polyhedrality which have been put forth by various authors were examined and it was determined that for infinite dimensional cubes only the Alfsen-Nordseth definition of polyhedrality was true. One notion of polyhedrality for finite dimensional convex sets is the existence of a finite simplicial subdivision. This forms the basis of much of piecewise linear topology in finite dimensions. It would be nice if it were possible to carry out the program of piecewise linear topology in infinite dimensions. One very important tool in infinite dimensional topology is the Hilbert cube manifold. We shall exhibit subdivisions of compact cubes by Bauer simplexes. From this it follows that Hilbert cube manifolds admit simplicial subdivision. We will also see that L^∞ spaces admit tesselation by Bauer simplexes. Hence, it would appear that L^∞ manifolds have Bauer simplicial subdivisions.

One reason that the possibility of simplicial subdivision of infinite dimensional convex sets hasn't been considered is the following. In [2] Alfsen asked whether there was a characterization of those convex compact sets K such that Caratheodory's theorem was valid in that given any $k \in K$ there was a Choquet simplex S_k containing k with the extreme points, $\xi(S_k)$ of S_k in those of K. Of course in finite dimensions this is equivalent to Caratheodory's theorem and is valid for any compact convex K hence for all compact polyhedra. This is used by piecewise linear topologists to show that any compact polyhedron admits a simplicial subdivision (which consists of simplexes of the same dimension as the polyhedron) whose simplexes have extreme points contained in the extreme points of the original polyhedron. Alfsen, [2], gives an example of an α-polytope K, [18], and $k \in K$ for which there is no simplex S_k of the type prescribed by Caratheodory's theorem. We shall see that, in addition to Choquet simplexes, compact cubes satisfy Caratheodory's theorem. We conjecture that any Alfsen-Nordseth polyhedron satisfies Caratheodory's theorem and in fact admits Caratheodory simplicial subdivisions.

2. Convex complexes and simplicial subdivisions.

Definition 2.1.

1. A convex *precomplex* is a collection $\{C_\alpha\}$ of closed convex subsets of a locally convex Hausdorff space such that $C_\alpha \cap C_\beta$ is a face both of C_α and C_β for all α, β.

2. A convex precomplex is *minimal* iff when $C_\alpha \neq C_\beta$ then C_α isn't a face of C_β.

3. A *convex complex* is a precomplex which contains along with each C_α all of its closed faces.

Remarks. 1. Nontopological definitions are obtained if the ambient vector space is given the finest locally convex topology. A precomplex then consists of linearly closed sets. Any face of a linearly closed convex set is linearly closed hence (3) of Definition 2-1 is the requirement that if C belongs to a complex so do all of its faces.

2. If A is a convex precomplex, adding all closed faces of its members to A yields the smallest convex complex containing the precomplex. If every element of a complex is in an element of the complex which is maximal with respect to inclusion the collection of maximal elements is a minimal precomplex which generates the complex.

3. By restricting the elements of precomplex to be in a certain class of convex sets one obtains the notions of Choquet simplicial complex, Bauer simplicial complex, cubical complex, compact cubical complex, convex compact complex etc. One must make sure that the class of convex sets one is using is closed upon taking closed faces. For instance if one considers cubes with their norm topology (as unit balls of $\mathscr{C}(X)$ spaces) not all norm closed faces are cubes [5] but compact cubes with their compact affine topology only have compact cubes as closed faces.

4. Any Hilbert cube manifold $\mathscr{H}$ is homeomorphic to $C_1 \times C_2$ where C_1 is the union of a finite dimensional simplicial complex and C_2 is a Hilbert cube [6]. Modulo orientability, there is a convex complex $\{S_\alpha \times C_2\}$ with $\bigcup_\alpha \{S_\alpha \times C_2\} = C_1 \times C_2$ with each S_α a finite dimensional simplex.

One may define the *skeleton* of a closed convex set to be the complex of all closed faces. For any cardinal number n one may define n-complex as all closed faces of topological dimension at most n and the n^--complex as all elements of the n-complex not of dimension n. The $\aleph_0^-$ complex is the finite dimensional skeleton. Fon any cardinal n the n-skeleton consists of the faces of dimension n. The 0-skeleton is the collection of extreme points. The 1-skeleton consisting of all edges has been studied in [14] for compact convex sets in Banach spaces.

DEFINITION 2.2. (a) A Choquet simplicial subdivision of a closed convex set C is a minimal Choquet simplicial precomplex whose union is C.

(b) A Caratheodory simplicial subdivision, $\{S_\alpha\}$, of a closed convex set C is a Choquet simplicial subdivision with $\xi(S_\alpha) \subset \xi(C)$ for all α.

REMARK. It is very easy to give Choquet simplicial subdivisions of closed convex sets but even α-polytopes needn't have Caratheodory simplicial subdivision.

3. Construction of subdivision of cubes. The construction of a Caratheodory Bauer simplicial subdivision of a compact cube that we are about to give is a straight forward generalization of a well known procedure for subdividing finite dimensional cubes as found in [16] and algorithmatized in [12].

Since a simplicial subdivision of one cube readily gives corresponding subdivisions of all affine homeomorphs we shall only consider subdivisions of the positive unit ball of $L^\infty(S, \Sigma, \mu)$ of a positive localizable measure space (S, Σ, μ) which is denoted by $\square^+$. The cube $\square^+$ is equipped with the topology $\sigma(L^\infty, L^1)$ under which all monotone nets are convergent. It is well known that the bounded linear functions on $\square^+$ which arise as restrictions to $\square^+$ of elements of $L^1(S, \Sigma, \mu)$ are just the $\sigma(L^\infty, L^1)$ continuous ones or the order continuous ones. By abuse of notation we may consider $\xi(\square^+)$ as $\{\chi_A: A \in \Sigma\}$ whereas it actually consists of equivalence class modulo locally μ-negligible sets. When ordered by $L^{\infty+}(X, \Sigma, \mu)$, $\xi(\square^+)$ is a Boolean algebra isomorphic with the hyperstonian measure algebra of μ hence is a complete Boolean algebra.

DEFINITION 3.1. (a) Chain $(\square^+)$ denotes all chains (i.e., linearly ordered subsets) of $\xi(\square^+)$.

(b) C-Chain $(\square^+)$ denotes all order complete chains in $\xi(\square^+)$.

(c) M-Chain $(\square^+)$ denotes all chains of $\square^+$ maximal with respect to inclusion.

