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A CLASS OF FUNDAMENTAL UNITS AND SOME CLASSES OF JACOBI-PERRON ALGORITHMS IN PURE CUBIC FIELDS

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In this paper, we consider some pure cubic fields of the form $K = \boldsymbol{Q}((D^3 \pm d)^{1/3})$ where $D, d \in N^*$ and $d \mid 3D^2$. Under certain conditions, we obtain the fundamental unit η of K by ruling out the case where η is the square of a unit. We also give three new classes of vectors, whose Jacobi-Perron Algorithms are periodic. The first ten vectors of a generalized Jacobi-Perron Algorithm are then written down. A generalization of the Bernstein formula is also achieved.

0. **Introduction and preliminaries.** Consider a pure cubic field of the form $K = \boldsymbol{Q}(\omega)$, where

(0.1) $\omega^3 = M = D^3 \pm d > 2$, with $D, d \in N^* = \{1, 2, \cdots\}$ and $d \mid 3D^2$.

H. J. Stender [9] showed that, when d is cube-free,

$$\varepsilon = \pm(\omega - D)^3 d^{-1} \tag{0.2}$$

is either the fundamental unit of K, its square or its cube, except for the fields $\boldsymbol{Q}(19^{1/3})$, $\boldsymbol{Q}(20^{1/3})$, $\boldsymbol{Q}(28^{1/3})$. For a certain class of integers M, he described explicitly the fundamental unit of K; see Theorem 1.1. In Chapter 1, we give other restrictions on M, under which the unit ε in (0.2) is the fundamental unit of K.

The rest of the work deals with the Jacobi-Perron Algorithm (JPA). The JPA is one of the generalizations of the ordinary continued fraction algorithm to higher dimensions [1]. Its definition and properties are recalled at the end of this introduction.

L. Bernstein [1] showed that the JPA of $a^{(0)} = (\omega, \omega^2)$ is periodic when

$$D \geqq d\ , \quad M = D^3 + d\ , \tag{0.3a}$$

$$(D, d) = (V, 3V)\ , \quad M = V^3 + 3V,\ V \in N^*\ , \quad V \geqq 2\ , \tag{0.3b}$$

$$(D, d) = (2V, 12V)\ , \quad M = 8V^3 + 12V,\ V \in N^*\ , \quad V \geqq 3\ , \tag{0.3c}$$

$$D \geqq 4d\ , \quad M = D^3 - d\ . \tag{0.3d}$$

In Chapter 2, we add to (0.3) the case

$$(D, d) = (2V, 3V)\ , \quad M = 8V^3 - 3V,\ V \in N^*\ . \tag{0.4}$$

Suppose now that a cube V^3 divides $\omega^3 = M = D^3 \pm d$ and let

$$\theta^3 = m = M/V^3 . \tag{0.5}$$

When

$$M=(tV^3+1)^3-1\ ,\quad \theta^3=M/V^3=t^3V^6+3t^2V^3+3t;\ t,\ V\in \boldsymbol{N}^*\ , \tag{0.6}$$

L. Bernstein [6] proved that the JPA of $(\theta, \theta^2)=(\omega/V, \omega^2/V^2)$ is periodic. Similarly, when

$$M = (8V-1)^3+1,\ \theta^3 = M/2^3 = 64V^3-24V^2+3V,\ V\in \boldsymbol{N}^*\ , \tag{0.7a}$$

$$M=(9tV^3+1)^3-1,\ \theta^3=M/(3V)^3=27t^3V^6+9t^2V^3+t;\ t,\ V\in \boldsymbol{N}^*\ , \tag{0.7b}$$

we prove in Chapters 3 and 4 that the JPA of $(\theta, \theta^2) = (\omega/2, \omega^2/4)$, resp. $(\theta, \theta^2) = (\omega/3V, \omega^2/9V^2)$, is periodic.

In Chapter 5, when

$$M=(tV^3+1)^3-(tV^3+1),\ \theta^3=M/V^3=t^3V^6+3t^2V^3+2t;\ t, V\in \boldsymbol{N}^*\ , \tag{0.8}$$

we write the first ten vectors of a generalized JPA of $(\theta, \theta^2) = (\omega/V, \omega^2/V^2)$; the case where t is a square was considered by L. Bernstein in [5], and we are using the same ideas.

Finally, in the concluding remarks, a generalization of the Bernstein formula is easily achieved.

Let $a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ be a vector of the real Euclidean vector space $\boldsymbol{R}^{n-1}$, $n \geqq 2$. A sequence $\langle a^{(v)}\rangle$ of vectors of $\boldsymbol{R}^{n-1}$ is called the JPA of $a^{(0)}$ if for all $v\in \boldsymbol{N}$,

$$a^{(v+1)} = \left(\frac{a_2^{(v)}-b_2^{(v)}}{a_1^{(v)}-b_1^{(v)}}, \cdots, \frac{a_{n-1}^{(v)}-b_{n-1}^{(v)}}{a_1^{(v)}-b_1^{(v)}}, \frac{1}{a_1^{(v)}-b_1^{(v)}}\right), \tag{0.9}$$

$$a_1^{(v)} \neq b_1^{(v)},\ b_i^{(v)} = [a_i^{(v)}] \quad (i = 1, \cdots, n-1)\ , \tag{0.10}$$

where $[\cdots]$ is the greatest integer function.

The JPA of $a^{(0)}$ is called periodic, if there exist two integers l, m with $l \geqq 0$, $m \geqq 1$ such that

$$a^{(v+m)} = a^{(v)} \quad (v = l, l+1, \cdots) . \tag{0.11}$$

The sequences

$$a^{(0)}, a^{(1)}, \cdots, a^{(l-1)} \quad \text{and} \quad a^{(l)}, a^{(l+1)}, \cdots, a^{(l+m-1)}$$

are called respectively the preperiod and the period of the periodic JPA, and l and m are their respective lengths. When l and m are minimal, the preperiod and the period are said to be primitive. If $l=0$, the JPA of $a^{(0)}$ is said to be purely periodic.

The following formula holds for any $v\in \boldsymbol{N}^*$:

$$(0.12)\quad \begin{vmatrix} 1 & A_0^{(v+1)} \cdots A_0^{(v+n-1)} \\ a_1^{(0)} & A_1^{(v+1)} \cdots A_1^{(v+n-1)} \\ \cdots & \cdots \qquad \cdots \\ a_{n-1}^{(0)} & A_{n-1}^{(v+1)} \cdots A_{n-1}^{(v+n-1)}) \end{vmatrix} = \frac{(-1)^{v(n-1)}}{\prod_{i=1}^{v} a_{n-1}^{(i)}}$$

$$= \frac{(-1)^{v(n-1)}}{A_0^{(v)} + A_0^{(v+1)}a_1^{(v)} + \cdots + A_0^{(v+n-1)}a_{n-1}^{(v)}}$$

where

$$(0.13)\qquad A_i^{(j)} = \delta_{ij} \quad (i, j = 0, 1, \cdots, n-1) ,$$

δ_{ij} being the Kronecker delta,

$$(0.14)\quad A_i^{(v+n)} = A_i^{(v)} + A_i^{(v+1)}b_1^{(v)} + \cdots + A_i^{(v+n-1)}b_{n-1}^{(v)}(i = 0, 1, \cdots, n-1) .$$

Let $\boldsymbol{K}_n = \boldsymbol{Q}(a_1^{(0)}, \cdots, a_{n-1}^{(0)})$. If the JPA of $a^{(0)}$ becomes periodic with primitive preperiod of length l and primitive period of length m, then H. Hasse and L. Bernstein [6] proved that

$$(0.15)\qquad \prod_{i=l}^{l+m-1} a_{n-1}^{(i)}$$

is a unit in $\boldsymbol{K}_n$. If the components of $a^{(0)}$ and $a^{(v)}$ for some $v \geqq 1$ are algebraic integers of $\boldsymbol{K}_n$, then L. Bernstein [2] proved that

$$(0.16)\qquad A_0^{(v)} + A_0^{(v+1)}a_1^{(v)} + \cdots + A_0^{(v+n-1)})a_{n-1}^{(v)}$$

is a unit in $\boldsymbol{K}_n$. Formulas (0.15) and (0.16) will be referred to as the Hasse-Bernstein and the Bernstein formula respectively. In this work, the units obtained with these formulas are fundamental, but it must be stressed that there is no known connection between the JPA and the fundamentality of units.

