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SPACES IN WHICH COMPACTA ARE UNIFORMLY REGULAR G_δ

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A space in which compacta are uniformly regular G_δ is said to be c-stratifiable. This concept turns out to be important in many reasons: c-stratifiability is a necessary and sufficient condition for regular wN-spaces to be Nagata, for regular $w\gamma$-spaces to be γ and for semimetrizable spaces to be K-semimetrizable. As applications, it is shown that a completely regular pseudocompact space is metrizable and that K-semimetrizable spaces are characterized by having semi-developments with the 3-link property.

The most important generalized metric spaces are Moore spaces, Nagata spaces and γ-spaces. Hodel and Kodake proved that being an α-space is a necessary and sufficient condition for regular $w\Delta$-spaces to be Moore, for regular wN-spaces to be Nagata. We will show that c-stratifiability plays the same role as α for wN and $w\gamma$. Spaces with regular G_δ-diagonal are c-stratifiable is proved using Zenor's characterization. Completely regular pseudocompact c-stratifiable spaces are γ. If a space has a semi-refined sequence or semi-development with the 3-link property, we can define a symmetric or semimetric in the usual way to show that such a symmetric or semimetric is characterized by $d(K_1, K_2) > 0$ for disjoint compacta K_1 and K_2. Since the 3-link property concerns with convergence of sequences, it is not surprising that the property is characterized by a concept of compactness. Going up to development with the 3-link property, we get a K-semimetric under which each point has arbitrarily small neighborhoods. The main method in this paper is another characterization of c-Nagataness (=first countable c-stratifiable) by a countable open covering map g: If $g(n, x) \cap g(n, x_n) \neq \emptyset$ for each n, then x is a cluster point of the sequence $\{x_n\}$ if there are any.

For a subset S of a space, we will denote the closure of S by S^-. For a point x and a sequence $\{x_n\}$, $\langle x_n \rangle$ will denote the point set $\{x_1, x_2, x_3, \cdots\}$ and $\langle x, x_n \rangle$ will denote the set $\{x, x_1, x_2, x_3, \cdots\}$. $Cp\{x_n\}$ denotes the set of all cluster points of $\{x_n\}$.

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1. c-Nagata spaces. Let X be a space and g a function from

$N \times X$ into the topology of X such that $x \in \cap \{g(n, x): n \in N\}$ for each $x \in X$ and the $g(n+1, x) \subset g(n, x)$ for each n and x. Note that if we let $\gamma_n = \{g(n, x): n \in N\}$, then $\{\gamma_1, \gamma_2, \gamma_3, \cdots\}$ is a sequence of open covers of X such that γ_{n+1} refines γ_n. Thus we call such a function a COC-*map* (=countable open covering map). For any subset S of X, we denote $g(n, S) = \cup \{g(n, x): x \in S\}$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be some families of subsets of X. g is said to *separate* (*separate regularly*) $\mathfrak{B}$ *from* $\mathfrak{A}$ if, for each $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ disjoint, there exists $n \in N$ such that $A \cap g(n, B) = \varnothing (A \cap g(n, B)^- = \varnothing)$. Consider the following conditions on g.

A. g separates closed compact sets from points,
B. g separates closed sets from points,
C. g separates points from closed sets,
D. g separates compact sets from closed sets,
E. g separates closed sets from compact sets, and
F. g separates regularly closed sets from points.

In [16] Martin defined *c-semistratifiable spaces* by A; *Semistratifiable spaces* [5] is characterized by B; C is the definition of *first countable spaces*; In [13] T_1, γ-*spaces* are characterized by D, which is precisely *coconvergent spaces* [20]; It is easily verified that *k-semistratifiable* spaces [14] can be characterized by E and that *stratifiable spaces* [2] by F.

We now introduce a new class of spaces which shares similar properties to these spaces.

DEFINITION. A T_1-space is said to be *c-stratifiable* if it has a COC-map that separates regularly compact sets from points. A space is *c-Nagata* if it is c-stratifiable and first countable.

