

ON THE ZEROS OF DERIVATIVES OF BALANCED TRIGONOMETRIC POLYNOMIALS

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The final set and initial set for a balanced 2π -periodic trigonometric polynomial with roots in a horizontal strip are obtained. Assuming the limits have finite order, we show that only trigonometric polynomials are uniform limits on compact sets of balanced 2π -periodic trigonometric polynomials that have only real zeros. The final set result is extended to balanced almost periodic trigonometric polynomials. Finally, we give a final set result for balanced exponential sums whose exponents are complex.

1. Introduction. The well known Gauss-Lucas theorem for algebraic polynomials states that the zeros of the derivative $P'(z)$ of a complex valued polynomial $P(z)$ lie in the convex hull of the zeros of $P(z)$.

Let $T(z)$ be a trigonometric polynomial having the form

$$T(z) = \sum_{k=-N}^{+N} a_k \exp(ikz)$$

where $a_{-N}a_N \neq 0$. Such trigonometric polynomials are called *balanced* by Genchev ([3], [4], [5]). He proves the following analogues of the Gauss-Lucas theorem for balanced trigonometric polynomials:

(1) If $T(z)$ has its zeros in the strip $a \leq \operatorname{Im} z \leq b$, then all the zeros of $T'(z)$ also lie in this strip.

(2) If $T(z)$ has zeros in the strip $a \leq \operatorname{Im} z \leq b$, then all the zeros of the trigonometric polynomial

$$L(T)(z) = \int_{z-i\lambda}^{z+i\lambda} T(t)dt, \lambda > 0,$$

also lie in the strip [3].

Repeated applications of Genchev's result in (1) shows that the zeros of all the successive derivatives $T^{(n)}(z)$, $n = 1, 2, 3, \dots$ of a balanced trigonometric polynomial likewise have their zeros in this strip.

Furthermore, these zeros must accumulate somewhere, since each $T^{(n)}(z)$ is 2π -periodic. The set of accumulation points of all the zeros of the successive derivatives form what we call the *final set* [11, page 197]. Included in the final set are points where infinitely many derivatives vanish.

We apply the same idea to Genchev's result referred to in (2),

by letting $L^n(T)(z)$ denote the composition of the operator L with itself n times. We determine the points of accumulation of the zeros of $L^n(T)(z)$, for $n = 0, 1, 2, \dots$. We shall call this set of accumulation points the *initial set*. We determine the final and initial sets in these cases.

THEOREM 1.1. *Given $T(z) = \sum_{k=-N}^{+N} a_k \exp(ikz)$, where $a_{-N}a_N \neq 0$, whose zeros lie in the strip $a \leq \operatorname{Im} z \leq b$, the final set consists of the zeros of the function $a_{-N} \exp(-iNz) + a_N \exp(iNz)$ together with the zeros of its derivative. That is, the final set consists of the discrete set of points*

$$z = (-i/2N) \log(a_{-N}/a_N) + (1/2N) \arg(\pm a_{-N}/a_N) + \pi t/N \text{ for } t$$

an integer and these points all lie on the horizontal line $\operatorname{Im} z = (-1/2N) \log |a_{-N}/a_N|$.