1. Fundamental units in some pure cubic fields. Let $\omega^3 = f^2g$ be a rational integer with fg square-free. Following Dedekind, we say that the pure cubic field $\boldsymbol{Q}(\omega)$ is of the first or second kind, according to whether $f^2g \not\equiv \pm 1 \pmod 9$ or $f^2g \equiv \pm 1 \pmod 9$. The numbers $1, \sqrt[3]{f^2g}, \sqrt[3]{fg^2}$ form an integral basis for $\boldsymbol{Q}(\omega)$ if it is of the first kind. The numbers $(1 + g\sqrt[3]{f^2g} + f\sqrt[3]{fg^2})/3, \sqrt[3]{f^2g}, \sqrt[3]{fg^2}$ form an integral basis for $\boldsymbol{Q}(\omega)$ if it is of the second kind, and each integer of $\boldsymbol{Q}(\omega)$ is representable in the form $(x + y\omega + zf^{-1}\omega)/3$ with $x, y, z \in \boldsymbol{Z}$. By Dirichlet's theorem, there is only one fundamental unit $\eta(<1)$ in this field, and any unit can be expressed as $\pm\eta^n(n \in \boldsymbol{Z})$. The following theorem of H. J. Stender [9] describes the fundamental unit of $\boldsymbol{K}$ for a certain class of integers M.

THEOREM 1.1. *Let*

$$\begin{cases} c = 1 \quad \text{or} \quad 3;\ D, d \in N^* \quad \text{with} \\ (c, D) = 1, d \quad \text{cube-free and } d \mid D^2 . \end{cases}$$

Assume

$$M = D^3 \pm cd > 1, \omega = M^{1/3} , \quad K = Q(\omega) .$$

(i) *Let $c = 1$ and further let $d = 1$ or M/d be a cube. Then*

$$\eta = \pm(\omega - D)/d^{1/3}$$

is the fundamental unit of K, except when $(D, d) = (8, 1), (19, 1), (3, 1)$ in the plus case and $(D, d) = (57, 19^2), (70, 2\cdot 5^2), (14, 2\cdot 7^2)$ in the minus case, where η is the square of the fundamental unit.

(ii) *Let $c = 1$ or 3; if $c = 1$, assume $d \neq 1$ and M/d different from a cube. If M or $D^3 d^{-1}$ is cube-free, then*

$$\varepsilon = \pm(\omega - D)^3/cd$$

is the fundamental unit of K, except when $(D, cd) = (2, 2), (1, 3)$ in the plus case and $(D, cd) = (5, 25), (2, 6)$ in the minus case, where ε is the square of the fundamental unit.

COROLLARY 1.2. *Let K be a pure cubic field. If there exists $D \in N^*$ such that*

$$K = Q(\sqrt[3]{D^3 + 1}) \quad \text{or} \quad K = Q(\sqrt[3]{D^3 - 1}) ,$$

then

$$\eta = -D + \sqrt[3]{D^3 + 1} , \quad \text{resp.} \quad \eta = D - \sqrt[3]{D^3 - 1} ,$$

is the fundamental unit of K, except when $D = 8, 19, 3$ in the plus case.

In Corollary 1.2, $D^3 + 1$, resp. $D^3 - 1$, need not be cube-free.

The purpose of this chapter is to prove the following theorem.

THEOREM 1.3. *Let $D, d, V \in \{2, 3, \cdots\}$ be such that $(D, d) = (D_0 d_1 d_2, d_1 d_2^2)$ with D_0, d_1, d_2 congruent to $+1$ or $-1 (\mathrm{mod}\, V^3)$. Suppose that*

$$\theta^3 = m = \omega^3/V^3 = M/V^3 = (D^3 \pm d)/V^3$$

is a cube-free integer >2. Then

$$\eta = \begin{cases} \mp(\theta^{1/2} - V) & \text{if} \quad (D, d) = (V^3 \mp 1, (V^3 \mp 1)^2) , \\ \pm(\omega - D)^3/d & \text{if} \quad (D, d) \neq (V^3 \mp 1, (V^3 \mp 1)^2) , \end{cases}$$

is the fundamental unit of $\boldsymbol{Q}(\theta) = \boldsymbol{Q}(\omega)$.

REMARK 1.4. Let $i = 1$ or -1; $a, b, c \in \boldsymbol{N}$; $V \in \{2, 3, \cdots\}$. The integer $M = D^3 \pm d = D_0^3 d_2^3 d_1^3 \pm d_2^2 d_1$ in Theorem 1.3 has one of the forms

$$(1.1) \qquad (cV^3 + 1)^3(aV^3 \mp 1)^3(bV^3 + i)^3 \pm (aV^3 \mp 1)^2(bV^3 + i)\ ,$$

$$(1.2) \qquad (cV^3 - 1)^3(aV^3 \pm 1)^3(bV^3 + i)^3 \pm (aV^3 \pm 1)^2(bV^3 + i)\ ,$$

with the following restrictions: a and b are not both 0; if $b = 0$, $i = 1$; $c \neq 0$ in (1.2); $a \neq 0$ in the upper case of (1.1) and in the lower case of (1.2).

Note that $(V^3, d) = 1$ and $d \mid m$, so that d is cube-free. Let

$$(1.3) \qquad d = u^2 s \quad \text{with} \quad us \quad \text{square-free}\ .$$

As $m = f^2 g$ with $(f, g) = 1$, then $u \mid f$; let $f = ut$ with $t \in \boldsymbol{N}^*$.

When $(D, d) = (V^3 \mp 1, (V^3 \mp 1)^2)$, which happens if and only if $M/d = V^3$, the conclusion follows from the first part of Theorem 1.1.

In what follows, let $M/d \neq V^3$. According to H. J. Stender [9, Chapter 3],

$$(1.4) \qquad \varepsilon = \pm(\omega - D)^3/d = 1 \pm 3D^2 d^{-1} V\theta \mp 3Dd^{-1}V^2\theta^2$$

is the fundamental unit of $\boldsymbol{Q}(\theta)$ or its square. The latter case will be ruled out by contradiction exactly as in [7] and [8].

(1) *Fields of the first kind.* Suppose ε is the square of a unit $x + y\theta + zf^{-1}\theta^2 (x, y, z \in \boldsymbol{Z})$ of $\boldsymbol{Q}(\theta)$. Then comparing coefficients and noting that $f = ut$, we obtain

$$(1.5) \qquad x^2 + 2utgyz = 1\ ,$$

$$(1.6) \qquad 2xy + gz^2 = \pm 3D^2 d^{-1} V\ ,$$

$$(1.7) \qquad uty^2 + 2xz = \mp 3t(D/us)V^2\ .$$

When $d \mid D$, we conclude from (1.5) and (1.7) that $(x, f) = 1$ and $f \mid 2z$, so $f^2 \leqq 4z^2$. Similarly, when $d \mid D^2$, we have $t^2 \leqq 4z^2$.

(A) *Plus case.* Let us consider the case where $\theta^3 = (D^3 + d)/V^3$. From (1.7), $xz < 0$ (in particular, $z \neq 0$). From (1.5), $yz \leqq 0$; but $y = 0$ implies

$$x^2 = 1 \quad \text{and} \quad m = f^2 g = 4d/3V^3\ ,$$

and this cannot happen. Therefore $yz < 0$, so that $xy > 0$. Finally, from (1.6) we obtain these crucial inequalities:

(1.8) $\quad Dd < 12V^4$ when $d \mid D$, and $Ds < 12V^4$ when $d \mid D^2$.

We want to show there is no pair (D, d) *satisfying* (1.8).

(i) Suppose $d \mid D$. Consider first the case $d_2 = 1$, where $d = d_2^2 d_1$. By Remark 1.4, we have

$$(D, d) = ((cV^3 - 1)(bV^3 + i), bV^3 + i);\ b, c, V \in N^*, V \neq 1, i = 1 \text{ or } -1,$$

so $Dd \geqq (V^3 - 1)^3 > 12V^4$ for $V \geqq 2$. Similarly, when $d_2 \neq 1$, $Dd > 12V^4$ for $V \geqq 2$. In brief, when $d \mid D$, there is no pair (D, d) for which $Dd < 12V^4$.

(ii) Suppose $d \mid D^2$ and $d \nmid D$, so that $a \neq 0$ in (1.1) and (1.2). Now the integer s defined in (1.3) is either equal to or is greater than 1.

