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SOME PROPERTIES OF THE SORGENFREY LINE AND RELATED SPACES

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Any finite power S^n of the Sorgenfrey line S has this covering property: if $\varphi(x)$ is a neighborhood of x for each $x \in S^n$, then there is a closed discrete subset D of S^n such that $\{\varphi(x): x \in D\}$ covers S^n. No finite power of the Sorgenfrey line is homeomorphic to finite power of the irrational Sorgenfrey line. The Sorgenfrey plane is not the union of countably many nice subspaces.

0. Introduction. All spaces considered are T_1. The *Sorgenfrey line*, S, is the set of all reals, retopologized by letting all half-open intervals $[a, b)$ be a base. The *irrational Sorgenfrey line*, T, is the subspace of S consisting of all irrational numbers. If κ is a cardinal, S^κ and T^κ are the product of κ copies of S and T, respectively. We refer to S^2 as the *Sorgenfrey plane*, and to T^2 as the *irrational Sorgenfrey plane*.

A *neighborhood assignment* for a space X is a function φ from X to the topology of X such that $x \in \varphi(x)$ for all $x \in X$, [2]. A space X is a *D-space* if for every neighborhood assignment φ for X there is a closed discrete subset D of X such that $\{\varphi(x): x \in D\}$ covers X, [2]. The space of countable ordinals is not a D-space since, as observed in [2]: every countably compact D-space is compact. Up to now no satisfactory example of a space which is not a D-space is known, where by satisfactory example we mean an example having a covering property at least as strong as metacompactness or subparacompactness.

Since the Sorgenfrey plane is subparacompact, [8, 3.1], it is a natural candidate for a satisfactory example of a non-D-space. However, we have the following theorem.

THEOREM 1. *Every finite power of S is a D-space.*

This leaves open the following three questions: Is every finite power of S hereditarily a D-space? Is S^ω a D-space? Is S^ω hereditarily a D-space? A negative answer to any of these questions would be welcome (but is not expected), since S^ω is hereditarily subparacompact, [8, 3.2]. We do not even know if T^2 is a D-space; however, it follows from results in [2] that S is hereditarily a D-space. In particular, T is a D-space.

The proof of Theorem 1 has suggested a new class of spaces, which generalizes the class of left-separated spaces. Recall that a space X is *left-separated* if there is a well-order $\leqq$ on X such that $\{y \in X: x \leqq y\}$ is open in X for each $x \in X$, [6, 0.4]. If $\leqq$ is any reflexive (not necessarily transitive) relation on a set E and $F \subseteq E$, then we shall call $m \in F$ a $\leqq$-minimal element of F whenever $x = m$ for each $x \in F$ with $x \leqq m$.

DEFINITION. A space X is called a *generalized left-separated* space (abbreviated GLS-space) if there is a reflexive binary relation $\leqq$ on X, called GLS-relation, such that

(1) every nonempty closed subset has a $\leqq$-minimal element

(2) $\{y \in X: x \leqq y\}$ is open for each $x \in X$.

THEOREM 2. *Every* GLS-*space is a* D-*space.*

THEOREM 3. *Every finite power of* S *is a* GLS-*space.*

Theorem 1 is an immediate consequence of these two theorems. Other applications of Theorem 2 will be mentioned in §2. Clearly a subspace of a left-separated space is again a left-separated space. The corresponding statement for GLS-spaces is false.

THEOREM 4. T *is not a* GLS-*space.*

In [1] it was shown that S and T are not homeomorphic. Since a closed subspace of a GLS-space is again a GLS-space [see (4.4a)], Theorems 3 and 4 yield the following strengthening of this result.

THEOREM 5. *For no positive integers* m *and* n *the spaces* S^m *and* T^n *are homeomorphic.*

It is apparently unknown if S^m and S^n, or T^m and T^n, can be homeomorphic for distinct positive integers m and n.