REMARKS. 1. If $C \in \mathrm{Chain}(\square^+)$ then $\bar{C}$, the $\sigma(L^\infty, L^1)$ closure of C, is the smallest element of C-Chain$(\square^+)$ containing it. Thus all element of C-Chain$(\square^+)$ are $\sigma(L^\infty, L^1)$ compact.

2. M-Chain$(\square^+) \subset C$-Chain$(\square^+)$.

3. C-Chain$(\square^+)$ is closed under arbitrary intersections.

4. The closure of M-Chain$(\square^+)$ under arbitrary intersections is all $C \in C$-Chain$(\square^+)$ with $\{0, 1\} \subset C$. The proof of this is analogous to the proof that every proper filter is the intersection of all ultrafilters containing it.

5. Both Chain$(\square^+)$ and C-Chain$(\square^+)$ are increasing families with respect to inclusion (i.e., they are filtering to the left).

6. On any chain the order topology and $\sigma(L^\infty, L^1)$ coincide.

DEFINITION 3.2. For $f \in \square^+$, $C_f \in C$-Chain(f) denotes the closure of the chain $\{\chi_{\{f \geqq \lambda\}}: 0 < \lambda \leqq \|f\|_\infty\}$.

REMARKS. 1. C_f contains $\chi_{\{f>\lambda\}} = \sup\{\chi_{\{f \geqq \lambda+1/n\}}: n \in N, \lambda + 1/n < \|f\|_\infty\}$ for any $0 \leqq \lambda < \|f\|_\infty$. C_f is the closure of $\{\chi_{\{f>\lambda\}} 0 \leqq \lambda < \|f\|_\infty\}$.

2. $0 \in C_f$ iff $\{f = \|f\|_\infty\} = \varnothing$ and $1 \in C_f$ iff $\{f > 0\} = X$.

DEFINITION 3.3. For any $C \in \mathrm{Chain}(\square^+)$, S_C denotes the $\sigma(L^\infty, L^1)$ closed convex hull of C.

REMARKS. 1. $S_C = S_{\bar{C}}$ and $\bar{C} = \xi(S_C)$ hence, we need only consider S_C with C complete.

2. If $C_1 \subset C_2$ are in C-Chain$(\square^+)$ then $\xi(S_{C_1}) \subset \xi(S_{C_2})$ so $S_{C_1} \subset S_{C_2}$.

PROPOSITION 3.1. *Let $C \in C$-Chain $(\square^+)$ have infimum χ_A and supremum χ_B in $\square^+$. Let S_C^n be the $\|\ \|_\infty$-closed convex hull of C and let S denote the class of $f \in \square^+$ with $\chi_A \leqq f \leqq \chi_B$ and with $C_f \subset C$. It is the case that $S = S_C^n = S_C$.*

Proof. Note that any $f \in S_C$ must satisfy $\chi_A \leqq f \leqq \chi_B$. Next note that if f is in the convex hull of C so that $f = \sum_{i=1}^n \lambda_i \chi_{A_i}$ where $\lambda_i > 0$ for all $\sum_{i=1}^n \lambda_i = 1$ and $\{\chi_{A_i}: i = 1, \cdots, n\} \subset C$ then $C_f \supset \{\chi_{A_i}: i = 1, \cdots, n\}$. Thus, S contains the convex hull of C.

If $f \in S$ set $f_n = 1/n \sum_{k=1}^n \chi_{\{f \geqq k/n\}}$ for $n \in N$. It is easily verified, even when $\|f\|_\infty < 1$, that f_n is in the convex hull of C. Since $\{f_n: n \in N\}$ converges uniformly to f we have $f \in S_C^n$ thus, $S \subset S_C^n$.

If $\{f_1, f_2\} \subset S$ then $\{f_1 \wedge f_2, f_1 \vee f_2\} \subset S$. That $\chi_A \leqq f_1 \wedge f_2 \leqq f_1 \vee f_2 \leqq \chi_B$ is immediate. If $\lambda > 0$ then $\{f_1 \vee f_2 \geqq \lambda\} = \{f_1 \geqq \lambda\} \cup \{f_2 \geqq \lambda\}$ and $\{f_1 \wedge f_2 \geqq \lambda\} = \{f_1 \geqq \lambda\} \cap \{f_2 \geqq \lambda\}$. From these observations the inclusions $C_{f_1 \wedge f_2} \subset C$ and $C_{f_1 \vee f} \subset C$ are apparent.

If $\{f_\alpha\} \subset S$ is any order convergent net in S with limit $f \in \square^+$ then $f \in S$. This need only be verified for increasing and decreasing nets. For instance, if $\{f_\alpha\}$ is increasing then $\chi_A \leqq f \leqq \chi_B$ and for any $0 < \lambda \leqq \|f\|_\infty$, $\chi_{\{f_\alpha \leqq \lambda\}} = \inf_{\varepsilon > 0} \sup_\alpha \chi_{\{f_\alpha \geqq \lambda - \varepsilon\}} \in C$.

Since any uniformly convergent net in $\square^+$ is order convergent, S is uniformly closed. Since the convex hull of C lies in S and $S \subset S_C^n$ we have $S = S_C^n$.

If (X, Σ, μ) were a finite measure space we could make use of a result of Grothendiek [9, 8.3.6] which asserts that for any element f of the $\sigma(L^\infty, L^1)$ closure of S_C^n there is a sequence in S_C^n which converges in $L^1(S, \Sigma, \mu)$ to f. From this sequence we could extract a subsequence convergent μ a.e. to f. Any such subsequence is easily seen to be order convergent. Hence, f would be an order limit of $S_C^n = S$ so f would be in S. Thus, when (X, Σ, μ) is a finite measure space $S = S_C$.

For any $E \in \Sigma$ with $\mu(E) < \infty$ let C_E be the complete chain in the extreme points of the unit ball of $L^\infty(E, \Sigma, \mu|_E)$ consisting of functions of the form $f|_E$ with $f \in C$. If $h \in S_C$ the remarks of the preceding paragraph show that $\chi_{A \cap E} \leqq h|_E \leqq \chi_{B \cap E}$ and that if $0 < \lambda \leqq \|h|_E\|_\infty$ then $\chi_{\{h|_E \geqq \lambda\}} \in C_E$. Note that

$$\{h|_E \geqq \lambda\} \cap F = \{h|_F \geqq \lambda\} \quad \text{when} \quad F \subset E$$

and that $\{h \geqq \lambda\} = \sup\{\{h|_E \geqq \lambda\}: \mu(E) < \infty\}$. If $\{A_E: \mu(E) < \infty\}$ is a collection with $\chi_{A_E} \in C_E$ for all E with $\mu(E) < \infty$ and with $A_E \cap F = A_F$ whenever $F \subset E$ it may be verified that there is one and only one A with $\chi_A \in C$ with $A \cap E = A_E$ for all $E \in \Sigma$ with $\mu(E) < \infty$. By applying this to the case where $A_E = \{h|_E \geqq \lambda\}$ for $E \in \Sigma$ with $\mu(E) < \infty$ and with $\|h\|_\infty = \|h|_E\|_\infty$ we may deduce that $\chi_{\{h \geqq \lambda\}} \in C$ if $0 < \lambda \leqq \|h\|_\infty$. This suffices to show that $S = S_C$ hence completes the proof.