From the definition, every compact set in a c-stratifiable space is a regular G_δ. Conversely, let X be a T_1-space such that: For each compact K, there exists a decreasing sequence $\{K_n\}$ of open sets with the properties

(1) $K = \bigcap_{n \in N} K_n = \bigcap_{n \in N} K_n^-$ for each compact K, and

(2) If K and H are compacta with $K \subset H$, then $K_n \subset H_n$ for every n. Define a COC-map g by $g(n, x) = \{x\}_n$. Then g is a COC-map that separates regularly compacta from points. Thus X is c-stratifiable.

Note that every Nagata space is c-Nagata and that every c-stratifiable space is c-semistratifiable. A c-stratifiable space is Hausdorff. But there exists a nonregular, nonperfect c-stratifiable space as 6.2 shows. This distinguishes c-stratifiable spaces from stratifiable spaces and from semistratifiable spaces. As can easily be shown, c-stratifiability is hereditary, countably productive.

LEMMA 1.1. *A* COC-*map g separates regularly T from compacta if g separates regularly T from points.*

Proof. Suppose g separates regularly T from points. Let K be a compact set disjoint from T. For each x in K, there exists $n(x) \in N$ such that $x \notin g(n(x), T)^-$. This implies that $\{X - g(n(x), T)^- : x \in K\}$ forms an open cover of the compact set K. Let $\{X - g(n(x_i), T)^- : x_i \in K, 1 \leqq i \leqq k\}$ be a finite subcover of K, and $n = \max\{n(x_i) : 1 \leqq i \leqq k\}$. Then $K \cap g(n, T)^- = \emptyset$.

LEMMA 1.2. *A first countable* COC-*map g for a T_1-space separates closed sets (compact sets, points, respectively) from compact a if and only if g separates regularly closed sets (compact sets, points, respectively) from points.*

Proof. First we show that a T_1-space which has a first countable COC-map g separating points from compacta is Hausdorff. Assume there are distinct points x and y such that $g(n, x) \cap g(n, y) \neq \emptyset$ for each n. There exists a sequence $\{x_n\}$ which converges to both x and y. We may assume $y \notin \langle x_n \rangle$. The point y cannot be separated from the compact set $\langle x, x_n \rangle$. This contradiction shows that the space is Hausdorff.

Assume there exists a subset T and a point $p \in X - T$ such that $p \in g(n, T)^-$ for every n. If $g(n, p) \cap g(n, T) - T = \emptyset$ for infinitely many n, then $g(n, p) \cap T \neq \emptyset$ for infinitely many n, and hence $p \in T^- = T$ if T is a closed (compact, singleton) set. Thus we can choose $y_n \in g(n, p) \cap g(n, T) - T$. Then T cannot be separated from the compactum $\langle p, y_n \rangle$. The converse is clear from 1.1.

COROLLARY (*Lutzer*). *A first countable k-semistratifiable T_1-space is stratifiable.*

THEOREM 1.3. *The following conditions on a first countable* COC-*map g for a T_1-space are equivalent:*

(1) *g separates regularly compacta from compacta,*

(2) *g separates regularly compacta from points,*

(3) *g separates compacta from compacta, and*

(4) *If $g(n, x) \cap g(n, x_n) \neq \emptyset$ for each n, then $Cp\{x_n\} \subset \{x\}$.*

Proof. By virtue of 1.1 and 1.2, it suffices to show that (3) $\Leftrightarrow$ (4).

Let g satisfy the third condition and let $g(n, x) \cap g(n, x_n) \neq \emptyset$ for each n. If $z \in Cp\{x_n\}$ with $z \neq x$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging to z. Let $\{y_n\}$ be a sequence such that $y_n \in$

$g(n, x) \cap g(n, x_n)$ for each n and that $\langle y_{n_i} \rangle$ is disjoint from $\langle x_{n_i} \rangle$. Now the compactum $\langle z, x_{n_i} \rangle$ cannot be separated from the compactum $\langle x, y_{n_i} \rangle$.