By a classical result of F. B. Jones, [5], a separable space which has a closed discrete subset of cardinality c is not normal. By a recent result of W. G. Fleissner, [4], such a space is not countably paracompact either. And it is obvious that such a space fails to have many other properties, like metacompactness and collectionwise Hausdorffness. Therefore, the following theorem shows that S^2, and hence S^K for $\kappa \geqq 2$, is not the union of countably many nice subspaces.

THEOREM 6. *For every countable family* $\mathscr{E}$ *of subspaces of* S^2 *that covers* S^2 *there is an* $E \in \mathscr{E}$ *which contains a closed separable subspace that has a closed discrete subset of cardinality* c.

Since S^2 is (weakly) θ-refinable, [8, 2.9], it follows that a weakly θ-refinable space need not be the union of countably many metacompact subspaces (the converse is patently true).

The organization of this paper is as follows. Theorem 2 is proved in §1, where we also consider some of its applications different from Theorem 1. Theorems 3 and 4 are proved in §2, and Theorem 6 is proved in §3. In the Appendix we collect some properties of GLS-spaces.

1. Proof of Theorem 2, and consequences.

1.1. *The proof.* Let X be a GLS-space with GLS-relation $\leqq$. Let φ be a neighborhood assignment for X. Define a new neighborhood assignment ψ for X by

$$\psi(x) = \{y \in \varphi(x) : x \leqq y\} .$$

It suffices to construct a closed discrete set D in X with $\bigcup \psi[D] = X$. With transfinite recursion construct, if possible, an $x_\xi \in X$ in such a way that for each x_ξ, when defined,

$$x_\xi \text{ is a } \leqq\text{-minimal element of } A_\xi = X - \bigcup\{\psi(x_\eta) : \eta < \xi\} .$$

We can find such an x_ξ if $A_\xi \neq \varnothing$, since clearly A_ξ is closed. Let α be the ordinal at which the construction breaks down because $A_\alpha = \varnothing$; α exists, for $x_\xi \neq x_\eta$ if $\xi \neq \eta$. Let $D = \{x_\xi : \xi < \alpha\}$. Then $\bigcup \psi[D] = X - A_\alpha = X$.

It remains to show that D is closed and discrete. To this end it suffices to prove that $\psi(x) \cap D = \{x\}$ for all $x \in D$, since $\bigcup \psi[D] = X$. Let $x_\xi \in \psi(x_\eta)$ for some $\xi, \eta < \alpha$. Clearly $\xi \leqq \eta$, and $x_\eta \leqq x_\xi$. Both x_η and x_ξ belong to A_ξ. Consequently $x_\eta = x_\xi$, as x_ξ is $\leqq$-minimal in A_ξ.

1.2. *Consequences.* Theorem 2 has been used in [3] for proving that certain spaces are D-spaces.

Let $\mathscr{K}(X)$ be the collection of nonempty compact subsets of a Hausdorff space X. Equip $\mathscr{K}(X)$ with the so-called Pixley-Roy topology, i.e., basic neighborhoods about $F \in \mathscr{K}(X)$ have the form $\{G \in \mathscr{K}(X) : F \subseteq G \subseteq U\}$, where U is an open neighborhood of F in X. It is noted in [3] that $\mathscr{K}(X)$ is a GLS-space (ordinary inclusion is a GLS-relation). Consequently $\mathscr{K}(X)$ is a D-space.

Let 2^N be the hyperspace of the integers, i.e., the nonempty subsets of N, equipped with the Vietoris (or finite, or exponential) topology. It is shown in [3] that 2^N is a GLS-space (reverse inclusion is a GLS-relation). Consequently 2^N is a D-space. The same argument would work for the hyperspace of an uncountable discrete space.

1.3. REMARK. Assume that a GLS-space X admits a GLS-relation satisfying

(1) $\leqq$ is a linear order (or, equivalently, every nonempty closed subset of X has a unique $\leqq$-minimal element);

(2) The intervals $[x, y) = \{z \in X: x \leqq z \prec y\}$ form a base for the topology of X.