Proof. The n th derivative of $T(z)$ is given by $T^{(n)}(z) = \sum_{k=-N}^{+N} i^n k^n a_k \exp(ikz)$, which is equal to $i^n N^n [(-1)^n a_{-N} \exp(-iNz) + a_N \exp(iNz)] + i^n \sum_{k=-N+1}^{N-1} k^n a_k \exp(ikz)$. We consider the sequence of functions $H_n(z) = T^{(n)}(z)/i^n N^n$. This is equal to $[(-1)^n a_{-N} \exp(-iNz) + a_N \exp(iNz)] + (1/N^n) \sum_{k=-N+1}^{N-1} k^n a_k \exp(ikz)$. Now as $n \rightarrow \infty$, the sum $(1/N^n) \sum_{k=-N+1}^{N-1} k^n a_k \exp(ikz) \rightarrow 0$ uniformly on compact sets. When n is odd (resp. even) the first term is $-a_{-N} \exp(-iNz) + a_N \exp(iNz)$ (resp. $a_{-N} \exp(-iNz) + a_N \exp(iNz)$). The sequence $\{H_n(z)\}$ for n odd is such that $H_n(z) \rightarrow -a_{-N} \exp(-iNz) + a_N \exp(iNz)$ as $n \rightarrow \infty$ uniformly on compact sets. By Hurwitz's theorem the zeros of $-a_{-N} \exp(-iNz) + a_N \exp(iNz)$ are points of accumulation of the zeros of these $H_n(z)$ and hence of the zeros of the corresponding $T^{(n)}(z)$. Similarly, for n even, the sequence $\{H_n(z)\}$ is such that $H_n(z) \rightarrow a_{-N} \exp(-iNz) + a_N \exp(iNz)$ as $n \rightarrow \infty$ uniformly on compact sets. Again by Hurwitz's theorem, the zeros of $a_{-N} \exp(iNz) + a_N \exp(iNz)$ are points of accumulation of the zeros of $T^{(n)}(z)$ (n even). Now, the zeros $-a_{-N} \exp(-iNz) + a_N \exp(+iNz)$ consists of the points $z = (-i/2N) \log |a_{-N}/a_N| + (1/2N) \arg(a_{-N}/a_N) + \pi t/N$, for t any integer, and the zeros of $a_{-N} \exp(-iNz) + a_N \exp(+iNz)$ consists of the points $z = (-i/2N) \log |a_{-N}/a_N| + (1/2N) \arg(-a_{-N}/a_N) + \pi t/N$, for t any integer.

Thus the final set of T does consist of that discrete set of points, which is contained in the line $\operatorname{Im} z = (-1/2N) \log |a_{-N}/a_N|$. This ends the proof of Theorem 1.1.

REMARK. The discrete points of the final set are precisely the accumulation points of the zeros of the successive derivatives of a balanced trigonometric polynomial. For if we are given the

zeros of $\pm a_{-N} \exp(-iNz) + a_N \exp(iNz)$, $a_{-N}a_N \neq 0$, then $\sum_{k=-N}^{+N} a_k \exp(ikz)$ has as its final set this collection of zeros.

Furthermore, since the imaginary part of the final set lies between a and b , we have

$$\begin{aligned} a &\leq (-1/2N) \log |a_{-N}/a_N| \leq b, \\ -2Na &\geq \log |a_{-N}/a_N| \geq -2Nb, \end{aligned}$$

and so

$$1/\exp(2Na) \geq |a_{-N}/a_N| \geq 1/\exp(2Nb).$$

As a special case of the above theorem, note that when $|a_{-N}| = |a_N|$, the zeros of the successive derivatives of $T(z)$ condense to discrete points on the real axis.

THEOREM 1.2. *Given $T(z) = \sum_{k=-N}^{+N} a_k \exp(ikz)$ a balanced trigonometric polynomial whose zeros lie in the strip $a \leq \operatorname{Im} z \leq b$, then the initial set consists of the discrete set of points $z = (-i/2N) \log |a_{-N}/a_N| + (1/2N) \arg(-a_{-N}/a_N) + \pi t/N$ (t any integer) all of which lie on the horizontal line $\operatorname{Im} z = (-1/2N) \log |a_{-N}/a_N|$.*

REMARK. Hence for balanced trigonometric polynomials with zeros in $a \leq \operatorname{Im} z \leq b$, the initial set is a subset of the final set.