Conversely, suppose g satisfies the fourth condition. Assume there exist disjoint compacta K_1 and K_2 such that K_2 cannot be separated from K_1. Choose a point x_n in $K_1 \cap g(n, K_2)$ for each n. Let $x \in Cp\{x_n\}$. Then $x \in \cap \{g(n, K_2)^-: n \in N\}$. This implies that there exists a sequence $\{y_n\}$ in K_2 such that $g(n, x) \cap g(n, y_n) \neq \varnothing$. Since $Cp\{y_n\} \cap K_2 \neq \varnothing$, x is the unique cluster point of $\{y_n\}$ which belongs to K_2. This implies $K_1 \cap K_2 \neq \varnothing$. This completes the proof.

2. Nagata spaces and γ-spaces. A space is called a *wN-space* [9] if it has a COC-map g such that: If $g(n, x) \cap g(n, x_n) \neq \varnothing$ for each $n \in N$, then $Cp\{x_n\} \neq \varnothing$. Similarly, a space is called a *wγ-space* [9] if it has a COC-map g such that: If $x_n \in g(n, y_n)$ and $y_n \in g(n, x)$ for each $n \in N$, then $Cp\{x_n\} \neq \varnothing$.

Martin [16] shows that a regular c-semistratifiable $w\Delta$-space is developable, and Hodel [9] proves that a Hausdorff γ, wN-space is metrizable. Since there exist nonmetrizable Nagata spaces, the condition being γ-spaces in Hodel's proposition cannot be weakened to c-Nagataness. However, as the following lemmas show, c-Nagataness is a necessary and sufficient condition for wN-spaces to be Nagata, and for $w\gamma$ to be γ.

LEMMA 2.1. *A space is Nagata if and only if it is a c-Nagata wN-space.*

LEMMA 2.2. *A Hausdorff space is γ if and only if it is c-Nagata and $w\gamma$.*

Proof. By Theorem 2.1 of [13], a γ-space has a COC-map which separates compacta from closed sets. But a Hausdorff space which has a first countable COC-map that separates compacta from compacta is c-Nagata by 1.3. These imply that Hausdorff γ-spaces are c-Nagata.

Conversely, let g be a first countable COC-map for a Hausdorff space satisfying the fourth condition of 1.3 and the condition of $w\gamma$. Let $x_n \in g(n, y_n)$ and $y_n \in g(n, x)$ for each n, and let $z \in Cp\{x_n\}$. There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \in g(i, z)$ for each $i \in N$. Thus $x_{n_i} \in g(i, z) \cap g(n_i, y_{n_i}) \subset g(i, z) \cap g(i, y_{n_i})$ for each $i \in N$, from which it follows that $Cp\{y_{n_i}\} \subset \{z\}$. Since g is a first countable COC-map, $\{y_n\}$ converges to x. Thus we have $z = x$.

If a T_1-space has a COC-map that separates compacta from closed sets, the COC-map separates compacta from points. This implies that every T_1, γ-space is c-semistratifiable. But such a space need not be c-stratifiable as 6.3 shows.

COROLLARY. *Every Nagata $w\gamma$-space is metrizable.*

Proof. A Nagata $w\gamma$-space is a γ-space by 2.2. Now apply [9, Theorem 4.3].

A regular $w\Delta$-space is developable if it is α, and a regular wN-space is Nagata if it is α. A natural substitute for α in the case of wN and $w\gamma$ is c-stratifiability as seen in the following.

THEOREM 2.3. (1) *A regular space is Nagata if and only if it is a c-stratifiable wN-space.* (2) *A regular space is γ if and only if it is a c-stratifiable $w\gamma$-space.*

Proof. Note that a wN-space or a $w\gamma$-space is a q-space [19] and that each point is a G_δ in a c-stratifiable space. Lutzer showed that a regular q-space in which each point is G_δ are first countable. The result follows from 2.1 and 2.2.