REMARK. The fact that $\square^+$ was a compact cube only used to show that $S = S_C$. Of course the weak* closed convex hull S_C of C only makes sense when $\square^+$ is a compact cube. If $\square^+$ is the positive unit ball of $\mathscr{C}(X)$ with X a Stonian compact Hausdorff space S_C could be defined as the closed conxex hull of C for the order topology. Although the order topology on $\square^+$ is an affine topology iff X is hyperstonian it is a compact T_1 topology. Each S_C is compact and T_1 in the order topology. If $\square^+$ is the positive unit ball of $\mathscr{C}(X)$ with X only a basically disconnected compact Hausdorff space, [10], a careful perusal of the proof of Proposition 3.1 shows that $S = S_C^n$. The validity of this identity when X is only totally disconnected is not known.

DEFINITION 3.4.

$$\Sigma(\square^+) = \{S_C: C \in M\text{-Chain}(\square^+)\} .$$

Proposition 3.1 shows that $\square^+ = \bigcup \Sigma(\square^+)$ and that $\xi(S_C) \subset \xi(\square^+)$ for all $C \in M$-Chain($\square^+$). To show that $\Sigma(\square^+)$ is a Caratheodory Bauer simplicial subdivision of $\square^+$ we need to establish two facts. First, we need to show that S_C is a Bauer simplex for each $C \in M$-Chain($\square^+$). This may be done just by showing that each S_C is affinely isomorphic to some simplex since then S_C is itself a simplex which is a Choquet simplex under the topology $\sigma(L^\infty, L^1)$ which is a Bauer simplex since $\xi(S_C) = C$ is $\sigma(L^\infty, L^1)$ compact. Second, we need to show that $S_{C_1} \cap S_{C_2} = S_{C_1}$ iff $C_1 \subset C_2$ when C_1 and C_2 are distinct elements of M-Chain($\square^+$).

PROPOSITION 3.2. *If $C \in C$-Chain($\square^+$) then S_C is a Bauer simplex.*

Proof. We must show that S_C is affinely asomorphic to some simplex. If the proposition is valid S_C will be affinely isomorphic to $\mathscr{P}(C)$, the Bauer simplex of all probability Radon measures on the compact Hausdorff space C. One affine isomorph of $\mathscr{P}(C)$ is the convex set $\mathscr{D}(C)$ of distribution functions on C. $\mathscr{D}(C)$ consists of all functions g from C to $[0, 1]$ which are decreasing, have $g(\inf(C)) = 1$ and are left continuous on C so that if $x \in C$ is the supremum of $\{y \in C: y < x\}$ then $g(x) = \inf\{g(y): y \in C, y < x\}$. The affine isomorphism between $\mathscr{P}(C)$ and $\mathscr{D}(C)$ is gotten by assigning to $p \in \mathscr{P}(C)$ its distribution function d_p which is defined by $d_p(x) = p\{y \in C: y \geqq x\}$. The details of this affine isomorphism is standard knowledge when C is order isomorphic to $[0, 1]$ and is folk lore otherwise. We shall establish an affine isomorphism between $\mathscr{D}(C)$ and S_C or, more precisely, between $\mathscr{D}(C)$ and S defined in Proposition 3.1. It is helpful to note that $\mathscr{D}(C)$, when given the pointwise ordering as a set of real functions on C, is a lattice with the usual lattice operations and actually is a complete lattice (the supremum of a family is the left continuous regularization of the pointwise supremum and the infimum is the pointwise infimum).

Let $x_1 \geqq x_2 \cdots \geqq x_n$ be a finite subchain of C and let $\lambda_1, \cdots, \lambda_n$ be positive reals with $\sum_{i=1}^n \lambda_i = 1$. Set $f = \sum_{i=1}^n \lambda_i x_i \in S_C$. Let $g \in \mathscr{D}(C)$ be d_p where $p = \sum_{i=1}^n \lambda_i \delta_{k_i}$ so that $g(y) = \sum_{i=1}^k \lambda_i$ iff $x_k \geqq y > x_{k+1}$ for $k = 1, \cdots n$ (where $x_{n+1} = \inf(C)$) and $g(y) = 0$ if $y > x_i$. The correspondence $\Phi: f \to g$ is $1-1$ and affine between the convex set of f in S_C with finitely many values and the convex set of g in $\mathscr{D}(C)$ with finitely many values. Furthermore, $\Phi(f) \geqq \Phi(\tilde{f})$ iff $f \geqq \tilde{f}$.

For any $f \in S_C$ let $\{f_n\}$ be the sequence of finite valued elements of S_C given in the second paragraph of the proof of Proposition 3.1. Define for $g \in \mathscr{D}(C)$ the analogous sequence $\{g_n\}$ of finite valued elements of $\mathscr{D}(C)$. The sequence $\{f_n\}$ increases uniformly to f and the sequence $\{g_n\}$ increases uniformly to g. If $\tilde{g}_n = \Phi(f_n)$ then $\{\tilde{g}_n\}$ increases uniformly to some $\Phi(f) \in \mathscr{D}(C)$ for which, as is easily verified, $\{[\Phi(f)]_n\} = \{\tilde{g}_n\}$. The correspondence $f \to \Phi(f)$ is affine (as the continuous extension of uniformly continuous Φ on the uniformly dense set of finite valued elements of S_C). If $g \in \mathscr{D}(C)$ the element $f = \lim_{n\to\infty} \Phi^{-1}(g_n)$ of S_C has $\Phi(f) = g$. Thus, Φ is surjective. Since both $f \in S_C$ and $g \in \mathscr{D}(C)$ are uniquely determined by the sequences $\{f_n\}$ and $\{g_n\}$, Φ is seen to be injective. Thus, $\Phi: S_C \to \mathscr{D}(C)$ is an affine isomorphism. This establishes the proposition.