Proof. We start with $T(z) = \sum_{k=-N}^{+N} a_k \exp(ikz) = a_0 + \sum'_{k=-N}^{+N} a_k \exp(ikz)$, where the prime (') indicates that the zeroth term is omitted. We get that $L(T)(z) = \sum_{k=-N}^{+N} (a_k/ik) [\exp(-k\lambda) - \exp(k\lambda)] \exp(ikz) + 2i\lambda a_0$. An elementary induction shows that

$$(L^n T)(z) = \sum_{k=1}^N [S(k)]^n \{a_{-k} e^{-ikz} + a_k e^{ikz}\} + (2i\lambda)^n a_0$$

where $S(x) = 2 \sinh(\lambda x)/ix$. Since $(\sinh(\lambda x))/x$ increases as $x \rightarrow \infty$ for fixed $\lambda > 0$, it follows that $L^n(T)(z)/(S(N))^n$ converges uniformly on compact sets to $a_{-N} \exp(-iNz) + a_N \exp(iNz)$.

By Hurwitz's theorem the zeros of that last function are points of accumulation of the zeros of $L^n(T)(z)/(S(N))^n$, and hence of the zeros of $L^n(T)(z)$. As we saw earlier, the zeros of $a_{-N} \exp(-iNz) + a_N \exp(iNz)$ are the discrete points $z = (-i/2N) \log |a_{-N}/a_N| + (1/2N) \arg(-a_{-N}/a_N) + \pi t/N$, t any integer.

These are a discrete set of points which lie on the horizontal line $\operatorname{Im} z = (-1/2N) \log |a_{-N}/a_N|$. When $|a_{-N}| = |a_N|$, the initial set consists of the discrete set of real numbers $(1/2N) \arg(-a_{-N}/a_N) + \pi t/N$ for t any integer. This ends the proof of Theorem 1.2.

2. Approximation by balanced trigonometric polynomials.

It is well known that the only entire functions that are uniform limits on compact sets of algebraic polynomials whose zeros are real are functions of the classical Laguerre-Pólya class that have the form $f(z) = Az^m \exp(-\alpha z^2 + \beta z) \pi_{k=1}^{\infty} E(z/\alpha_k, 1)$, where $\alpha \geq 0$, β and the α_k are real and $\sum_{k=1}^{\infty} 1/(\alpha_k)^2 < \infty$ [9]. Since $f(z) = \lim_{n \rightarrow \infty} P_n(z)$, where the polynomials $P_n(z)$ have only real zeros, we have $f^{(k)}(z) = \lim_{n \rightarrow \infty} P_n^{(k)}(z)$. By Rolle's theorem, the $P_n^{(k)}(z)$ likewise only have real zeros.

Theorem 1 of Genchev says that if the zeros of a balanced trigonometric polynomial $T(z)$ are real, then the zeros of the successive derivatives $T^{(k)}(z)$, $k = 1, 2, \dots$ are also real. It is then natural to ask "What are the uniform limits on compact sets of balanced 2π -periodic trigonometric polynomials whose zeros are real?" We partially answer this by the next result.

THEOREM 2.1. *The only entire functions $f \not\equiv 0$ of finite order that can be approximated uniformly on compact sets by balanced 2π -periodic trigonometric polynomials having only real zeros are themselves trigonometric polynomials.*

LEMMA 1. *The only entire functions of finite order that are periodic with no real zeros are functions of the form $\exp(az + b)$.*

Proof. Since f is of finite order without zeros, $f(z) = \exp(g(z))$ for g a polynomial by the Hadamard factorization theorem. Then as $\exp(g(z+a)) = \exp(g(z))$, we get $\exp(g(z+a) - g(z)) = 1 \Rightarrow g(z+a) - g(z) = 2\pi i K$ for some integer $K \Rightarrow g'(z+a) = g'(z)$. So g' is a periodic polynomial $\Rightarrow g'(z)$ is constant $\Rightarrow g(z)$ is linear.

Proof of Theorem 2.1. The limit function f is 2π -periodic. Therefore the counting function $n(r)$ of the zeros is $O(1)$ and $\sum_{|a_n| \leq r} 1/a_n = O(1)$ as $r \rightarrow \infty$, since the positive and negative zeros $\{a_n\}$ are "essentially" symmetric (a shift of the origin to the beginning of a 2π -period makes them symmetric). Lindelöf's theorem [1, page 27] implies that the canonical product for the $\{a_n\}$ is of exponential type. Since f and P are periodic, so is e^g and since f is of finite order, g is a polynomial. By Lemma 1, g is linear. Hence f is of exponential type. A well known theorem [1, page 109] gives that f must be a trigonometric polynomial.