The technique of the proof in 1.1 can be used to show that X is ultraparacompact. (Recall that a space is *ultracompact* if every open cover has a disjoint open refinement.) Indeed, let $\mathscr{U}$ be an open cover of X. For each $x \in X$, choose $\psi(x) = [x, y_x)$ so that $\psi(x) \subset U$ for some $U \in \mathscr{U}$. If D is as in 1.1, then $\psi[D]$ is a disjoint open refinement of $\mathscr{U}$.

In the proof of Theorem 3, we will see that S and $H = S \cap [0, \infty)$ are homeomorphic, and that $\leqq$ is a GLS-relation on H. So we have a somewhat unusual proof of the well-known fact that S is ultraparacompact.

2. **Proof of Theorems 3 and 4.** For the proof of Theorem 3 we will need the following simple lemma.

LEMMA 2.1. *Let $\leqq$ be a reflexive and transitive binary relation on a space X such that for every nonempty $\leqq$-chain K in X there is an $m \in K^-$ with $m \leqq x$ for all $x \in K$. Then each nonempty closed subset of X has a $\leqq$-minimal element.*

Proof. Let F be a nonempty closed subset of X. For every nonempty $\leqq$-chain K in F there is an $m \in K^-$ with $m \leqq x$ for all $x \in K$; clearly $m \in F$. It follows from the Kuratowski-Zorn lemma that F has a $\leqq$-minimal element.

2.2. *Proof of Theorem* 3. Let H be $S \cap [0, \infty)$, half the Sorgenfrey line. Then H and S are homeomorphic, for both admit a disjoint open cover by countably many copies of $S \cap [0, 1)$. We shall consider H instead of S. Let n be nonnegative integer. As usual, the ith coordinate of $x \in H^n$ is x_i, $1 \leqq i \leqq n$. Define a reflexive and transitive binary relation $\leqq$ on H^n by

$$x \leqq y \quad \text{if} \quad x_i \leqq y_i \quad \text{for all} \quad 1 \leqq i \leqq n\,.$$

Clearly $\{y \in H^n: x \leqq y\}$ is open in H^n for each $x \in H^n$. Let $K \subseteq H^n$ be a $\leqq$-chain, and define $m \in H^n$ by

$$m_i = \inf\{x_i \colon x \in K\}\ .$$

Then $m \leqq x$ for each $x \in K$, and since K is a $\lesseqq$-chain, it is easy to see that for each $\varepsilon > 0$ there is an $x \in K$ such that $m_i \leqq x_i < m_i + \varepsilon$ for $1 \leqq i \leqq n$. Consequently $m \in K^-$. It follows from the above lemma that $\lesseqq$ is a GLS-relation.

2.3. *Proof of Theorem* 4. Let $\lesseqq$ be a reflexive binary relation on T such that $\{y \in T \colon x \lesseqq y\}$ is open in T for each $x \in T$. Now T is a Baire space as a subspace of S (since it is a Baire space as a subspace of R). Hence one can easily find $\varepsilon > 0$ and a, b with $a < b$, and a dense subset E of $(a, b) \cap T$, such that $[x, x + \varepsilon) \cap T \subseteq \{y \in T \colon x \lesseqq y\}$ for each $x \in E$. Let q be a *rational* number, $a < q < b$. There is a set $F = \{x_n \colon n \geqq 1\} \subseteq E$ such that

$$q < x_{n+1} < x_n < q + \min\{\varepsilon, 1/n\} \quad \text{for } n \geqq 1\ .$$

Then F is closed in T but has no $\lesseqq$-minimal element, since $x_{n+1} \prec x_n$ (i.e., $x_{n+1} \lesseqq x_n$ and $x_{n+1} \neq x_n$) for all $n \geqq 1$.

3. Proof of Theorem 6. Let R be the real line with its usual topology. For $x \in R$ let $R_x = R \times \{x\}$.

3.1. *The proof*. For each $z \in R$ the set

$$V_z = \{\langle x, y\rangle \in S^2 \colon x + y = z\}$$

is closed and discrete in S^2. Hence it suffices to show that there are an $E \in \mathscr{E}$ and a $z \in R$ such that some separable, not necessarily closed, subspace intersects V_z in a set of cardinality c.