REMARKS. 1. The affine isomorphism between S_C and $\mathscr{P}(C)$ is still valid even if $\square^+$ is just the positive unit ball of $\mathscr{C}(X)$ with X Stonian.

2. If X is only basically disconnected S_C^n is affinely isomorphic to the (nonChoquet) simplex of all probability Borel measures on the topological space C equipped with the order topology. If S_C^n is given the norm $\| \ \|_\infty$ and $\mathscr{D}(C)$ is given the uniform on C then S_C^n and $\mathscr{D}(C)$ are isometric under Φ.

PROPOSITION 3.3. *Let $C \in C$-Chain$(\square^+)$.*

(i) *If C' is a complete subchain of C then $S_{C'}$ is a closed face of S_C.*

(ii) *If F is a closed face of S_C then $F = S_{C'}$ for the complete subchain $C' = F \cap C$ of C.*

(iii) *$S_C \cap S_{C'} = S_{C \cap C'}$ is a face both of S_C and $S_{C'}$ when $C' \in$ C-Chain $(\square^+)$.*

Proof. (i) and (ii) The closed faces of a Bauer simplex are the closed convex hulls of closed sets of extreme points. The closed sets of extreme points of S_C are the complete subchains of C.

(iii) That $S_{C \cap C'} \subset S_C \cap S_{C'}$ is immediate. When $f \in S_C \cap S_{C'}$ then $C_f \subset C \cap C'$ so $f \in S_{C \cap C'}$ thus $S_{C \cap C'} = S_{C \cap} S_{C'}$.

COROLLARY 3.3.1. *$\Sigma(\square^+)$ is a Caratheodory Bauer Bauer simplicial subdivision of $\square^+$.*

COROLLARY 3.3.2. *If $S \in \Sigma(\square^+)$ its $\sigma(L^\infty, L^1)$ closed faces and its $\| \ \|_\infty$-closed faces agree and each such face is the $\| \ \|_\infty$-closed convex hull of its extreme points.*

4. **Barycentric subdivisions of cubes. Simplicial tesselations of L^∞.** If $\square^+$ is the positive unit ball of $L^\infty(S, \Sigma, \mu)$ it is the order interval $\{x: 0 \leqq x \leqq 1\}$. Given any $e \in \xi(\square)$ we may order $L^\infty (S, \Sigma, \mu)$ so that it becomes an M-space with e as order unit and with $\square$ as unit ball. The positive unit ball, $\square_e^+$, under this ordering is $(e - \square) \cap \square$ and the positive cone is $\bigcup_{n=1}^\infty n \cdot \square_e^+$. The mapping $R_e: f \to f \cdot e$ is an isometry of $L^\infty(S, \Sigma, \mu)$ taking $\square^+$ onto $\square_e^+$. The map R_e is involutory and is a $\sigma(L^\infty, L^1)$ isomorphism.

PROPOSITION 4.1.

$$R_e(\Sigma(\square^+)) = \Sigma(\square_e^+) .$$

Proof. $R_e(\Sigma(\square^+))$ is a simplicial subdivision of $\square_e^+$ consisting of $\| \ \|_\infty$-closed convex hulls of sets of the form $R_e(C)$ where $C \in M$-Chain $(\square^+)$. Note that if $\{f, g\} \subset \square$ and $\geqq_e$ is the order on $L^\infty(S, \Sigma, \mu)$ with e as order unit then $f \geqq_e g$ iff $fe \geqq ge$. Since $R_e(\xi(\square^+)) = \xi(\square_e^+)$ it follows that $R_e(C\text{-Chain}(\square^+)) = C\text{-Chain}(\square_e^+)$ and that

$$R_e(M\text{-Chain}(\square^+)) = M\text{-Chain}(\square_e^+) .$$

That $R_e(\Sigma(\square_e^+)) = \Sigma(\square_e^+)$ is now immediate.

PROPOSITION 4.2. 1. *$\tilde{\Sigma}(\square) = \bigcup \{\Sigma(\square_e^+): e \in \xi(\square)\}$ is a simplicial subdivision of $\square$.*

2. *The zero skeleton of* $\tilde{\Sigma}(\square)$ *is the set of geometric centers of* $\sigma(L^\infty, L^1)$ *closed faces of* $\square$.

Proof. 1. $\tilde{\Sigma}(\square)$ consists of Bauer simplexes. If $f \in \square$ and $e = \chi_{\{f \geqq 0\}} - \chi_{\{f<0\}}$ then $f \in \square_e^+$ hence is in some element of $\Sigma(\square_e^+)$. Thus $\tilde{\Sigma}(\square)$ covers $\square$.

To show that $\tilde{\Sigma}(\square)$ is a subdivision it only remains to show that if S_1 and S_2 are distinct elements of $\tilde{\Sigma}(\square)$ then $S_1 \cap S_2$ is a proper face both of S_1 and S_2. If $S_1 \in \Sigma(\square_e^+)$ then $S_1 \cap S_2$ is a proper face of S_1 and of S_2 iff $R_e(S_1) \cap R_e(S_2)$ is a proper face of $R_e(S_1)$ and of $R_e(S_2)$. Consequently we may assume that $S_1 \in \Sigma(\square^+)$ and that $S_2 \in \Sigma(\square_f^+)$ where $f = \chi_A - \chi_{A}c \in \xi(\square)$. Note that $g \in \square^+ \cap \square_f^+$ iff $R_f g = g$ iff $0 \leqq g \leqq \chi_A$. For any such g, $R_f(\chi_B) = \chi_B$ if $B \in C_g$. Thus $R_f(C_g) = C_g$. We have $g \in S_2$ iff $C_g = R_f(C_g) \subset \xi(S_2)$ and

$$\inf(R_f(\xi(S_2))) \leqq R_f(g) = g \leqq \sup(R_f(\xi(S_2)))\ .$$

Since $R_f[\xi(S_2)]$ is maximal the last condition is vacuous so $g \in S_2$ iff $C_g \subset \xi(S_2)$. Thus, $g \in S_1 \cap S_2$ iff $C_g \subset \xi(S_2) \cap \xi(S_1) = C'$. Thus, $S_1 \cap S_2$ is the face $S_{C'}$ of S_1 which is proper since $1 \notin S_{C'}$. This establishes 1).