REMARKS.

(A) Presumably the assumption of finite order in this theorem can be removed and the result will remain valid.

(B) Let $\{T_n(z)\}_{n=1}^\infty$ be a sequence of balanced trigonometric polynomials that converges uniformly on compact sets to f . Since T_n and all its derivatives have only real zeros, the same is true for f and its derivatives. If f is real entire, then a result of Hellerstein and Williamson [8] implies that f is in the Laguerre-Pólya class and in particular of finite order. Theorem 2.1 then implies that f is a trigonometric polynomial. If, however, f is not a real entire function, then if f is assumed to have finite order, as in the hypothesis of Theorem 2.1, it follows from a theorem of Alander (reproved in the paper of Hellerstein and Williamson) and the periodicity of f that f is a trigonometric polynomial of the form $f(z) = a(e^{icz} - e^{id})$, a arbitrary, b, c , and d real.

3. Some Gauss-Lucas results. We now prove a Gauss-Lucas result for any canonical product of genus 1 having zeros in any horizontal strip.

THEOREM 3.1. *Let $P(z)$ be a canonical product of genus 1 whose zeros lie in $|\operatorname{Im} z| < A$. Then the zeros of $P'(z)$ also lie in $|\operatorname{Im} z| < A$.*

REMARK. Proving the stated theorem suffices to prove this result when the zeros lie in any horizontal strip, since by a change of variable we can always vertically translate any horizontal strip so that it is symmetric about the real axis.

Proof. We will prove this in a series of steps, by demonstrating the following two facts:

(I) Given $P(z) = \pi_{k=1}^\infty E(z/\alpha_k, 1)$, where $\alpha_k = \psi_k + i\tau_k$ and the α_k satisfy $0 < b \leq \tau_k \leq c$, then for $y > c$, it is true that $\operatorname{Im}[f'/f] < 0$.

(II) Given $P(z)$ as in (I) and the α_k satisfy $b \leq \tau_k \leq c < 0$, then for $y < b$, it is true that $\operatorname{Im}[f'/f] > 0$.

Claim. It suffices to prove (I) and (II) to prove this Theorem. For if $P(z)$ is given as above $P(z-iA)$ has zeros in $0 < \operatorname{Im} z < 2A$ and by (I) $P'(z-iA) \neq 0$ for $y \geq 2A$. Similarly, $P(z+iA)$ has zeros in $-2A < \operatorname{Im} z < 0$ and by (II) $P'(z+iA) \neq 0$ for $y \leq -2A$. But the zeros of $P'(z)$ are just vertical translations of the zeros of $P'(z-iA)$ and $P'(z+iA)$. Therefore, if the zeros of $P(z)$ lie in $|\operatorname{Im} z| < A$, so do the zeros of $P'(z)$ lie in $|\operatorname{Im} z| < A$.

Now for the proofs of (I) and (II): We let $\alpha_k = \psi_k + i\tau_k$ and $z = x + iy$. Then an easy computation shows that

$$\operatorname{Im}[P'(z)/P(z)] = \sum_{k=1}^\infty \{(\tau_k - y)/[(x - \psi_k)^2 + (y - \tau_k)^2] - \tau_k/(\psi_k^2 + \tau_k^2)\}.$$

If as in (I) $0 < b \leq \tau_k \leq c$ and $y > c$, then $\tau_k - y < 0$ and $-\tau_k < 0$ so that $\operatorname{Im}[f'/f] < 0$. If as in (II), $B \leq \tau_k \leq c < 0$ and $y < b$, then $\tau_k - y > 0$ and $-\tau_k > 0$ so that $\operatorname{Im}[f'/f] > 0$. This completes the proof.

It is now possible to give an alternate proof of Genchev's theorem for almost periodic trigonometric polynomials. For some basic properties of general almost periodic trigonometric polynomials, see [2] or [8, page 246].