Suppose first $s > 1$, so that $b \neq 0$. If $c \neq 0$, then

$$Ds \geqq D \geqq (V^3 - 1)^3 > 12V^4 \quad \text{for} \quad V \geqq 2. \tag{1.9}$$

Let $c = 0$. Then

$$Ds \geqq D \geqq (V^3 - 1)^2 > 12V^4 \quad \text{for} \quad V \geqq 4. \tag{1.10}$$

When $V = 3$

$$Ds \geqq D \geqq 675ab > 972 = 12V^4 \quad \text{for} \quad ab \geqq 2, \tag{1.11}$$

and the only case where $a = 1 = b$ and where m is cube-free is $(D, d) = (26 \cdot 28, 26^2 \cdot 28)$, but $Ds = 26 \cdot 28 \cdot 7 > 972$. When $V = 2$,

$$Ds \geqq D \geqq 48ab \geqq 192 = 12V^4 \quad \text{for} \quad ab \geqq 4, \tag{1.12}$$

and for all cases where $ab \leqq 3$, it is directly checked that $Ds > 192 = 12V^4$.

Consider now the case $s = 1$. Hence $bV^3 + i$ is a square > 1 or $b = 0$. First, let $bV^3 + i$ be a square > 1. If $c \neq 0$, we obtain (1.9). If $c = 0$, then we get (1.10). When $V = 3$, (1.11) holds, and the case $a = 1 = b$ is ruled out because $27 + i$ is not a square. When $V = 2$, (1.12) holds, so that when $ab \leqq 3$, we need only take care of the cases $(D, d) = (7 \cdot 9, 7^2 \cdot 9), (7 \cdot 25, 7^2 \cdot 25)$; in each case, a contradiction is obtained from (1.6). Finally, suppose $b = 0$. If $c = 0$, then $D^3 d^{-1} = aV^3 - 1$ is cube-free, and we can apply Theorem 1.1 (ii). Therefore suppose $c \neq 0$. Then (1.10) holds. Proceeding as above, we can show that for $V = 3$ or 2, the only cases permitted by (1.8) are $(D, d, V) = (26 \cdot 28, 26^2, 3), (9 \cdot 7, 7^2, 2), (17 \cdot 7, 7^2, 2), (15 \cdot 9, 15^2, 2)$, but, in each case, a contradiction is again obtained from (1.6).

(B) *Minus case.* Let us consider the case $\theta^3 = (D^3 - d)/V^3$. Similarly, $xy < 0$ and $yz \leqq 0$. But if $z = 0$, then

$$x^2 = 1 \quad \text{and} \quad 3D^3 = 4d ,$$

which cannot occur. Hence $yz < 0$, so that $xz > 0$. From (1.6), we obtain

$$8|xy| \geqq \begin{cases} m + 12D^2d^{-1}V & \text{if } d|D , \\ msd^{-1} + 12D^2d^{-1}V & \text{if } d|D^2 . \end{cases} \tag{1.13}$$

We can proceed as in Chapter 4 of [8] to show

$$|xy| < (1 + 24D^2d^{-1}V\theta + 4D\sqrt{6d^{-1}V\theta})/9\theta . \tag{1.14}$$

Combining (1.13) and (1.14), we obtain

$$\begin{aligned} \frac{(D-1)h}{V^3} &< \frac{8d}{9D^2\theta} + \frac{28V}{3} + \frac{32}{9}\sqrt{\frac{6dV}{D^2\theta}} \\ &< \frac{8}{9\theta} + \frac{28V}{3} + \frac{32}{9}\sqrt{\frac{6V}{\theta}} , \end{aligned} \tag{1.15}$$

where $h = d$ if $d|D$ and $h = s$ if $d|D^2$. It is easy to show that $\theta > V^2 - 1$, so that we obtain from (1.15) these crucial inequalities:

$$(D-1)d < 13V^4 \text{ when } d|D , \text{ and } (D-1)s < 13V^4 \text{ when } d|D^2 . \tag{1.16}$$

(i) Suppose $d|D$. As D and d are integers $\geqq 2$ congruent to $+1$ or $-1 (\operatorname{mod} V^3)$, then $(D-1)d \geqq (V^3 - 2)(V^3 - 1) > 13V^4$ for $V \geqq 4$. As is easily seen, the only cases to consider are $(D, d, V) = (28, 28, 3)$, $(26, 26, 3)$, $(9, 9, 2)$, $(7, 7, 2)$. For example, if $m = 2^2 \cdot 7 \cdot 29$, then, from (1.6), (1.5), (1.7) and the fact that $xz > 0$, we obtain

$$7|xy, 7|y \quad \text{and} \quad y^2 < 27 ,$$

which is absurd. We have a similar contradiction when $m = 3^2 \cdot 2 \cdot 5$. Because D^3d^{-1} is cube-free, one takes care of the two other cases with the help of Theorem 1.1 (ii).

(ii) Suppose $d|D^2$ and $d \nmid D$, so that $a \neq 0$ in (1.1) and (1.2). Suppose first $s > 1$ so that $b \neq 0$. Corresponding to inequalities (1.9), (1.10), (1.11), and (1.12), we obtain

$$(D-1)s \geqq D - 1 \geqq (V^3 - 1)^3 - 1 > 13V^4 \quad \text{for} \quad V \geqq 2 , \tag{1.17}$$

$$(D-1)s \geqq D - 1 \geqq (V^3 - 1)^2 - 1 > 13V^4 \quad \text{for} \quad V \geqq 4 , \tag{1.18}$$

$$(D-1)s \geqq D - 1 \geqq 675ab - 1 > 1053 = 13V^4 \quad \text{for} \quad ab \geqq 2 , \tag{1.19}$$

$$(D-1)s \geqq D - 1 \geqq 48ab - 1 > 208 = 13V^4 \quad \text{for} \quad ab \geqq 5 , \tag{1.20}$$

respectively. Finally, it is easy to check that (1.16) excludes all possible cases.

Suppose $s = 1$. First, let $bV^3 + i$ be a square >1. If $c \neq 0$, we obtain (1.17). Suppose then $c = 0$ so that we get (1.18). When $V = 3$, (1.19) holds and the case $a = 1 = b$ is ruled out because $V^3 + i$ is not a square. When $V = 2$, (1.20) holds, so that when $ab \leqq 4$, we need only take care of $(D, d) = (17 \cdot 9, 17^2 \cdot 9)$. Finally, suppose $b = 0$. When $c = 0$, D^3d^{-1} is cube-free and one can apply Theorem 1.1 (ii). Therefore suppose $c \neq 0$. Then (1.18) holds. Proceeding as above, we show that the only cases permitted by (1.16) (when $Q(\theta)$ is a field of the first kind and when $d \mid D^2$ but $d \nmid D$) are $(D, d, V) = (7 \cdot 15, 7^2, 2)$, $(7 \cdot 23, 7^2, 2)$, $(7 \cdot 15, 15^2, 2)$; for each case, a contradiction is again obtained from (1.5), (1.6), and (1.7).

(2) *Fields of the second kind.* When $Q(\theta)$ is a field of the second kind, the proof is about the same and we shall omit many details. Suppose ε is the square of a unit $(x + y\theta + zf^{-1}\theta^2)/3$ of $Q(\theta)$, where $x, y, z \in Z$. Then comparing coefficients, we obtain

$$x^2 + 2utgyz = 9\,, \tag{1.21}$$

$$2xy + gz^2 = \pm 27D^2d^{-1}V\,, \tag{1.22}$$

$$uty^2 + 2xz = \mp 27t(D/us)V^2\,. \tag{1.23}$$

As f^2g is congruent to $+1$ or $-1 (\mathrm{mod}\ 9)$, then (1.21) implies $(x, f) = 1$; therefore, when $d \mid D$, (1.23) implies $f \mid 2z$, so $f^2 \leqq 4z^2$. Similarly, when $d \mid D^2$, we have $t^2 \leqq 4z^2$.

(A) *Plus case.* Again $xy > 0$ and we have

$$Dd < 108V^4 \text{ when } d \mid D, \text{ and } Ds < 108V^4 \text{ when } d \mid D^2\,. \tag{1.24}$$

(i) Let $d \mid D$. The only cases allowed by (1.24) when m is cube-free and when $Q(\theta)$ is of the second kind are $(D, d, V) = (7 \cdot 7, 7, 2)$, $(31 \cdot 7, 7, 2)$; a contradiction is obtained as usual.