2. In [5] it is shown that the $\sigma(L^\infty, L^1)$ closed faces of are the order intervals $\{f \in \square : \chi_A - \chi_{A}c \leqq f \leqq \chi_B - \chi_{B}c\}$ where $A \subset B$ are in Σ. The center of this face is $\chi_{B \cap A} - \chi_{A \cap B}c$. If D_1 and D_2 are disjoint $\chi_{D_2} - \chi_{D_1}$ is the center of a unique $\sigma(L^\infty, L^1)$ closed face of $\square$. $\chi_{D_2} - \chi_{D_1}$ is an element of the 0-skeleton of $\Sigma(\square_e^+)$ where $e = \chi_{D_1}c - \chi_{D_1}$ hence is an element of the 0-skeleton of $\tilde{\Sigma}(\square)$. Conversely, any element of 0-skeleton of $\tilde{\Sigma}(\square)$ is in the zero skeleton of $\Sigma(\square_e^+)$ for some $e = \chi_A - \chi_{A}c$ hence is of the form $(\chi_A - \chi_{A}c)\chi_B = \chi_{D_2} - \chi_{D_1}$ for some B and some disjoint D_2 and D_1. This suffices to establish (2).

REMARKS. 1. This proposition remains valid even if $\square$ is the unit ball of $\mathscr{C}(X)$ basically disconnected.

2. $\{\square_e^+ : e \in \xi(\square)\}$ is a compact cubical subdivision of $\square$. It is only a cubical subdivision if $\square$ isn't compact but is the unit ball of $\mathscr{C}(X)$ with X basically disconnected.

3. $\tilde{\Sigma}(\square)$ isn't a Caratheodory subdivision of $\square$.

If $\square$ is a compact cube, order the closed faces by inclusion and let the centers of the closed faces be given the induced order so that if c_j is the center of face F_j for $j = 1, 2$ then $c_1 \leqq c_2$ iff $F_1 \subset F_2$. Consider maximal chains of centers under this ordering. If C is such a chain it has an infimum e which is easily verified to be the center of a 0-dimensional face hence $e \in \xi(\square)$. If $e = 1$ then $1 \in \text{face}(c)$ for any $c \in C$. Consequently $\text{face}(c) = \{f : \chi_A - \chi_{A}c \leqq f \leqq 1\}$ for some $A = A(c)$ and $c = \chi_{A(c)}$. It is easily verified that $c_1 \leqq c_2$ in C iff

$A(c_1) \supset A(c_2)$ thus, the maximal chains C for $\leqq$ with $1 \in C$ are in $1-1$ correspondence with the maximal chains in $\square^+$. The closed convex hulls of such chains are just the element of $\Sigma(\square^+)$. Similarily, the closed convex hulls of the maximal chains C for $\leqq$ containing $e \in \xi(\square)$ are the elements of $\Sigma(\square_e^+)$. Consequently, we have the following proposition whose terminology in self explanatory.

PROPOSITION 4.3. *$\tilde{\Sigma}(\square)$ is the barycentric subdivision of $\square$.*

We recall that all closed faces of compact cubes are compact cubes.

COROLLARY 4.3.1. *If F is a closed face of the compact cube $\square$ then $\{S \cap F : S \in \tilde{\Sigma}(\square)\}$ is the barycentric subdivision of F.*

Proof. If $S \in \tilde{\Sigma}(\square)$ then $\xi(S) \cap F$ is a chain of centers of closed faces of F for $\leqq$. If not maximal it could be enlarged. But this would mean that the maximal chain $\xi(S)$ could be enlarged. Thus, $\xi(S) \cap F$ is maximal hence has as its convex hull an element of the barycentric subdivision of F. Since F is a face of $\square$, $S \cap F$ is a face of S hence is the closed convex hull of a closed subset of $\xi(S)$ which is contained in $\xi(S) \cap F$. It readily follows that $S \cap F$ is the convex hull of $\xi(S) \cap F$ which establishes the corollary.

DEFINITION 4.1. A *convex tessellation* of a locally convex space E is a convex subdivision of E.

REMARK. The meaning of a simplicial or of a cubical tesselation is immediate.

If X is a compact Hausdorff space those elements of $\mathscr{C}(X)$ whose values are even integers $I_2(X)$ forms a subring or sublattice of $\mathscr{C}(X)$. We may define an action of $I_2(X)$ in $\mathscr{C}(X)$ by translation so that, if $f \in I_2(X)$, $T_f : \mathscr{C}(X) \to \mathscr{C}(X)$ is defined by $T_f(g) = f + g$ for $g \in \mathscr{C}(X)$. The set $\{T_f : f \in I_2(X)\}$ is a ring of homeomorphisms of $\mathscr{C}(X)$ for any locally convex topology on $\mathscr{C}(X)$.

PROPOSITION 4.5. *$\{T_f(\square) : f \in I_2(X)\}$ is a cubical tesselation of $\mathscr{C}(X)$ if X is a basically disconnected compact Hausdorff space and $\square$ is the unit ball of $\mathscr{C}(X)$.*

Proof. $\Gamma = \{T_f(\square) : f \in I_2(X)\}$ consists of cubes. To establish the proposition it is necessary to show that Γ covers $\mathscr{C}(X)$ and to show that when f_1 and f_2 are distinct elements of $I_2(X)$ then $T_{f_1}(\square) \cap$

$T_{f_2}(\square)$ is a proper face both of $T_{f_1}(\square)$ and of $T_{f_2}(\square)$. For the latter it suffices to show that $\square \cap T_f(\square)$ is a proper face of $\square$ for any non-zero f in $I_2(\bar{X})$.

Let $g_1 \in \mathscr{C}(X)$ with $n+1 \geqq \sup(g_1) > n$ and with $-m > \inf(g_1) \geqq -(m+1)$ for $n, m \in Z$. Let A be the clopen set $\overline{\{g_1 > n\}}$ and let B be the clopen set $\overline{\{g_1 < -m\}}$. Set g_2 equal to $g_1 - \chi_A + \chi_B$ if $n \geqq 1$ and $m \geqq 1$. Set g_2 equal to $g_1 - \chi_A$ if $m < 1$ and $m \leqq 1$. Set q_2 equal to $g_1 + x_B$ if $m \geqq 1$ and $n < 1$. Repeat this process inductively obtaining a sequence $\{g_j : j \in N\}$. There is a least integer k so that $g_k = g_{k+1}(k \leqq n \vee m)$. This occurs iff $g_k \in \square$. For this k, $g_1 = g_k + f$ with $f \in I_2(X)$ hence $g_1 \in T_f(\square)$. This shows, since g_1 was arbitrary, that $\mathscr{C}(X) = \bigcup \Gamma$.