DEFINITION. We say that an almost periodic trigonometric polynomial $T(z) = \sum_{k=-N}^{+N} c_k \exp(i\lambda_k z)$ is balanced if the exponents λ_k are contained in the symmetric interval $[-D, +D]$ with $c_{-N}c_N \neq 0$ and where $\lambda_{-N} = -D$ and $\lambda_N = +D$.

THEOREM 3.2. *Let $T(z) = \sum_{k=-N}^{+N} c_k \exp(i\lambda_k z)$ be a balanced almost periodic sum whose roots lie in a horizontal strip $|\operatorname{Im} z| \leq M$. Then the roots of $T'(z)$ also lie in $|\operatorname{Im} z| \leq M$.*

REMARK. Note that this also includes Genchev's theorem for the derivative operator, since a 2π -periodic balanced trigonometric polynomial is clearly almost periodic.

Proof. By definition, we have that the λ_k lie in $[-D, +D]$, with $\pm D$ occurring. It is a fact that $f(z)$ is an entire function of exponential type D by [8, Theorem 1 and its corollary, pages 266-268]. Moreover, it is true that $[-iD, +iD]$ is its indicator diagram [1, Corollary 6.94, page 108]. We therefore know that the indicator diagram has length $2D$. This, by definition, makes f a function of the class $[D]$ (see [8, page 211]). B. Ja. Levin and M. G. Krein characterized functions of the class $[D]$ by the following theorem [8, Appendix 6, Theorem 4, page 453]: $T(z)$ is an almost periodic function of the class $[D] \leftrightarrow 1) T(z)$ can be expressed in the form $T(z) = c \lim_{N \rightarrow \infty} \prod_{k=-N}^{+N} (1 - z/\alpha_k)$, where c is a constant. There are two other properties that T satisfies in this characterization. Since they won't be used, we omit them.

Hence by (1) $T(z) = c \lim_{N \rightarrow \infty} \prod_{k=-N}^{+N} (1 - z/\alpha_k)$ as $N \rightarrow \infty$; so $T'(z) = c \lim_{N \rightarrow \infty} [\prod_{k=-N}^{+N} (1 - z/\alpha_k)]'$ and since $\prod_{k=-N}^{+N} (1 - z/\alpha_k)$ has its roots in $|\operatorname{Im} z| \leq M$, by hypothesis, by Theorem 3.1, the zeros of its derivative $[\prod_{k=-N}^{+N} (1 - z/\alpha_k)]'$ also lie in $|\operatorname{Im} z| \leq M$. By Hurwitz's theorem, $T'(z)$ also has its roots in $|\operatorname{Im} z| \leq M$.

REMARK. Theorem 3.2 is true when the zeros lie in any horizontal strip $a \leq \operatorname{Im} z \leq b$, since Theorem 3.1 is true in this case.

(4) A final set result. Now that Genchev's theorem has been demonstrated for (finite sums of) almost periodic balanced trigonometric polynomials, we also have established a final set result for this class of polynomials analogous to Theorem 1.1.

THEOREM 4.1. *Given $T(z) = \sum_{k=-N}^{+N} a_k \exp(i\lambda_k z)$, a balanced almost periodic trigonometric polynomial whose zeros lie in $a \leq \operatorname{Im} z \leq b$, with $\min [\lambda_k]_{k=-N}^{+N} = \lambda_{-N} = -D$ and $\max [\lambda_k]_{k=-N}^{+N} = \lambda_N = +D$, then the final set consists of the zeros of the function $a_{-N} \exp(-iDz) + a_N \exp(iDz)$ together with the zeros of its derivative. That is, the final set consists of the discrete set of points.*

$z = (-i/2D) \log |a_{-N}/a_N| + (1/2D) \arg(\pm a_{-N}/a_N) + \pi t/D$ (t any integer).

The proof of this is similar to the proof of Theorem 1.1.

(5) Final sets for balanced exponential sums. We determine the final set for balanced finite exponential sums whose exponents are complex.