(ii) Suppose $d \mid D^2$ and $d \nmid D$. When $s > 1$ and $c \neq 0$, one verifies that (1.24) excludes all cases. Let $s > 1$ and $c = 0$. We have $Ds > D \geqq (V^2 - 1)^7 > 108V^4$ for $V \geqq 11$. When $V = 10, 9, 8, 7, 6, 5, 4, 3, 2$, then $Ds > D \geqq (V^3 - 1)(bV^3 - 1) \geqq 108V^4$ for $b \geqq 2, 2, 2, 3, 4, 5, 7, 13, 31$ respectively. Let us explain the case $V = 2$. For each b, $1 \leqq b \leqq 30$, the value of s related with the value of $8b + i$ can be used in order to obtain a better bound for Ds, so that only a few possibilities for the integer a remain to investigate by hand. After having done this for $V = 2, 3, \cdots, 10$, we conclude that (1.24) excludes all possible cases.

When $s = 1$ and when $bV^3 + i$ is a square >1, we easily get a contradiction for $(D, d, V) = (7 \cdot 121, 7^2 \cdot 121, 2)$, the only case permitted by (1.24).

Finally, suppose $s = 1$ and $b = 0$. We can assume $c \neq 0$ by Theorem 1.1(ii). We then have to take care of the following cases:

(D, d, V)	$m(=f^2g)$	(D, d, V)	$m(=f^2g)$
$(215\cdot 433, 433^2, 6)$	$433^2\cdot 23\cdot 37\cdot 41\cdot 571$	$(9\cdot 55, 55^2, 2)$	$2^2\cdot 5^2\cdot 11^2\cdot 7\cdot 179$
$(217\cdot 431, 431^2, 6)$	$431^2\cdot 20389319$	$(9\cdot 71, 71^2, 2)$	$71^2\cdot 2\cdot 5\cdot 647$
$(249\cdot 251, 251^2, 5)$	$251^2\cdot 2^2\cdot 7\cdot 683\cdot 1621$	$(9\cdot 127, 127^2, 2)$	$127^2\cdot 71\cdot 163$
$(65\cdot 383, 383^2, 4)$	$383^2\cdot 29\cdot 56671$	$(9\cdot 143, 143^2, 2)$	$11^2\cdot 13^2\cdot 83\cdot 157$
$(129\cdot 127, 127^2, 4)$	$127^2\cdot 2^2\cdot 7\cdot 167\cdot 911$	$(7\cdot 193, 193^2, 2)$	$193^2\cdot 5^2\cdot 331$
$(127\cdot 193, 193^2, 4)$	$193^2\cdot 5\cdot 1235431$	$(17\cdot 15, 17^2, 2)$	$2^2\cdot 17^2\cdot 11\cdot 163$
$(28\cdot 53, 53^2, 3)$	$53^2\cdot 41\cdot 1051$	$(15\cdot 73, 73^2, 2)$	$73^2\cdot 13\cdot 23\cdot 103$
$(26\cdot 55, 55^2, 3)$	$5^2\cdot 11^2\cdot 35803$	$(15\cdot 89, 89^2, 2)$	$89^2\cdot 37547$
$(26\cdot 109, 109^2, 3)$	$109^2\cdot 5\cdot 23\cdot 617$	$(17\cdot 95, 95^2, 2)$	$5^2\cdot 19^2\cdot 2\cdot 31\cdot 941$
$(28\cdot 107, 107^2, 3)$	$107^2\cdot 5\cdot 127\cdot 137$	$(25\cdot 31, 31^2, 2)$	$31^2\cdot 191\cdot 317$
$(26\cdot 298, 298^2, 3)$	$2^2\cdot 149^2\cdot 17\cdot 11411$	$(23\cdot 41, 41^2, 2)$	$2^2\cdot 41^2\cdot 7\cdot 17\cdot 131$
$(55\cdot 134, 134^2, 3)$	$2^2\cdot 67^2\cdot 7\cdot 117959$	$(39\cdot 17, 17^2, 2)$	$17^2\cdot 233\cdot 541$
$(82\cdot 26, 26^2, 3)$	$2^2\cdot 13^2\cdot 530947$	$(41\cdot 23, 23^2, 2)$	$2^2\cdot 23^2\cdot 49537$
$(109\cdot 53, 53^2, 3)$	$53^2\cdot 2\cdot 1271047$	$(49\cdot 31, 31^2, 2)$	$31^2\cdot 2\cdot 5\cdot 45589$
$(107\cdot 55, 55^2, 3)$	$5^2\cdot 11^2\cdot 2\cdot 7\cdot 178247$	$(63\cdot 17, 17^2, 2)$	$5^2\cdot 17^2\cdot 2\cdot 10627$
$(163\cdot 26, 26^2, 3)$	$2^2\cdot 13^2\cdot 4170349$	$(65\cdot 23, 23^2, 2)$	$23^2\cdot 11\cdot 71777$
$(244\cdot 26, 26^2, 3)$	$2^2\cdot 13^2\cdot 5\cdot 11\cdot 167\cdot 1523$	$(87\cdot 17, 17^2, 2)$	$17^2\cdot 1399319$
$(325\cdot 26, 26^2, 3)$	$2^2\cdot 13^2\cdot 19\cdot 1739827$		

For example, when $m = 251^2\cdot 2^2\cdot 7\cdot 683\cdot 1621$, we have from (1.22)

$$z^2 = 1,\ xy = 311\cdot 997$$

and any choice for x and y contradicts (1.21). For the other cases, a contradiction is more easily obtained.

(B) *Minus case.* Again $xz > 0$ and we have

$$(1.25)\qquad \begin{cases} (D-1)d < 113V^4 & \text{when } d\,|\,D\,, \\ \text{and } (D-1)s < 113V^4 & \text{when } d\,|\,D^2\,. \end{cases}$$

(i) When $d|D$, the only case to take care of is $(D, d, V) = (17\cdot 7, 7, 2)$.

(ii) Suppose $d|D^2$ and $d\nmid D$. When $s > 1$, there is no case permitted by (1.25). The same conclusion holds when $s = 1$ and $bV^3 + i$ is a square >1.

Finally, if $s = 1$ and $b = 0$, the only individual cases to consider are the following ones:

(D, d, V)	$m(= f^2g)$	(D, d, V)	$m(= f^2g)$
$(342\cdot 685, 685^2, 7)$	$5^2\cdot 137^2\cdot 83\cdot 193\cdot 4987$	$(9\cdot 17, 17^2, 2)$	$17^2\cdot 1549$
$(649\cdot 217, 217^2, 6)$	$7^2\cdot 31^2\cdot 2\cdot 277\cdot 495713$	$(7\cdot 23, 23^2, 2)$	$23^2\cdot 2\cdot 17\cdot 29$
$(647\cdot 215, 215^2, 6)$	$5^2\cdot 43^2\cdot 2\cdot 113\cdot 137\cdot 8707$	$(9\cdot 73, 73^2, 2)$	$72^2\cdot 2^2\cdot 1663$
$(126\cdot 251, 251^2, 5)$	$251^2\cdot 5\cdot 79\cdot 10169$	$(9\cdot 89, 89^2, 2)$	$89^2\cdot 2\cdot 5\cdot 811$
$(124\cdot 374, 374^2, 5)$	$2^2\cdot 11^2\cdot 17^2\cdot 29\cdot 229\cdot 859$	$(7\cdot 95, 95^2, 2)$	$5^2\cdot 19^2\cdot 4073$
$(63\cdot 127, 127^2, 4)$	$127^2\cdot 496187$	$(9\cdot 145, 145^2, 2)$	$5^2\cdot 29^2\cdot 73\cdot 181$
$(65\cdot 193, 193^2, 4)$	$193^2\cdot 2\cdot 414083$	$(9\cdot 161, 161^2, 2)$	$7^2\cdot 23^2\cdot 17\cdot 863$
$(26\cdot 134, 134^2, 3)$	$2^2\cdot 67^2\cdot 19\cdot 4591$	$(7\cdot 239, 239^2, 2)$	$239^2\cdot 10247$
$(28\cdot 190, 190^2, 3)$	$2^2\cdot 5^2\cdot 19^2\cdot 179\cdot 863$	$(15\cdot 55, 55^2, 2)$	$5^2\cdot 11^2\cdot 23203$
$(55\cdot 109, 109^2, 3)$	$109^2\cdot 2\cdot 41\cdot 8191$	$(15\cdot 71, 71^2, 2)$	$71^2\cdot 7\cdot 11\cdot 389$
$(53\cdot 107, 107^2, 3)$	$107^2\cdot 2\cdot 294997$	$(23\cdot 31, 31^2, 2)$	$31^2\cdot 47147$
$(80\cdot 26, 26^2, 3)$	$2^2\cdot 13^2\cdot 229\cdot 2153$	$(25\cdot 41, 41^2, 2)$	$41^2\cdot 2\cdot 40039$
$(134\cdot 53, 53^2, 3)$	$53^2\cdot 17\cdot 277829$	$(33\cdot 17, 17^2, 2)$	$17^2\cdot 2\cdot 38183$
$(161\cdot 26, 26^2, 3)$	$2^2\cdot 13^2\cdot 5\cdot 17\cdot 47279$	$(31\cdot 23, 23^2, 2)$	$23^2\cdot 41\cdot 2089$
$(242\cdot 26, 26^2, 3)$	$2^2\cdot 13^2\cdot 181\cdot 75401$	$(47\cdot 31, 31^2, 2)$	$31^2\cdot 2\cdot 11\cdot 18287$
$(323\cdot 26, 26^2, 3)$	$26^2\cdot 32450183$	$(57\cdot 17, 17^2, 2)$	$17^2\cdot 5\cdot 78707$
		$(105\cdot 17, 17^2, 2)$	$17^2\cdot 2459953$