Let $f \in I_2(X)$ so that $f = \sum_{i=-m}^{n} (2i)\chi_{A_i}$ where $\{A_{-m}, \cdots, A_0, \cdots A_n\}$ is a finite partition of X consisting of clopen sets. The set

$$\square \cap T_f(\square) = \{g \in \mathscr{C}(X) : -1 \vee (-1+f) \leqq g \leqq 1 \wedge (1+f)\} .$$

This is empty unless $-2 \leqq f \leqq 2$ so that $f = -2\chi_{A_{-1}} + 2\chi_{A_1}$. In this case,

$$\square \cap T_f(\square) = \{g \in \mathscr{C}(X) : \chi_{A_1} - \chi_{A_1}c \leqq g \leqq \chi_{A_{-1}}\}$$

which by Lemma 3 of [5], is a face of $\square$. This completes the proof of this proposition.

REMARK. It may be verified that a compact Hausdorff space X is totally disconnected iff $\cup\{T_f(\square) : f \in I_2(X)\}$ is a dense subset of $\mathscr{C}(X)$ for $\| \ \|_X$. It may also be verified that $\{T_f(\square) : f \in I_2(X)\}$ is a minimal cubical precomplex even if X is not disconnected. We conjecture that this precomplex is a cubical tesselation of $\mathscr{C}(X)$ iff X is basically disconnected.

DEFINITION 4.2. If X is a hyperstonian compact Hausdorff space we let $\Sigma^\#(X)$ be $\cup\{T_f(\tilde{\Sigma}(\square)) : f \in I_2(X)\}$.

PROPOSITION 4.3. *If X is a hyperstorian compact Hausdorff space then $\Sigma^\#(X)$ is a Bauer simplicial tesselation of $\mathscr{C}(X)$.*

Proof. $\Sigma^\#(X)$ covers $\mathscr{C}(X)$ with Bauer simplexes. To establish the proposition it suffices to show that if S_1 and S_2 are distinct elements of $\Sigma^\#(X)$ then $S_1 \cap S_2$ is a proper face of both. We may assume that $S_1 \in \tilde{\Sigma}(\square)$ and $S_2 \in \tilde{\Sigma}(T_f(\square))$ for some f in $I_2(X)$. If $f = 0$ the assertion is immediate. Otherwise, $F = \square \cap T_f(\square)$ is a proper face of $\square$ and of $T_f(\square)$. $S_1 \cap S_2$ is equal to $(S_1 \cap F) \cap (S_2 \cap F)$. Since $0 \in S_1 \backslash F$, $S_1 \cap S_2$ is a proper subset of S_1 (and also of S_2). By Corollary 4.3.1, $S_1 \cap F$ is an element of the barycentric subdivision of F as is $S_2 \cap F$. Consequently, $(S_1 \cap F) \cap (S_2 \cap F)$ is a face both of $S_1 \cap F$ and $S_2 \cap F$. Thus, $S_1 \cap S_2$ is a proper face both of S_1 and of S_2.

5. **Non-Coherence.** A simplicial subdivision Σ of a convex set S of a t.v.s. E is *coherent* iff there is a unique topology on S inducing on each simplex in Σ its t.v.s. topology.

PROPOSITION 5.1. *Let (S, Σ, μ) be a positive localizable measure space. The Hausdorff locally convex topologies T on $L^\infty(S, \Sigma, \mu)$ which induce $\sigma(L^\infty, L^1)$ on each simplex in $\Sigma(\square^+)$ are precisely those compatible with the duality (L^∞, L^1).*

Proof. Let T be such a topology and $A = (L^\infty, T)'$. For any $\mathfrak{l} \in A$ and $E \in \Sigma$ set $\nu_\mathfrak{l}(E) = \mathfrak{l}(\chi_E)$. The set function $\nu_\mathfrak{l}$ on Σ is finitely additive and absolutely continuous with respect to μ in that when $\mu(E) = 0$ then $\nu_\mathfrak{l}(E) = 0$. If $\{E_n : n \in N\} \subset \Sigma$ is decreasing with empty intersection we choose an $S_C \in \Sigma(\square^+)$ with $\{\chi_{E_n} : n \in N\} \subset C$. Since $\mathfrak{l} \mid S_C$ is $\sigma(L^\infty, L^1)$ continuous $\lim_{n\to\infty} \nu_\mathfrak{l}(E_n) = \lim_{n\to\infty} \mathfrak{l}(\chi_{E_n}) = \mathfrak{l}(0) = 0$. Thus $\nu_\mathfrak{l}$ is countably additive on Σ. Theorem A, §29 of [11] shows that there is an $F^+ \in \Sigma$ such that if $E \subset F^+$ then $\nu_\mathfrak{l}(E) \geqq 0$ and if $E \subset F^- = F \mid F^+$ then $\nu_\mathfrak{l}(E) \leqq 0$. On F^+, $\nu_\mathfrak{l}$ has variation $\nu_\mathfrak{l}(F^+) < \infty$ and, on F^-, $\nu_\mathfrak{l}$ has variation $-\nu_\mathfrak{l}(F^-) < \infty$. Thus, $\nu_\mathfrak{l}$ is of bounded variation. The Radon-Nikodym theorem implies the existence of $g \in L^1(S, \Sigma, \mu)$ such that $\mathfrak{l}(f) = \int f d\nu_\mathfrak{l} = \int f \cdot g d\mu$ for all $f \in L^\infty(S, \Sigma, \mu)$. Thus, A is a subspace of $L^1(S, \Sigma, \mu)$. The topology $\sigma(L^\infty, A)$ is a coarser Hausdorff topology than $\sigma(L^\infty, L^1)$ hence equals $\sigma(L^\infty, L^1)$. It follows that $A = L^1(S, \Sigma, \mu)$ hence that T is compatible with the duality $\langle L^\infty, L^1 \rangle$.

If we show that the finest topology, $\tau(L^\infty, L^1)$, compatible with the duality $\langle L^\infty, L^1 \rangle$ induces $\sigma(L^\infty, L^1)$ on each element of $\Sigma(\square^+)$ we will be done. $\tau(L^\infty, L^1)$, the Mackey topology, is the topology of convergence uniform on $\sigma(L^\infty, L^1)$ compact sets (i.e., uniformly integrable subsets) in $L^1(S, \Sigma, \mu)$. Equivalently, since K is uniformly integrable iff $\{|g| : g \in K\}$ is uniformly integrable, $\tau(L^\infty, L^1)$ is the topology of convergence uniform on uniformly integrable subsets of $L^{1+}(S, \Sigma, \mu)$.