3. Spaces with regular G_δ-diagonals. A space is said to have a *$G_\delta(k)$-diagonal* if it has a sequence $\{\mathfrak{Y}_1, \mathfrak{Y}_2, \mathfrak{Y}_3, \cdots\}$ of open covers such that: For any distinct two points x and y, there is n such that $y \notin st^k(x, \mathfrak{Y}_n)$. $G_\delta(1)$-diagonal coincides with G_δ-diagonal. A space is said to have a *regular G_δ-diagonal* if the diagonal of $X \times X$ is a regular G_δ-set in the product space. For more properties of these, see [2, 10, 16 and 21].

LEMMA 3.1 (*Zenor*). *A space X has a regular G_δ-diagonal if and only if there is a sequence $\{\gamma_n\}$ of open covers of X such that if x and y are distinct points of X, then there exists an integer n and open sets U and V containing x and y respectively such that no member of γ_n intersects both U and V.*

PROPOSITION 3.2. *Any space with a reqular G_δ-diagonal is c-stratifiable.*

Proof. Let $\{\gamma_n\}$ be a sequence of open covers of a space X metioned in 3.1. We may assume that each γ_n is refined by γ_{n+1}. Define a COC-map g by $g(n, x) = st(x, \gamma_n)$. Let a compactum K and a point $p \in X - K$ be given. For each $x \in K$, there exists an integer $n(x)$ and open sets $U(x)$ and $V(x)$ containing p and x, respectively, such that $U(x) \cap st(V(x), \gamma_{n(x)}) = \emptyset$. Since K is compact, we can find a finite number of points $x_1, x_2, \cdots, x_k$ of K such that $\{V(x_i): i = 1, 2, \cdots, k\}$ covers K. Let $n = \max\{n(x_i): i = 1, 2, \cdots, k\}$, and $U = \cap\{U(x_i): i = 1, 2, \cdots, k\}$. Then $U \cap st(K, \gamma_n) = \emptyset$. That is, $U \cap g(n, K) = \emptyset$. This implies $p \notin g(n, K)^-$.

4. **Pseudocompact spaces.** A space X is *pseudocompact* if every real-valued continuous function on X is bounded. The following characterization of completely regular pseudocompact spaces is well known.

LEMMA 4.1. *Let X be a completely regular space. X is pseudocompact if and only if for every sequence $G_1 \subset G_2 \subset G_3 \subset \cdots$ of nonvoid open subsets of X, $\bigcap_{n \in N} G_n^- \neq \emptyset$.*

In [18], it is proved that a completely regular pseudocompact space with a regular G_δ-diagonal is metrizable. Even though the condition that the space have a regular G_δ-diagonal cannot be weakened to c-stratifiability (see 3.2) as shown in 6.6, we are able to prove the following.

THEOREM 4.2. *Completely regular pseudocompact c-stratifiable spaces are γ.*

Proof. Let g be a c-stratifiable COC-map for a completely regular pseudocompact space X, and K a compact subset and G an open set containing K. Assume $g(n, K) \cap (X - G) \neq \emptyset$ for every n. Since X is regular there is an open set H such that $K \subset H \subset H^- \subset G$. Then $\{g(n, K) \cap (X - H^-): n \in N\}$ is a decreasing sequence of nonvoid open sets so that $\emptyset \neq \bigcap_{n \in N}\{g(n, K) \cap (X - H^-)\}^- \subset \bigcap_{n \in N} g(n, K)^- \cap (X - H) = K \cap (X - H) = \emptyset$. This contradiction implies that g separates compacta from closed sets. Thus X is a γ-space.

COROLLARY. *A completely regular pseudocompact stratifiable space is metrizable.*

Proof. Note that a stratifiable space is c-stratifiable. Now apply 4.2 and corollary to 2.2.