For example, when $m = 2^2\cdot 11^2\cdot 17^2\cdot 29\cdot 229\cdot 859$, we have from (1.21), (1.22) and (1.23)

$$2\mid z,\ 4\mid 2xy,\ 2\mid y;\ 4\nmid z,\ 4\nmid y;\ |y| \leqq 14\ .$$

Therefore $y^2 = 4$, 36, 100 or 196; corresponding to each case, we obtain the value of xz from (1.23), and each possible value of x contradicts (1.21). We have similar contradictions for the other cases. This terminates the proof of Theorem 1.3.

Theorem 1.3 provides us with another class of integers m for which as in [3] the Delaunay-Nagell Diophantine equation $x^3 + my^3 = 1$ is not solvable.

2. The case $M = 8V^3 - 3V$. We want to show that the JPA of $a^{(0)} = (\omega, \omega^2)$, when

$$\omega^3 = M = 8V^3 - 3V\ , \quad V \in N^*\ , \tag{2.1}$$

yields the following vectors:

$$a^{(0)} = (\omega, \omega^2)\ ;$$

$$b^{(0)} = (2V - 1, 4V^2 - 2)\ .$$

$$a^{(1)} = \left(\frac{(-4V + 3)\omega^2 + (4V^2 + V - 2)\omega + (8V^3 - 2V^2 - 5V + 2)}{12V^2 - 9V + 1}\ , \right.$$
$$\left. \frac{\omega^2 + (2V - 1)\omega + (4V^2 - 4V + 1)}{12V^2 - 9V + 1}\right)\ ;$$

$$b^{(1)} = (1, 1)\ .$$

$$a^{(2)} = \left(\frac{(V-1)\omega^2 + (2V^2+V-1)\omega + (4V^3-4V^2+2V)}{12V^3-9V^2+1}, \right.$$
$$\left.\frac{(4V^2-3V+1)\omega^2 + (8V^3-6V^2-2V+2)\omega + (16V^4-12V^3-V+1)}{12V^3-9V^2+1}\right);$$

$$b^{(2)} = (1, 4V-1) .$$

$$a^{(3)} = \left(\frac{-\omega^2 + V\omega + (12V^3-7V^2-4V+2)}{3V^2-1}, \right.$$
$$\left.\frac{V\omega^2 + (2V^2-1)\omega + (4V^3-3V^2-V+1)}{3V^2-1}\right);$$

$$b^{(3)} = (4V-4, 4V-2) .$$

$$a^{(4)} = \left(\frac{(-V+2)\omega^2 + (10V^2+V-4)\omega + (-16V^3+11V^2+6V-4)}{21V^2-8}, \right.$$
$$\left.\frac{2\omega^2 + V\omega + (11V^2-4)}{21V^2-8}\right);$$

$$b^{(4)} = (0, 1) .$$

$$a_1^{(5)} = \frac{(-4V+2)\omega^2 + (4V^2-V-1)\omega + (8V^3-2V^2-3V+1)}{48V^4-32V^3-15V^2+12V-1},$$

$$a_2^{(5)} = \frac{\left(\begin{array}{c}(4V^2+3V-3)\omega^2 + (8V^3-6V^2-2V+2)\omega \\ +(16V^4-24V^3-V^2+9V-2)\end{array}\right)}{48V^4-32V^3-15V^4+12V-1};$$

$$b^{(5)} = (0, 1) .$$

$$a^{(6)} = (\omega - 1, \omega^2 + (2V-1)\omega + (4V^2-1)) ;$$
$$b^{(6)} = (2V-2, 12V^2-2V-3) .$$

$$a^{(7)} = \left(\frac{(-6V+4)\omega^2 + (5V-3)\omega + (24V^3-20V^2+2)}{12V^2-9V+1}, \right.$$
$$\left.\frac{\omega^2 + (2V-1)\omega + (4V^2-4V+1)}{12V^2-9V+1}\right);$$

$$b^{(7)} = (0, 1) .$$

$$a_1^{(8)} = \frac{(3V-2)\omega^2 + (12V^2-V-4)\omega + (-8V^2+5V)}{72V^4-63V^2+9V+8},$$

$$a_2^{(8)} = \frac{\left(\begin{array}{c}(12V^2-9V+1)\omega^2 + (24V^3-24V^2-4V+6)\omega \\ +(48V^4-24V^3-14V^2+4)\end{array}\right)}{72V^4-63V^2+9V+8};$$

$$b^{(8)} = (0, 1) .$$

$$a^{(9)} = \left(\frac{-\omega^2 + V\omega + (8V^2-6V)}{3V}, \quad \frac{2\omega^2 + V\omega + (-4V^2+3V)}{3V}\right);$$

$$b^{(9)} = (2V - 2, 2V) .$$

$$a^{(10)} = \left(\frac{(V-1)\omega^2 + (2V^2 + V - 1)\omega + (4V^3 - 7V^2 - V + 2)}{3V^2 - 1}, \right.$$
$$\left. \frac{V\omega^2 + (2V^2 - 1)\omega + (4V^3 - V)}{3V^2 - 1}\right) ;$$

$$b^{(10)} = (4V - 4, 4V - 1) .$$

$$a_1^{(11)} = \frac{(-4V+3)\omega^2 + (4V^2 + V - 2)\omega + (48V^4 - 48V^3 - 7V^2 + 18V - 4)}{48V^4 - 56V^3 - 3V^2 + 21V - 8},$$

$$a_2^{(11)} = \frac{\begin{pmatrix}(4V^2 - 3V + 1)\omega^2 + (8V^3 - 6V^2 - 2V + 2)\omega \\ +(16V^4 - 24V^3 + 5V^2 + 9V - 4)\end{pmatrix}}{48V^4 - 56V^3 + 3V^2 + 21V - 8} ;$$

$$b^{(11)} = (1, 1) .$$

$$a^{(12)} = (\omega + 1, \omega^2 + (2V - 1)\omega + (4V^2 - 2)) ;$$
$$b^{(12)} = (2V, 12V^2 - 2V - 4) .$$

$$a^{(13)} = a^{(7)} .$$

Because the a's are obtained from the recursion formulas defined in [6], we simply have to prove that the b's are correctly stated.

It is easy to show that the inequalities

$$2V - 1/4V - 1/8V^3 < \omega < 2V , \tag{2.2}$$

$$4V^2 - 1 - 1/2V^2 < \omega^2 < 4V^2 - 1 \tag{2.3}$$

hold, so that we have

$$1 < \frac{12V^2 - 6V - 5/4 - 1/8V^2 + 1/4V^3}{12V^2 - 9V + 1} < a_1^{(1)}$$
$$< \frac{12V^2 - 5V - 1 + 2/V - 3/2V^2}{12V^2 - 9V + 1} < 2 .$$

This implies $b_1^{(1)} = 1$. The other b's are similarly obtained. However, we need for $b_1^{(3)}$ the more refined inequality $\omega < 2V - 1/4V$, and sometimes we have to check directly the case $V = 1$.