DEFINITION. The exponential sum $T(z) = \sum_{k=1}^n a_k \exp(\lambda_k z)$, where $\lambda_k \in \mathbb{C}$ and the conjugate indicator diagram I of T (i.e., the convex hull of $\{\lambda_1, \dots, \lambda_n\}$) is a polygon where the exponents $\neq 0$, is said to be *balanced* when with $\Delta = \max_{1 \leq k \leq n} |\lambda_k|$, the circle $|z| = \Delta$ contains 2 or more vertices of I and the coefficients corresponding to these vertices are nonzero.

THEOREM 5.1. *Let $T(z) = \sum_{k=1}^n a_k \exp(\lambda_k z)$ be balanced with $|z| = \Delta$ containing $\lambda_1, \dots, \lambda_p$, for $1 < p \leq n$ and a_1, \dots, a_p all nonzero. Then the final set for T consists of the roots of either finitely many trigonometric sums (which happens when any two $\arg(\lambda_k)$ differ by a rational multiple of π for all $k = 1, \dots, p$) or limits of roots of infinitely many (which happens when some two $\arg(\lambda_k)$ differ by an irrational multiple of π). Each sum contains terms corresponding to one of the factors $\exp(\lambda_k z)$, $k = 1, \dots, p$.*

Proof. Since $|\lambda_1| = \dots = |\lambda_p|$, let $\lambda_k = e^{i\theta_k} \lambda_1$ for $k = 1, \dots, p$ (with $\theta_1 = 0$). Let z belong to a compact set. Then

$$\begin{aligned} T^{(m)}(z)/(\lambda_1)^m &= \sum_{k=1}^p a_k e^{im\theta_k} \exp(\lambda_k z) + (1/(\lambda_1)^m) \sum_{k=p+1}^n (\lambda_k)^m \exp(\lambda_k z) \\ &= \sum_{k=1}^p a_k e^{im\theta_k} \exp(\lambda_k z) + (\text{term which converges to 0 in} \\ &\quad \text{modulus as } m \rightarrow \infty). \end{aligned}$$

By Hurwitz's theorem, when θ_k is a rational multiple of π , for $k =$

$1, \dots, p$, the final set consists of the roots of finitely many trigonometric sums $\sum_{k=1}^p a_k e^{im\theta_k} \exp(\lambda_k z)$ as $m \rightarrow \infty$ ¹. Otherwise, the final set consists of limits of roots of infinitely many $\sum_{k=1}^p a_k e^{im\theta_k} \exp(\lambda_k z)$, as $m \rightarrow \infty$.

The roots of each such sum were investigated by Pólya [10] and Schwengeler [12]. They show that the roots of each such sum are asymptotic to the rays which emanate from the origin and are parallel to the normals to the sides of \bar{I} , the indicator diagram of I (in other words, the convex hull of $\{\bar{\lambda}_1, \dots, \bar{\lambda}_p\}$, where $\bar{\lambda}_j$ denotes the conjugate of λ_j). By asymptotic to is meant that for any sectors containing any these rays of angle ε , for any $\varepsilon > 0$, all but finitely many zeros of $T^{(m)}$ lie in these sectors.

REMARK. For the case when $|z| = \Delta$ contains only two vertices λ_1 and λ_k , say, and $\arg(\lambda_k) - \arg(\lambda_1)$ is an irrational multiple of π , then the final set for the balanced trigonometric sum $\sum_{k=1}^n a_k \exp(\lambda_k z)$ consists of the roots of infinitely many sums of the form $a_k e^{im\theta_k} \exp(\lambda_k z) + a_1 e^{im\theta_1} \exp(\lambda_1 z)$ as $m \rightarrow \infty$. The roots of these infinitely many sums are dense along the rays that they lie on. It would be interesting to know if this happens in general, that is, when $|z| = \Delta = \max_{1 \leq k \leq n} |\lambda_k|$ contains more than 2 vertices.

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¹ Since $\exp(im\theta_k)$ is periodic in m .

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