5. **K-symmetrics and semi-refined sequences of covers satisfing the 3-link property.** Let X be a space and d a real-valued nonnegative function defined on $X \times X$ such that $d(x, y) = d(y, x)$, and $d(x, y) = 0$ if and only if $x = y$. The function d is called a *symmetric* [1] for X provided that a set $M \subset X$ is closed if and

only if $d(x, M) > 0$ for any $x \in X - M$. The function d is called a *semimetric* for X provided that for a set $M \subset X$, $x \in M^-$ if and only if $d(x, M) = 0$. In [1] and [15] it is proved that a space is metrizable if and only if it has a compatible symmetric d such that $d(F, K) > 0$ for any disjoint closed F and compact K. The following condition on symmetrics is due to Arhangel'skii.

(K) For any disjoint compacta K_1 and K_2 in X, $d(K_1, K_2) > 0$.

DEFINITION. A symmetric satisfying (K) is called a *K-symmetric*. Similarly, a semimetric satisfying (K) is called a *K-semimetric* [17]. A space is said to be K-symmetrizable (K-semimetrizable) if it is symmetrizable (semimetrizable) via a K-symmetric (K-semimetric).

We can no longer claim that a K-symmetrizable space is metrizable even if it is paracompact. Arhangel'skii's conjecture saying that every symmetrizable space is K-symmetrizable seems to be unsolved yet. It is known that a paracompact semimetric space is K-semimetrizable.

A sequence $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ of covers of a space X such that each γ_{n+1} refines γ_n is called a *semi-refined sequence of covers* [3] if $\mathfrak{W}_x = \{st(x, \gamma_n): n \in N\}$ forms a *weak-base* [1], a *semi-development* if it is a semi-refined sequence of covers such that each $st(x, \gamma_n)$ is a neighborhood of x, and a *development* if it is a semi-development such that each γ_n is an open cover of X.

A development $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ for a space X is said to satisfy the *3-link property* [7] if it is true that for any distinct points x and y, there exists an integer n such that no member of γ_n intersects both $st(x, \gamma_n)$ and $st(y, \gamma_n)$. We generalize this concept to arbitrary sequence of covers.

DEFINITION. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ be a sequence of covers of a space. γ is said to satisfy the *3-link property* if for any distinct points x and y, there exists an integer n such that $y \notin st^3(x, \gamma_n)$.

Note that this definition coincides with the original definition of the 3-link property for developments, and that a space has a sequence of open covers with the 3-link property if and only if it has a $G_\delta(3)$-diagonal.

A continuously semimetrizable space is a Moore space that admits a development with the 3-link property. A space admits a development with the 3-link property if and only if it is a $w\Delta$-space with a regular G_δ-diagonal. A locally connected developable space with a regular G_δ-diagonal has a K-semimetric d such that the space is d-spherically connected, a locally connected rim compact space is K-semimetrizable if and only if it is a developable γ-space. See [4], [17] and [21] for details.

The following three theorems show that the condition (K) has a close relation to the 3-link property and the c-stratifiability.

THEOREM 5.1. *A space X is Hausdorff K-symmetrizable if and only if it admits a semi-refined sequence of covers satisfying the 3-link property.*

Proof. Let d be a K-symmetric for X. For each n, put γ_n be the set of all subsets of X with diameter less than $1/n$. Then $d(x, y) < 1/n$ if and only if $y \in st(x, \gamma_n)$. This implies that $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ is a semi-refined sequence of covers of X. If there exist distinct points x and y such that $y \in st^3(x, \gamma_n)$ for every n, there are sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \in st(x, \gamma_n)$, $y_n \in st(y, \gamma_n)$ and $y_n \in st(x_n, \gamma_n)$. We may assume $\langle x, x_n\rangle \cap \langle y, y_n\rangle = \varnothing$ with both sets compact. But $d(\langle x, x_n\rangle, \langle y, y_n\rangle) = 0$, a contradiction.