It is easy to see that if $A \subseteq R^2$ is dense in some open subspace of R^2, then A, considered as subspace of S^2, is separable. Since all sets V_z are lines of slope -1, the theorem follows from the following lemma.

LEMMA 3.2. *Let $\mathscr{A}$ be any countable family of subsets of R^2 that covers R^2. Then there are an $A \in \mathscr{A}$ and a nonempty open U in R^2 such that*

$$\bigcup\{R_x \cap A \cap U \colon x \in R,\ |R_x \cap A \cap U| = c\}$$

is dense in U.

Proof. Let $\mathscr{B}$ be a countable base for R^2. We argue by contradiction, and suppose that for each $A \in \mathscr{A}$ and $B \in \mathscr{B}$ we can find a nonempty open $V(A, B) \subseteq B$ such that

$$|R_x \cap A \cap V(A, B)| < c \quad \text{for all } x \in R .$$

For $A \in \mathscr{A}$ let $W(A) = \bigcup\{V(A, B): B \in \mathscr{B}\}$. Then $W(A)$ is a dense open set in R^2, and since the union of countably many sets of cardinality less than c has again cardinality less than c,

$$|R_x \cap A \cap W(A)| < c \quad \text{for all } x \in R .$$

If $G = \bigcap\{W(A): A \in \mathscr{A}\}$, then $|R_x \cap A \cap G| < c$ for all $x \in R$. Since $G \subseteq \bigcup \mathscr{A}$, the same argument shows that

$$|R_x \cap G| < c \quad \text{for all } x \in R .$$

However, this is impossible. For by [7], or [9, 15.1], every intersection of countably many dense open sets in R^2 intersects R_x in a set that is dense in R_x with respect to the subspace topology for at least one x. But then $R_x \cap G$ is a dense G_δ in R_x, hence has cardinality c, cf. [9, 5.1].

4. Appendix: GLS-spaces. GLS-spaces are useful in proving that certain spaces are D-spaces. Here we collect some of their basic properties. We omit straightforward proofs.

We did not postulate that a GLS-relation be transitive; an easy example shows that it need not be: define a GLS-relation $\preceq$ on the nonnegative integers by $k \preceq n$ iff $k \leqq n \leqq k + 1$. (We did not attempt to decide whether or not every GLS-space admits a transitive GLS-relation.) On the other hand, since all spaces considered are T_1, we have the following.

PROPOSITION 4.1. *Every* GLS-*relation is antisymmetric.*

This can be used to prove the following propositions.

PROPOSITION 4.2. *A compact Hausdorff space without isolated points is not a* GLS-*space.*

The example of a countable space with the cofinite topology shows that the Hausdorff condition is essential. A collection $\mathscr{A}$ of subsets of a space X is called a *network* for X if for every open U in X and every $x \in U$ there is an $A \in \mathscr{A}$ with $x \in A \subseteq U$.

PROPOSITION 4.3. *If the space X has a network $\mathscr{A}$ with $|\mathscr{A}| < |X|$, then X is not a* GLS-*space. In particular, no uncountable separable metrizable space is a* GLS-*space.*

The following proposition should be compared with analogous results for D-spaces [2].

PROPOSITION 4.4. (a) *A closed subspace of a* GLS-*space is a* GLS-*space.*

(b) *If* $X = X_1 \cup X_2$, *with* X_1 *and* X_2 GLS-*spaces and* X_1 *closed, then* X *is a* GLS-*space.*

(c) *A space is a* GLS-*space if it is the union of countably many closed subspaces, each of which is a* GLS-*space.*

Note that it follows from 4.4 (a), and 4.2 or 4.3 that no infinite product of nontrivial spaces is a GLS-space. In particular, S^ω is not a GLS-space.

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