The components of $a^{(6)}$ are algebraic integers so that the Bernstein formula will produce in $\boldsymbol{Q}(\omega)$ the unit

$$\begin{aligned} e_1 &= A_0^{(6)} + A_0^{(7)}a_1^{(6)} + A_0^{(8)}a_2^{(6)} \\ &= (64V^4 - 24V^2 + 1) + (32V^3 - 8V)\omega + (16V^2 - 2)\omega^2 \\ &= -3V(\omega - 2V)^{-3} . \end{aligned}$$

The unit obtained from the Hasse-Bernstein formula is

$$e_2 = \prod_{i=7}^{12} a_2^{(i)} .$$

The time needed to calculate the above expression is reduced if we note that the JPA of $c^{(0)} = (a_1^{(12)}, a_2^{(12)})$ is purely periodic with primitive length 6; the A's in relation with the JPA of $c^{(0)}$ can be calculated and the last equality of formula (0.12) is used to get

$$\begin{aligned} e_2 &= \prod_{i=7}^{12} a_2^{(i)} = \prod_{i=1}^{6} c_2^{(i)} = (4V-1) + (16V^2-4V-2)(\omega+1) \\ &\quad + (16V^2-2)(\omega^2+(2V-1)\omega+(4V^2-2)) \\ &= -3V(\omega-2V)^{-3} = e_1 . \end{aligned}$$

In conclusion, we obtain the same unit. By Theorem 1.1, this is the inverse of the fundamental unit of $\boldsymbol{Q}(\sqrt[3]{M})$ when M is cube-free.

3. The case $m = 64V^3 - 24V^2 + 3V$. When

$$\text{(3.1)} \qquad \begin{cases} \theta^3 = m = 64V^3 - 24V^2 + 3V = M/2^3 , \\ \quad \text{with } M = (8V-1)^3 + 1,\ V \in N^* , \end{cases}$$

the JPA of (θ, θ^2) yields the following vectors.

$$a^{(0)} = (\theta, \theta^2) ;$$
$$b^{(0)} = (4V-1, 16V^2-4V).$$

$$a^{(1)} = \left(\frac{(-4V+1)\theta^2 + (8V^2-V)\theta + (32V^3-12V^2+V)}{24V^2-9V+1}, \frac{\theta^2 + (4V-1)\theta + (16V^2-8V+1)}{24V^2-9V+1}\right);$$

$$b^{(1)} = (0, 1) .$$

$$a_1^{(2)} = \frac{(8V^2-3V)\theta^2 + (32V^3-4V^2)\theta + (128V^4-80V^3+12V^2)}{192V^4-80V^3+9V^2} ,$$

$$a_2^{(2)} = \frac{(8V^2-V)\theta^2 + (32V^3-20V^2+3V)\theta + (128V^4-48V^3+4V^2)}{192V^4-80V^3+9V^2} ;$$

$$b^{(2)} = (1, 1) .$$

$$a_1^{(3)} = \frac{(-2V+1)\theta^2 + (16V^2-3V)\theta + (192V^4-96V^3+15V^2)}{192V^4-64V^3+15V^2-3V} ,$$

$$a_2^{(3)} = \frac{(8V^2+V)\theta^2 + (32V^3-12V^2+3V)\theta + (-64V^4+48V^3-3V^2)}{192V^4-64V^3+15V^2-3V};$$

$$b^{(3)} = (1, 1) .$$

$$a^{(4)} = (\theta + 4V, \theta^2 + (4V-1)\theta + (16V^2-6V)) ;$$
$$b^{(4)} = (8V-1, 48V^2-16V) .$$

$$a^{(5)} = \left(\frac{(-6V+2)\theta^2 + (5V-1)\theta + (96V^3 - 40V^2 + 4V)}{24V^2 - 9V + 1}, \right.$$
$$\left. \frac{\theta^2 + (4V-1)\theta + (16V^2 - 8V + 1)}{24V^2 - 9V + 1}\right);$$

$$b^{(5)} = (1, 1).$$

$$a_1^{(6)} = \frac{\left(\begin{array}{c}(24V^2 - 15V + 2)\theta^2 + (96V^3 - 60V^2 + 14V - 1)\theta \\ +(384V^4 - 384V^3 + 122V^2 - 11V)\end{array}\right)}{576V^4 - 432V^3 + 93V^2 - 1},$$

$$a_2^{(6)} = \frac{\left(\begin{array}{c}(24V^2 - 15V + 3)\theta^2 + (96V^3 - 84V^2 + 21V - 1)\theta \\ +(384V^4 - 288V^3 + 78V^2 - 11V + 1)\end{array}\right)}{576V^4 - 432V^3 + 93V^2 - 1};$$

$$b^{(6)} = (1, 1).$$

$$a_1^{(7)} = \frac{(24V^2 - 9V + 1)\theta + (576V^4 - 528V^3 + 189V^2 - 31V + 2)}{576V^4 - 432V^3 + 129V^2 - 18V + 1},$$

$$a_2^{(7)} = \frac{\left(\begin{array}{c}(24V^2 - 9V + 1)\theta^2 + (96V^3 - 60V^2 + 13V - 1)\theta \\ +(-192V^4 + 240V^3 - 95V^2 + 16V - 1)\end{array}\right)}{576V^4 - 432V^3 + 129V^2 - 18V + 1};$$

$$b^{(7)} = (1, 1).$$

$$a^{(8)} = (\theta + (8V - 2), \theta^2 + (4V - 1)\theta + (16V^2 - 8V + 1));$$
$$b^{(8)} = (12V - 3, 48V^2 - 18V + 1).$$

$$a^{(9)} = a^{(5)}.$$

Here again the a's are obtained from the recursion formulas defined in [6]. The above-mentioned b's are obtained from the inequalities

$$(3.2) \quad 4V - 1 < \theta < 4V \quad \text{and} \quad 16V^2 - 4V < \theta^2 < 16V^2 - 4V + 1,$$

though we need for $b_1^{(3)}$, $b_2^{(4)}$, and $b_2^{(8)}$ the inequalities

$$(3.3) \qquad 4V - 1/2 < \theta < 4V - 1/2 + 1/64V^2,$$

$$(3.4) \qquad 16V^2 - 4V + 1/4 < \theta^2 < 16V^2 - 4V + 1/4 + 1/8V.$$

The components of $a^{(4)}$ are algebraic integers, so that the Bernstein formula produces in $\boldsymbol{Q}(\theta)$ the unit

$$e_1 = A_0^{(4)} + A_0^{(5)}a_1^{(4)} + A_0^{(6)}a_2^{(4)} = ((-8V + 1) + 2\theta)^{-1}.$$

Finally, the Hasse-Bernstein formula provides us with the unit

$$e_2 = a_2^{(5)}a_2^{(6)}a_2^{(7)}a_2^{(8)} = ((-8V + 1) + 2\theta)^{-1} = e_1$$

which is by Corollary 1.2 the inverse of the fundamental unit of $\boldsymbol{Q}(\theta)$.

4. **The case** $m=27t^3V^6+9t^2V^3+t$. The purpose of this chapter is to carry out the JPA of $a^{(0)}=(\theta, \theta^2)$ where

$$\text{(4.1)}\qquad \begin{cases} m=\theta^3=27t^3V^6+9t^2V^3+t=M/(3V)^3\,, \\ \quad \text{with } M=(9tV^3+1)^3-1;\ t,\ V\in \boldsymbol{N}^*\,. \end{cases}$$