Conversely, let $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ be a semi-refined sequence of covers of X satifying the 3-link property. Define a symmetric d by $d(x, y) = 1/\inf\{j \in N: x \notin st(y, \gamma_j)\}$. From the definition $d(x, y) < 1/n$ if and only if $x \in st(y, \gamma_n)$. Assume there exist disjoint compacta K_1 and K_2 such that $d(K_1, K_2) = 0$. We can find two sequences $\{x_n\}$ and $\{y_n\}$ in K_1 and K_2 respectively, such that $d(x_n, y_n) < 1/n$. Note that X is sequential and Hausdorff so that $\{x_n\}$ and $\{y_n\}$ have convergent subsequences. Let $\{x_{n_i}\}$ and $\{y_{n_i}\}$ be subsequence of $\{x_n\}$ and $\{y_n\}$ converging to x and y, respectively. Without loss of generality, we may assume $d(x, x_{n_i}) < 1/i$ and $d(y, y_{n_i}) < 1/i$ for each $i \in N$. Since $d(x_{n_i}, y_{n_i}) < 1/i$, it follows that there is no k such that $y \notin st^3(x, \gamma_k)$. This contradiction completes the proof.

THEOREM 5.2. *For a space X, the following are equivalent:* (1) *X is a semimetrizable c-stratifiable space,* (2) *X is K-semimetrizable, and* (3) *X admits a semi-development with the* 3-*link property. Furthermore, if the space is regular, each of these is equivalent to* (4) *X is a c-Nagata β-space.*

Proof. Note that spaces satisfying one of these conditions are Hausdorff. In a semimetric space, any compatible symmetric is actually a compatible semimetric. Also it is easily verified that any semi-refined sequence of covers of a semimetric space forms a semi-development. Applying these remarks to 5.1, we have (2) $\Leftrightarrow$ (3).

For (1) $\Leftrightarrow$ (2), let g be a semistratifiable, first countable, c-stratifiable COC-map for X. Define a semimetric d by $d(x, y) = 1/\inf\{j \in N: x \notin g(j, y)$ and $y \notin g(j, x)\}$. Let K_1 and K_2 be disjoint compacta. By 1.3, there is $m \in N$ such that $K_1 \cap g(m, K_2) = \emptyset$ and $K_2 \cap g(m, K_1) = \emptyset$. It follows that $d(K_1, K_2) \geqq 1/m > 0$. Conversely, let d be a K-semimetric on X. Define a first countable COC-map g by $g(n, x) =$ Interior of $1/n$ sphere centered at x. It is easily verified that g satisfies the third condition of 1.3.

For (4) note that a c-Nagata space is c-semistratifiable. Martin shows that a regular c-semistratifiable β-space is semistratifiable.

COROLLARY. *A semistratifiable γ-space is K-semimetrizable.*

The following lemma is due to Alexandrov-Nemitskii and Morton Brown. Analogous result for symmetrics can be found in [12].

LEMMA 5.3. *A Hausdorff space is developable if and only if it is semimetrizable via a semimetric d satisfying one of the following equivalent conditions*: (1) *Every convergent sequence is d-Cauchy,* (2) *If $\{x_n\}$ and $\{y_n\}$ are sequences both converging to x, then* $\lim d(x_n, y_n) = 0$, *and* (3) (AN) *Each point has a neighborhood of arbitrarily small diameter.*

THEOREM 5.4. *For a space X, the following are equivalent*: (1) *X is a $w\Delta$-space with a regular G_δ-diagonal,* (2) *X is K-semimetrizable via a semimetric satisfying* (AN), (3) *X admits a development with the 3-link property, and* (4) *There is a semimetric d on X such that*: (a) *If $\{x_n\}$ and $\{y_n\}$ are sequences both converging to the same point, then* $\lim d(x_n, y_n) = 0$, *and* (b) *If x and y are distinct points of X and $\{x_n\}$ and $\{y_n\}$ are sequences converging to x and y, respectively, then there are integers L and M such that if $n > L$, then $d(x_n, y_n) > 1/M$.*

Proof. Zenor proved the equivalence of (1), (3) and (4). (2) $\Leftrightarrow$ (4) is easy.