Let K be a $\sigma(L^1, L^\infty)$ compact subset of $L^{1+}(S, \Sigma, \mu)$. Let $S_C \in \Sigma(\square^+)$. For $g \in K$ set $h_g(\chi_A) = \int_A g d\mu$ when $\chi_A \in C$. Each function h_g is continuous on C for $\sigma(L^\infty, L^1)$. The collection $h(K) = \{h_g : g \in K\}$ is a compact subset of $\mathscr{C}(C)$ for the topology of pointwise convergence on C since K is $\sigma(L^1, L^\infty)$ compact. The set $h(K)$ is actually compact for the topology of uniform convergence for it is uniformly bounded and equicontinuous. Uniform boundedness is immediate. If $h(K)$ weren't equicontinuous at $\chi_A \in C$ we would either be able to find a net $\{\chi_{A_\alpha}\}$ decreasing to χ_A or a net $\{\chi_{A_\alpha}\}$ increasing to χ_A in C, a net $\{g_\alpha\} \subset K$ and an $\varepsilon > 0$ such that $\int |\chi_{A_\alpha} - \chi_A| g_\alpha d\mu \geqq \varepsilon$. Let us assume that $\chi_{A_\alpha} \downarrow \chi_A$. Then $\int \chi_{A_\beta} \backslash \chi_A g_\alpha d\mu \geqq \varepsilon$ for any $\alpha \geqq \beta$. Assume that $\{g_\alpha\}$ converges to $g \in K$ for $\sigma(L^1, L^\infty)$. We then have $\int \chi_{A_\beta} \backslash \chi_A g d\mu \geqq \varepsilon$ for all β hence $\lim_\beta h_g(\chi_{A_\beta}) \neq h_g(\chi_A)$ which is impossible. Thus, $h(K)$ is compact for the uniform topology. If $\{\mu_\alpha\}$ is any $\sigma(\mathscr{M}(C), \mathscr{C}(C))$ convergent net of probability measures on C it is convergent uniformly on norm compact sets in $\mathscr{C}(C)$ hence it converges uniformly on any $h(K)$. Thus, if $\{f_\alpha\}$ is a $\sigma(L^\infty, L^1)$ convergent net in S_C, it converges uniformly on $\sigma(L^1, L^\infty)$ compact sets in $L^1(S, \Sigma, \mu)$ hence is $\tau(L^\infty, L^1)$ convergent. Thus, S_C is $\tau(L^\infty, L^1)$ compact so $\tau(L^\infty, L^1)$ and $\sigma(L^\infty, L^1)$ agree on S_C. This suffices to establish the proposition.

REMARKS. The finest locally convex topology on $L^\infty(S, \Sigma, \mu)$ inducing $\sigma(L^\infty, L^1)$ on $\square^+$ is the Arens topology $\kappa(L^\infty, L^1)$ of convergence uniform on norm compact subsets of $L^1(S, \Sigma, \mu)$, [21, p. 150], [9, p. 505].

If $\Sigma^\#$ is the tesselation of $L^\infty(S, \Sigma, \mu)$ in §4 all of the topologies compatible with the duality $\langle L^\infty, L^1 \rangle$ induce the same topology on each tesselation simplex. Since these are all distinct $\Sigma^\#$ would be called a *noncoherent* tesselation of $L^\infty(S, \Sigma, \mu)$. If $\sigma(L^\infty, L^1) \subset \tau \subset \kappa(L^\infty, L^1)$ then τ induces the topology $\sigma(L^\infty, L^1)$ on $\square^+$. It might appear that $\Sigma(\square^+)$ has a chance to be coherent. For this we would need to have $\tau(L^\infty, L^1)$ induce $\sigma(L^\infty, L^1)$ on $\square^+$. However, $\kappa(L^\infty, L^1)$ is the finest such topology and is coarser that $\tau(L^\infty, L^1)$.

6. Homogenity. If Γ is a Caratheodory simplicial subdivision of an n-dimensional convex compact set all elements of Γ are $n+1$-simplexes hence all are affinely homeomorphic. We say that a convex precomplex is *homogeneous* iff all elements are affinely homeomorphic. Thus, all Caratheodory simplicial subdivisions of an n-dimensional convex compact set are homogeneous. If $\square^+$ is the positive unit ball of an infinite dimensional space $L^\infty(S, \Sigma, \mu)$ is $\Sigma(\square^+)$ homogeneous? Let C_1 and C_2 be in M-Chain($\square^+$). Since S_{C_1} and S_{C_2} are Bauer simplexes they are affinely homeomorphic iff C_1 and C_2 are homeomorphic. Our question reduces to examination of elements of M-Chain($\square^+$) as topological spaces.

We first note the measure space (S, Σ, μ) has an atom iff there is a $C \in M$-Chain($\square^+$) with a gap, i.e., there are elements $\chi_A < \chi_B$ in is C such that there is no element of C between them and, if this the case, then $B \backslash A$ is an atom. Actually, (S, Σ, μ) has an atom iff every $C \in M$-Chain($\square^+$) has gaps. If (S, Σ, μ) has no atoms every $C \in M$-Chain($\square^+$) has no isolated points and is connected hence C looks like a "long line segment".

PROPOSITION 6.1. *If the positive localizable measure space* (S, Σ, μ) *is neither atomic nor nonatomic the subdivision* $\Sigma(\square^+)$ *of the positive unit ball of* $L^\infty(S, \Sigma, \mu)$ *is not homogeneous.*

Proof. Let $S_a \neq \varnothing$ be the supremum of the atoms of (S, Σ, μ) and let $S_n = S \backslash S_a \neq \varnothing$. Consider the following chains C_1 and C_2. Let C_a be a maximal chain for the measure space (S_a, Σ, μ) and let C_n be one for (S_n, Σ, μ). Let $C_1 = C_1^a \cap \{f + \chi_{S_a}: f \in C_n\}$. To construct C_2 break S_n into two pieces $S_{n_1} \neq \varnothing$ and $S_{n_2} \neq \varnothing$. Construct maximal chains C_{n_1} in (S_{n_1}, Σ, μ) and C_{n_2} in (S_{n_2}, Σ, μ). Let $C_2 = C_{n_1} \cup \{f + x_{S_{n_1}}: f \in C_a\} \cup \{f + \chi_{S_{n_2}} + \chi_{S_a}: f \in C_{n_2}\}$.