Proceeding as in Chapters 3 and 4, and using the inequalities

$$\text{(4.2)}\qquad 3tV^2+1/3V-1/81tV^4<\theta<3tV^2+1/3V\,,$$

$$\text{(4.3)}\qquad 9t^2V^4+2tV+1/27V^2<\theta^2<9t^2V^4+2tV+1/9V^2\,,$$

we obtain the following vectors:

$$a^{(0)}=(\theta, \theta^2)\,;$$
$$b^{(0)}=(3tV^2, 9t^2V^4+2tV)\,.$$

$$a^{(1)}=\left(\frac{-2tV\theta^2+(3t^2V^3+t)\theta+(9t^3V^5+3t^2V^2)}{9t^2V^3+t}, \quad \frac{\theta^2+3tV^2\theta+9t^2V^4}{9t^2V^3+t}\right);$$
$$b^{(1)}=(0, 3V)\,.$$

$$a^{(2)}=\left(\frac{-V\theta^2+(6tV^3+1)\theta+(-9t^2V^5-2tV^2)}{27t^3V^6+10t^2V^3+t}\,,\right.$$
$$\left.\frac{(3tV^3+1)\theta^2+(9t^2V^5+tV^2)\theta+(27t^3V^7+15t^2V^4+2tV)}{27t^3V^6+10t^2V^3+t}\right);$$
$$b^{(2)}=(0, 3V)\,.$$

$$a^{(3)}=(\theta+3tV^2, \theta^2+3tV^2\theta+(9t^2V^4+tV))\,;$$
$$b^{(3)}=(6tV^2, 27t^2V^4+4tV)\,.$$

$$a^{(4)}=\left(\frac{-3tV\theta^2+t\theta+(27t^3V^5+6t^2V^2)}{9t^2V^3+t}\,, \quad \frac{\theta^2+3tV^2\theta+9t^2V^4}{9t^2V^3+t}\right);$$
$$b^{(4)}=(0, 3V)\,.$$

$$a^{(5)}=\left(\frac{(9tV^3+1)\theta+(-27t^2V^5-3tV^2)}{81t^3V^6+18t^2V^3+t}\,,\right.$$
$$\left.\frac{(9tV^3+1)\theta^2+(27t^2V^5+3tV^2)\theta+(81t^3V^7+36t^2V^4+3tV)}{81t^3V^6+18t^2V^3+t}\right);$$
$$b^{(5)}=(0, 3V).$$

$$a^{(6)}=(\theta+6tV^2, \theta^2+3tV^2\theta+9t^2V^4)\,;$$
$$b^{(6)}=(9tV^2, 27t^2V^4+3tV)\,.$$

$a^{(7)} = a^{(4)}$.

As the components of $a^{(3)}$ are algebraic integers, the Bernstein formula gives the unit

$$e_1 = A_0^{(3)} + A_0^{(4)}a_1^{(3)} + A_0^{(5)}a_2^{(3)} = (9tV^3 + 1 - 3V\theta)^{-1} .$$

From the Hasse-Bernstein formula, we obtain the same unit; by Corollary 1.2, this is the inverse of the fundamental unit of $\boldsymbol{Q}(\theta)$.

When $t = 3s$, $s \in N^*$, then

$$m = s^3U^6 + 3s^2U^3 + 3s , \quad \text{with} \quad U = 3V .$$

This shows that if $3|t$, the JPA of (θ, θ^2) is a particular case considered by L. Bernstein. (See (0.6) in the introduction.)

5. The case $m = t^3V^6 + 3t^2V^3 + 2t$. The notion of a generalized JPA was introduced by L. Bernstein [5], the only difference from a JPA being in the fact that the b's are defined arbitrarily.

In this chapter, we show that a generalization of the JPA of (θ, θ^2) where

$$\text{(5.1)} \qquad \begin{cases} m = \theta^3 = t^3V^6 + 3t^2V^3 + 2t = M/V^3 , \\ \qquad \text{with } M = (tV^3 + 1)^3 - (tV^3 + 1) ; \quad t,\ V \in N^* , \end{cases}$$

yields the following vectors:

$a^{(0)} = (\theta, \theta^2)$;

$b^{(0)} = (tV^2, t^2V^4 + 2tV)$.

$$a^{(1)} = \left(\frac{-2tV\theta^2 + (t^2V^3 + 2t)\theta + (t^3V^5 + 2t^2V^2)}{3t^2V^3 + 2t} , \ \frac{\theta^2 + tV^2\theta + t^2V^4}{3^2tV^3 + 2t}\right) ;$$

$b^{(1)} = (0, V)$.

$$a^{(2)} = \left(\frac{\theta}{t^2V^3 + 2t} , \ \frac{\theta^2 + tV^2\theta + (t^2V^4 + 2tV)}{t^2V^3 + 2t}\right) ;$$

$b^{(2)} = (0, 3V - 1)$.

$$a^{(3)} = \left(\frac{(-2V + 1)\theta^2 + (tV^3 + 1)\theta + (t^2V^5 + tV^2)}{tV^3 + 1} , \ \frac{\theta^2}{tV^3 + 1}\right) ;$$

$b^{(3)} = (tV - 1, tV)$.

$$a_1^{(4)} = \frac{\begin{pmatrix} (2tV - 2t + 1)\theta^2 + (3t^2V^4 - t^2V^3 + t^2V^2 + 5tV)\theta + (-3t^3V^6 \\ + 8t^3V^5 - 8t^3V^4 - t^2V^4 + 3t^3V^3 - 5t^2V^3 + 12t^2V^2 - 12t^2V \\ - tV + 4t^2 - 2t) \end{pmatrix}}{\begin{pmatrix} 9t^3V^7 - 18t^3V^6 + 21t^3V^5 + 3t^2V^5 - 12t^3V^4 + 18t^2V^4 + 3t^3V^3 \\ - 30t^2V^3 + tV^3 + 33t^2V^2 + 3tV^2 - 18t^2V + 9tV + 4t^2 - 4t + 1 \end{pmatrix}} ,$$

$$a_2^{(4)} = \frac{\begin{pmatrix}(3tV^3 - 3tV^2 + tV + 2V)\theta^2 + (3t^2V^5 - 3t^2V^4 + t^2V^3 - tV^3 \\ + 7tV^2 - 7tV + 2t - 1)\theta + (3t^3V^7 - 3t^3V^6 + t^3V^5 + 2t^2V^5 \\ + 5t^2V^4 - 5t^2V^3 + tV^3 + t^2V^2 + 2tV^2 + 2tV - 2t + 1)\end{pmatrix}}{M_4};$$

$$b^{(4)} = (0, 1) .$$

$$a_1^{(5)} = \frac{-2\theta^2 + tV^2\theta + (4t^2V^4 - 3t^2V^3 + 6tV - 4t)}{3t^2V^3 + 4t},$$

$$a_2^{(5)} = \frac{\begin{pmatrix}(V + 2)\theta^2 + (tV^3 - tV^2 + 2)\theta \\ + (t^2V^5 - 4t^2V^4 + 3t^2V^3 + tV^2 - 6tV + 4t)\end{pmatrix}}{3t^2V^3 + 4t};$$

$$b^{(5)} = (V - 2,\ V^2 - V + 1) .$$

$$a_1^{(6)} = \frac{\begin{pmatrix}(-tV^2 + 2tV + 2t + 1)\theta^2 + (2t^2V^4 + 2t^2V^3 - t^2V^2 + 3tV + 4t)\theta \\ + (-t^3V^6 - 4t^3V^5 - t^3V^4 + t^2V^4 - 7t^2V^2 - 2t^2V + 2tV + 4t)\end{pmatrix}}{3t^3V^5 + 3t^3V^4 + 3t^3V^3 - t^2V^3 + 6t^2V^2 + 6t^2V + 4t^2 - 2t},$$

$$a_2^{(6)} = \frac{\begin{pmatrix}(tV + 2t)\theta^2 + (t^2V^3 - t^2V^2 + 2t)\theta \\ + (t^3V^5 + 2t^3V^4 + 3t^3V^3 + 2t^2V^2 + 4t^2V + 4t^2)\end{pmatrix}}{M_6};$$

$$b^{(6)} = (0, 1) .$$

$$a_1^{(7)} = \frac{(-V - 1)\theta^2 + (tV^3 + 1)\theta + (t^2V^5 + 2t^2V^4 + t^2V^3 + 2tV^2 + 4tV + 2t)}{2t^2V^6 + 3t^2V^5 + 3t^2V^4 + t^2V^3 + 4tV^3 + 6tV^2 + 6tV + 2t + 1},$$

$$a_2^{(7)} = \frac{\begin{pmatrix}(tV^3 + tV^2 + tV + V + 2)\theta^2 + (t^2V^5 + t^2V^4 + t^2V^3 - tV^3 \\ + 2tV^2 + 3tV + 2t - 1)\theta + (t^3V^7 + t^3V^6 + t^3V^5 - t^2V^5 \\ - t^2V^4 + 2t^2V^2 - 2tV^2 - 5tV - 4t)\end{pmatrix}}{M_7};$$

$$b^{(7)} = (0,\ 3tV/2 - t) .$$

$$a_1^{(8)} = \frac{\begin{pmatrix}(-tV + 2t)\theta^2 + (2t^2V^3 + 4t)\theta + (-t^3V^5 - t^3V^4 + 2t^3V^3 - 2t^2V^3 \\ -2t^2V^2 - 2t^2V + 4t^2 - 4t)\end{pmatrix}}{2t^2V^3 + 4t},$$

$$a_2^{(8)} = \frac{2\theta^2 + (2t^2V^4 + 2t^2V^3 + 4tV + 4t)}{2t^2V^3 + 4t};$$

$$b^{(8)} = (tV/2 + t - 1,\ 2V + 1) .$$

$$a_1^{(9)} = \frac{\begin{pmatrix}(-12V - 8)\theta^2 + (12tV^3 - 8tV^2 + 16)\theta \\ + (16t^2V^4 + 2tV^2 + 24tV + 8t)\end{pmatrix}}{18t^3V^6 + 24t^3V^4 + 51t^2V^3 + 6t^2V^2 + 36t^2V + 8t^2 + 32t},$$

$$a_2^{(9)} = \frac{\begin{pmatrix}(6tV + 8tV + 16)\theta^2 + (6t^2V^5 + 8t^2V^3 + 10tV^2 + 8tV + 8t)\theta \\ + (6t^3V^7 + 8t^3V^5 + 16t^2V^4 - 8t^2V^3 + 16t^2V^2 + 8tV - 16t)\end{pmatrix}}{M_9};$$

$$b^{(9)} = (0,\ V) .$$

$$a^{(10)} = (\theta + (tV^2/2 - tV),\ \theta^2 + tV^2\theta + (t^2V^4 + 3tV/2 + t)) .$$

