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THE CENTRALIZER OF TENSOR PRODUCTS OF BANACH SPACES (A FUNCTION SPACE REPRESENTATION)

EHRHARD BEHRENDS

Let X, Y be real Banach spaces, $X \hat{\otimes}_\varepsilon Y$ their usual ε-tensor product. We represent $Z(X \hat{\otimes}_\varepsilon Y)$, the centralizer of $X \hat{\otimes}_\varepsilon Y$, as a space of real-valued functions on a suitable compact Hausdorff space. As a corollary we obtain Wickstead's result: $Z(X \hat{\otimes}_\varepsilon Y)$ is the closure with respect to the strong operator topology of $Z(X) \otimes Z(Y)$. In addition it is shown that $Z(X \hat{\otimes}_\varepsilon Y)$ is in fact the uniform closure of $Z(X) \otimes Z(Y)$ provided the norm topology and the strong operator topology coincide on the centralizers of X and Y.

1. Introduction. Let X be a real Banach space. By $Z(X)$, the *centralizer* of X, we denote the set of M-bounded operators on X, i.e., the collection of those continuous linear operators $T: X \to X$ for which there is a $\lambda \in \boldsymbol{R}$ such that Tx is contained in every open ball which contains $\pm\lambda x$ (for $x \in X$); cf [2], [3], [4], [5], [8]. $Z(X)$ is, as a Banach algebra, isometrically isomorphic to the space $C(K_X)$ of continuous real-valued functions on a suitable compact Hausdorff space K_X: $C(K_X) \cong Z(X)$ ([2], 4.8).

For example, if L is a locally compact Hausdorff space and $X: = C_0L: = \{f \mid f: L \to \boldsymbol{R}, f \text{ continuous}, f \text{ vanishes at infinity}\}$, provided with the supremum norm, then it is easy to see that $Z(X)$ is identical with the space of all multiplication operators M_h, $f \mapsto hf$ (all $f \in C_0L$), h a bounded continuous function. Therefore $Z(X)$ is isometrically isomorphic with $C^bL: = \{h \mid h: L \to \boldsymbol{R}, h \text{ continuous and bounded}\}$ so that $K_X = \beta L =$ the Stone-Čech compactification of L (up to homeomorphism).

Centralizers of Banach spaces play an important role in a great number of papers (cf. for example the references in [2]). We will investigate the centralizer of tensor products. In particular we are interested in the relation between the centralizer of the tensor product and the centralizers of the factors. Let X and Y be real Banach spaces, $X \otimes Y$ their algebraic tensor product. For $\sum_{i=1}^r x_i \otimes y_i \in X \otimes Y$ we define

$$\left\| \sum_{i=1}^r x_i \otimes y_i \right\| : = \sup \left\{ \sum_{i=1}^r f(x_i)\tilde{f}(y_i) \;\middle|\; f \in X',\ \|f\| \leqq 1,\ \tilde{f} \in Y',\ \|\tilde{f}\| \leqq 1 \right\}$$

$$\left(= \sup \left\{ \left\| \sum_{i=1}^r f(x_i) y_i \right\| \;\middle|\; f \in X',\ \|f\| \leqq 1 \right\} \right.$$

$$= \sup \left\{ \left\| \sum_{i=1}^{r} \tilde{f}(y_i)x_i \right\| \,\middle|\, \tilde{f} \in Y', \|\tilde{f}\| \leq 1 \right\} \Big) ;$$

we will use the same symbol $\| \quad \|$ to denote the norm in all tensor products of Banach spaces which will appear in this paper — this is justified because we will not consider any other tensor product norms. $X \hat{\otimes}_\varepsilon Y$ means the completion of $X \otimes Y$ provided with this norm.

It is not hard to see that, for $T \in Z(X)$ and $S \in Z(Y)$ we have $S \otimes T \in Z(X \hat{\otimes}_\varepsilon Y)$ ([8], p. 564; note that Wickstead uses another but equivalent definition of M-boundedness and that he writes $\otimes_\lambda$ instead of $\hat{\otimes}_\varepsilon$). Therefore $Z(X) \otimes Z(Y)$ may be thought of as a subspace of $Z(X \hat{\otimes}_\varepsilon Y)$. We note that the tensor product norm of the operators in $Z(X) \otimes Z(Y)$ is exactly their operator norm. Wickstead proves ([8], Th. 3) that $Z(X \hat{\otimes}_\varepsilon Y)$ is the strong closure of $Z(X) \otimes Z(Y)$. In general the strong closure may not be replaced by the uniform closure in this theorem. There are, however, important classes of Banach spaces for which $Z(X \hat{\otimes}_\varepsilon Y)$ is the uniform closure of $Z(X) \otimes Z(Y)$. We will prove in §4 that this is the case if the strong operator topology and the norm topology are equivalent on the centralizers of X and Y.