REMARK. In view of 5.3 and 5.4, one may conjecture that a developable K-semimetrizable space satisfies the above four conditions. This can be rephrased as: If a space X satisfies one of the following equivalent conditions, it satisfies the conditions of 5.4: (1) X is semimetrizable via a semimetric satisfying (AN) and is K-semimetrizable, (2) X is developable and admits a semidevelopment with the 3-link property, and (3) X has two semimetrics satisfying (a) and (b) of (4) in 5.3, respectively.

If this were true, any developable space that does not admit a development with the 3-link property would be semimetrizable but not K-semimetrizable. But we have an example (see 6.6).

6. Examples.

6.1. There exists a perfect, hereditarily Lindelöf, hereditarily separable γ-space which is not a β-space. Such a space is c-stratifiable but not semistratifiable. The space of reals with the upper limit topology.

6.2. There exists a Hausdorff γ-space which is neither regular nor perfect. Such a space is c-stratifiable but not semistratifiable. Let X be the space of all real numbers equipped with the topology generated by a first countable COC-map g

$$g(n, x) = \begin{cases} (x - 1/n, x + 1/n), \text{ if } x \text{ is rational} \\ (x - 1/n, x + 1/n) \cap (X - Q), \text{ otherwise} \end{cases}$$

where Q denotes the rationals. It is easy to check this space is a γ-space. The point $\sqrt{2}$ and the closed set Q cannot be separated by disjoint open sets, which shows that X is not regular. By the Baire Category Theorem, it is easily shown that Q is not a G_δ.

6.3. There exists a T_1, γ-space which is not Hausdorff and hence, is not c-stratifiable. Let $X = R \cup \{-\infty, +\infty\}$, where R is the reals, with the topology generated by a first countable COC-map g

$$g(n, x) = \begin{cases} (x - 1/n, x + 1/n), \text{ if } x \in R , \\ (-\infty, -n) \cup (n, +\infty) \cup \{+\infty\}, \text{ if } x = +\infty , \\ (-\infty, -n) \cup (n, +\infty) \cup \{-\infty\}, \text{ if } x = -\infty . \end{cases}$$

Then g separates compacta from closed sets.

6.4. There exists a first countable Hausdorff wN-space which is not c-semistratifiable, and hence is not c-stratifiable. Let Ω be the first uncountable ordinal, and consider the space $[0, \Omega)$ with the order topology. Since it is countably compact, it is a wN-space. Now Corollary 5 of [16] ensures that this space is not c-semistratifiable.

6.5. Nonmetrizable Nagata spaces ([2], Examples 9.1 and 9.2) are c-stratifiable spaces which are not $w\gamma$. See corollary to 2.2.

6.6. The space Ψ of [6, 5I] is completely regular, pseudocompact, c-Nagata and developable but does not have a regular G_δ-diagonal, and hence is not metrizable. See the remark following 4.1, 4.2, 5.3 and 5.4. We define a COC-map g for Ψ as follows. $g(n, x) = \{x\}$ for every $n \in N$ and for every $x \in N$. Note that for each point ω_E, there corresponds an infinite sequence $E = \{x_1, x_2, x_3, \cdots\}$ of distinct natural numbers. Let $g(n, \omega_E) = \{x_n, x_{n+1}, x_{n+2}, \cdots\}$. Since any compact set can contain only finitely many points of D, we can easily verify that g separates regularly compacta from points. That is, Ψ is c-stratifiable. Also, g satisfies: If $x, x_n \in g(n, y_n)$ for each n, then $\{x_n\}$ converges to x. This is a characterization of developable spaces proved in [8].

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