Here M_i denotes the denominator of $a_1^{(i)} (i = 4, 6, 7, 9)$.

When t is a square, this is L. Bernstein's result [5]; however there are calculation mistakes which are consequences of his wrong value of $a_2^{(5)}$. For arbitrary t, the technique is exactly the same. We replace θ and θ^2 in $a^{(v)}$ by $[\theta]$ and $[\theta^2]$ respectively so that we obtain approximations for the components $a_1^{(v)}, a_2^{(v)}$, and then we find the greatest integer contained in these approximations. If we suppose that $2 | tV$, then the value we gave for $b_2^{(7)}$ is correct for $t = 1$ or 2, and the values of the other b's are correct for any t, as easily verified. This leads us to choose the above-mentioned b's for any value of t of V.

Using formula (0.14), we obtain for the A's the following values:

$$\begin{aligned}
A_0^{(10)} &= 9t^2V^6/2 - 3t^2V^5 + 15tV^3/2 - 3tV^2 + 1 ; \\
A_0^{(11)} &= 9t^2V^7 + 15tV^4 + 3V ; \\
A_1^{(11)} &= 9t^3V^9 + 24t^2V^6 + 15tV^3 + 1 ; \\
A_2^{(11)} &= 9t^4V^{11} + 33t^3V^8 + 36t^2V^5 + 11tV^2 ; \\
A_0^{(12)} &= 9t^2V^8 + 18tV^5 + 6V^2 ; \\
A_1^{(12)} &= 9t^3V^{10} + 27t^2V^7 + 21tV^4 + 3V ; \\
A_2^{(12)} &= 9t^4V^{12} + 36t^3V^9 + 45t^2V^6 + 18tV^3 + 1 .
\end{aligned}$$

We can now apply the following theorem for $n = 3$.

THEOREM 5.1. *Assume* $\boldsymbol{K}_n = \boldsymbol{Q}(a_1^{(0)}, \cdots, a_{n-1}^{(0)})$. *Let* $\langle a^{(v)} \rangle$ *be any generalized* JPA *of* $a^{(0)}$ *where the components of* $a^{(0)}$ *are algebraic integers, and let the vectors* $b^{(v)}$ *have any rational components. If, for some* $v \geqq 1$,

$$A_0^{(v)} + A_0^{(v+1)}a_1^{(v)} + \cdots + A_0^{(v+n-1)}a_{n-1}^{(v)} \tag{5.2}$$

is an algebraic integer of $\boldsymbol{K}_n$ *and if*

$$A_i^{(v+1)}, A_i^{(v+2)}, \cdots, A_i^{(v+n-1)} \quad (i = 0, 1, \cdots, n-1) \tag{5.3}$$

are rational integers, then (5.2) *is a unit.*

Proof. By (0.12), the proof is immediate.

As

$$\begin{aligned}
e_1 &= A_0^{(10)} + A_0^{(11)}a_1^{(10)} + A_0^{(12)}a_2^{(10)} \\
&= (9t^4V^{12} + 36t^3V^9 + 45t^2V^6 + 18tV^3 + 1) + (9t^3V^9 \\
&\quad + 27t^2V^6 + 21tV^3 + 3)V\theta + (9t^2V^6 + 18tV^3 + 6)V^2\theta^2
\end{aligned}$$

is an algebraic integer, as and $A_0^{(11)}, A_1^{(11)}, A_2^{(11)}, A_0^{(12)}, A_1^{(12)}, A_2^{(12)}$ are integers, then e_1 is a unit. It turns out that, when θ^3 is cube-free,

$$e_1^{-1} = -(V\theta - tV^3 - 1)^3/(tV^3 + 1)$$

is the fundamental unit of $\boldsymbol{Q}(\theta)$; this result follows from Theorem 1.1, when $V = 1$, and from Theorem 1.3, when $V \geqq 2$.

6. Concluding remarks. L. Bernstein proved in [2] that the JPA of $a^{(0)} = (\omega, \omega^2)$, where

$$\omega^3 = 8K^3 + 12K, K \in \boldsymbol{N}^* , \tag{6.1}$$

is periodic with primitive lengths $l = 4$ and $m = 8$; here

$$\begin{aligned} &a^{(4)} = ((2\omega + 4K - 4)/4 , \quad (\omega^2 + 2K\omega + 4K^2 + 2)/4) , \\ &A_0^{(4)} = K = A_0^{(5)} ; \quad A_0^{(6)} = 2K^2 + 1 ; \quad A_2^{(6)} = 8K^4 + 12K^2 + 2 ; \\ &A_2^{(5)} = 4K^3 + 4K . \end{aligned}$$

Because the components of $a^{(8)}$ are algebraic integers, L. Bernstein obtained the unit

$$e_1 = (12K)^2(\omega - 2K)^{-6} , \tag{6.2}$$

which is the square of the inverse of the fundamental unit of $\boldsymbol{Q}(\omega)$, when ω^3 is cube-free.

If we calculate $\prod_{i=4}^{11} a_2^{(i)}$ by proceeding in Chapter 2, we obtain the same unit (6.2). Nonetheless, a unit which turns out to be the fundamental unit of $\boldsymbol{Q}(\omega)$, can be obtained from the JPA of (ω, ω^2), from the following theorem.

THEOREM 6.1. *Let $\langle a^{(v)} \rangle$ be the* JPA *of $a^{(0)}$, where the components of $a^{(0)}$ are algebraic integers of $\boldsymbol{K}_n = \boldsymbol{Q}(a_1^{(0)}, \cdots, a_{n-1}^{(0)})$. Suppose that for a $v \geqq 1$, there exists a rational integer $N_v \geqq 1$ such that the components of $N_v a^{(v)}$ are algebraic integers of $\boldsymbol{K}_n$, and such that one of the following conditions holds*:

(i) *either there exists k, where $1 \leqq k \leqq n - 1$, such that*

$$N_v \,|\, A_i^{(v+k)} \qquad (i = 0, 1, \cdots, n - 1) ;$$

(ii) *or there exists i, where $1 \leqq i \leqq n - 1$, such that*

$$N_v \,|\, A_i^{(v+k)} \quad (k = 1, \cdots, n - 1)$$

and such that $a_i^{(0)}/N_v$ is an algebraic integer of $\boldsymbol{K}_n$.
Then

$$N_v(A_0^{(v)} + A_0^{(v+1)}a_1^{(v)} + \cdots + A_0^{(v+n-1)}a_{n-1}^{(v)}) \tag{6.3}$$

is a unit in $\boldsymbol{K}_n$.

Proof. If we multiply both members of identity (0.12) by N_v^{-1}, it follows immediately that (6.3) and its inverse are algebraic integers.

To apply the last theorem, take $N_4 = 2$. Then $2a_1^{(4)}$ and $2a_2^{(4)}$ are algebraic integers; for a proof that $\omega^2/2$ is an algebraic integer, see [2]; moreover $2 \mid A_2^{(5)}$ and $2 \mid A_2^{(6)}$. Therefore

$$e_3 = 2(A_0^{(4)} + A_0^{(5)}a_1^{(4)} + A_0^{(6)}a_2^{(4)}) = (12K)(\omega - 2K)^{-3}$$

is a unit in $\boldsymbol{Q}(\omega)$.

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