Construct the derived sets $C_1^{(1)}$ and $C_2^{(1)}$ of C_1 and C_2 by deleting isolated points and repeat by transfinite induction getting derived sets $C_1^{(\alpha)}$ and $C_2^{(\alpha)}$ for all ordinals α. When $C_1^{(\alpha)} = C_1^{(\alpha+1)}$ and $C_2^{(\alpha)} = C_2^{(\alpha+1)}$ then $C_1^{(\alpha)}$ and $C_2^{(\alpha)}$ have no isolated points. This occurs at some ordinal α, and at this point $C_1^{(\alpha)}$ has one connectivity component homeomorphic to C_n and $C_2^{(\alpha)}$ has two, one homeomorphic to C_{n_1} and one to C_{n_2}. It is well known that if C_1 and C_2 are homeomorphic so are $C_1^{(\alpha)}$ and $C_2^{(\alpha)}$. Thus, C_1 isn't homeomorphic to C_2 which establishes the proposition.

If C is an element of M-Chain $(\square^+)$ it is *scattered space*, [23], iff the chain of derived sets $\{C^{(\alpha)}: \alpha \text{ an ordinal}\}$, as used in the preceding proof, terminates with $C^{(\alpha)} = \varnothing$. This is easily verified to hold iff (S, Σ, μ) is purely atomic. In this case $L^\infty(S, \Sigma, \mu)$ is Banach lattice isomorphic to $l^\infty(m)$ where m is the cardinality of the set of atoms. In this case we may enumerate the set of atoms as $\{A_\lambda: \lambda \in \Gamma\}$ where card $(\Gamma) = m$. It is easy to see that the elements of M-Chain$(\square^+)$ are in $1-1$ correspondence with the linear orderings of Γ. If $C \in M$-Chain$(\square^+)$ set $\lambda \leqq_C \gamma$ iff $\chi_{A_\gamma} \leqq f \in C$ implies that $\chi_{A_\lambda} \leqq f$ for $\{\lambda, \gamma\} \in \Gamma$. The correspondence $C \leftrightarrow S_C$ is the desired bijection between M-Chain$(\square^+)$ and linear orderings of Γ.

PROPOSITION 6.2. *Let* (S, Σ, μ) *be a purely atomic positive localizable measure space, with infinitely many atoms. If* $\square^+$ *is the positive unit ball of* $L^\infty(S, \Sigma, \mu)$ *then* $\Sigma(\square^+)$ *isn't homogeneous.*

Proof. Let $\{A_\lambda: \lambda \in \Gamma\}$ be an enumeration of the atoms of (S, Σ, μ). Let $\leqq_1$ and $\leqq_2$ be the linear orderings of Γ obtained by putting Γ in $1-1$ correspondence with the sets $D_1 = \{\alpha: 0 \leqq \alpha \leqq \omega_1\}$ and $D_2 = \{\alpha: 0 \leqq \alpha \leqq \omega_2\}$ where ω_1 is the first ordinal of cardinal card (Γ) and $\omega_2 = \omega_1 + \omega_1$. Let C_1 and C_2 be the corresponding elements of M-Chain$(\square^+)C_i$ is homeonorphic D_i for $i = 1, 2$. D_1 has only one point q which is not a limit of a net in $D_1 \backslash \{q\}$ of cardinality less than card (Γ) whereas there are two such points in D_2. Thus D_1 isn't homeomorphic with D_2 hence C_1 isn't homeomorphic to C_2.

To find examples of positive localizable measure space (S, Σ, μ) such that the space $L^\infty(S, \Sigma, \mu)$ is infinite dimensional and the positive unit ball $\square^+$ has $\Sigma(\square^+)$ homogeneous we must assume that μ is purely nonatomic. We show that μ must be a σ-finite. The σ-finiteness of μ is equivalent to the countable chain condition or to the assertion that any $C \in \mathrm{Chain}(\square^+)$ is separable in the order topology, [19]. In this case any $C \in M\text{-}\mathrm{Chain}(\square^+)$ is a compact separable linearly ordered set with no isolated points thus is homeomorphic to $[0, 1]$. Consequently, if μ is σ-finite, $\Sigma(\square^+)$ is homogeneous.

PROPOSITION 6.4. *If (S, Σ, μ) is a localizable purely nonatomic positive measure space with $\square^+$ the positive unit ball of $L^\infty(S, \Sigma, \mu)$ then $\Sigma(\square^+)$ is homogeneous iff μ is σ-finite.*

Pooof. If μ is σ-finite we have seen that $\Sigma(\square^+)$ is homogeneous. If μ isn't σ-finite there exist compacts $\{X_\lambda : \lambda \in \Gamma\}$ and nonatomic probabilities $\{\mu_\lambda : \lambda \in \Gamma\}$ on these compacts such that $\square^+$ is affinely homeomorphic with the positive unit ball of $L^\infty(X, \hat{\mu})$ where $\dot{\mu}$ is the Radon measure on the locally compact disjoint union X of $\{X_\lambda : \lambda \in \Gamma\}$ with $\dot{\mu}|_{X_\lambda} = \mu_\lambda$ for $\lambda \in \Gamma$. This is an immediate consequence of Kakutani's Representation Theorem for L-space [23, 26.3.3]. The measure μ is σ-finite iff Γ is countable. To establish the proposition it suffices to find for Γ uncountable a $C \in M\text{-}\mathrm{Chain}(\square^+)$ which is separable in the order topology. This is because the uncountability of Γ guarantees the nonσ-finiteness of μ, hence, implies that the countable chain condition is violated, hence, implies the existence of a nonseparable $C_0 \in M\text{-}\mathrm{Chain}(\square^+)$ which can't be homeomorphic to C.

To construct C first construct maximal chains C_λ in the extreme points of the unit ball of $L^\infty(X_\lambda, \mu_\lambda)$ for $\lambda \in \Gamma$. Define $f_{\lambda,t} \in C_\lambda$ by the requirement that $\int f_{\lambda,t} d\mu_\lambda = t$ for $\lambda \in \Gamma$. The map $t \to f_{\lambda,t}$ is a homeomorphic order isomorphism of $[0, 1]$ onto C_λ for $\lambda \in \Gamma$. Set f_t equal to $f_{\lambda,t}$ on X_λ for all $\lambda \in \Gamma$. The map $t \to f_t$ is an order isomorphic homeomorphism of $[0, 1]$ onto the chain $C = \{f_t : 0 \leqq t \leqq 1\}$. The maximality of the separable chain C is readily verified.

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