We will proceed as follows: In §2 we will state without proof those results of the function module representation theory introduced in [5] which we will need in the sequel. We will show that $X \hat{\otimes}_\varepsilon Y$ has a function module representation which is related to the function module representations of X and Y in a natural way, a theorem which will be of fundamental importance for the following considerations. Section 3 contains a discussion of those Banach spaces X for which the norm topology and the strong operator topology on $Z(X)$ are equivalent. In §4 we will show that $Z(X \hat{\otimes}_\varepsilon Y)$ is isometrically isomorphic to a space of real-valued bounded(not necessarily continuous) functions on a suitable compact Hausdorff space. Finally, we investigate some consequences of this representation theorem. For example, we derive Wickstead's result as a corollary.

Note. In the first version of this paper Wickstead's theorem was used at a crucial point in the proof of Theorem 4.2. We are grateful to the referee for suggesting that we give an independent proof using the theory of function modules.

2. A function module representation of $X \hat{\otimes}_\varepsilon Y$.

DEFINITION 2.1 ([5]). Let K be a compact Hausdorff space, $(W_k)_{k \in K}$ a family of Banach spaces indexed by the points of K. A closed subspace W of

$$\prod_{k\in K}^{\infty} W_k := \left\{(w(k))_{k\in K} \,\middle|\, (w(k))_{k\in K} \in \prod_{k\in K} W_k\ , \right.$$
$$\left. \|(w(k))_{k\in K}\| := \sup_{k\in K} \|w(k)\| < \infty \right\}$$

is called a *function module* in $\prod_{k\in K}^{\infty} W_k$ if the following conditions are satisfied:

(a) $hw \in W$ for $h \in CK$, $w \in W$ ($(hw)(k) := h(k)w(k)$ for $k \in K$)

(b) $k \mapsto \|w(k)\|$ is upper semi-continuous on K for $w \in W$

(c) $W_k = \{w(k) \mid w \in W\}$ for $k \in K$.

Note. By [5], p. 621, $\{w(k) \mid w \in W\}$ is closed for each $k \in K$ if W is a closed subspace of $\prod_{k\in K}^{\infty} W_k$ and (a) and (b) are satisfied.

PROPOSITION 2.2. *Let W be as in the preceding definition. For $h \in CK$, the multiplication operator $M_h: W \to W$, $w \mapsto hw$, is well-defined by* 2.1(a). *We claim that $M_h \in Z(W)$. More generally, if $\alpha: K \to \boldsymbol{R}$ is a bounded function such that $M_\alpha(W) \subset W$, then $M_\alpha \in Z(W)$. In addition, M_α is contained in the strong operator closure of $\{M_h \mid h \in CK\}$.*

Proof. It is easy to see that $M_\alpha: W \to W$ is linear and continuous with $\|M_\alpha\| \leqq \|\alpha\| := \sup\{|\alpha(k)| \mid k \in K\}$($\alpha: K \to \boldsymbol{R}$ a bounded function such that $M_\alpha(W) \subset W$). M_α obviously satisfies the condition for M-bounded operators with $\lambda = \|\alpha\|$.

Let $w_1, \cdots, w_n \in W$, $\varepsilon > 0$ be arbitrarily given. For every $k \in K$, $\alpha(k)w_i - \alpha w_i$ is in W and vanishes at k, so that, by 2.1(b), there is an open neighborhood U_k of k such that $\|(\alpha(k)w_i - \alpha w_i)(1)\| \leqq \varepsilon$ for 1 in U_k (all $i \in \{1, \cdots, n\}$). Let $U_{k_1}, \cdots, U_{k_r}$ be a finite covering of K. Then $\|hw_i - \alpha w_i\| \leqq \varepsilon$ for $i = 1, \cdots, n$, where $h := \sum_{j=1}^{r} \alpha(k_j)h_j$ and $h_1, \cdots, h_r$ is a suitable partition of unity subordinate to $U_{k_1}, \cdots, U_{k_r}$. This proves that M_α is in the strong closure of $\{M_h \mid h \in CK\}$.

THEOREM 2.3. *Let X be a real Banach space, K_X a compact Hausdorff space such that $Z(X) \cong CK_X$ (note that K_X is uniquely determined up to homeomorphism). X can be identified with a function module in $\prod_{k\in K_X}^{\infty} X_k$ ($(X_k)_{k\in K_X}$ a family of Banach spaces, the component spaces) such that the operators in $Z(X)$ correspond to multiplication operators associated with the elements of CK_X. More precisely, there is a linear isometry $\omega: X \to \prod_{k\in K_X}^{\infty} X_k$ such that*

(i) *$\omega(X)$ is a function module in $\prod_{k\in K_X}^{\infty} X_k$.*

(ii) *for $T \in Z(X)$, $x \in X$ we have $\omega(Tx) = \tilde{T}\omega(x)$, where $\tilde{T} \in CK_X$ corresponds to T according to the isometry $Z(X) \cong CK_X$.*

In addition we have

(iii) *$\{k \mid X_k \neq 0\}$ is dense in K_X.*

Proof. (i) and (ii) are proved in [5] (Theorem 6 and Theorem 3; note that the maximal M-structure of X is just $Z(X)$ by [2], 4.8). (iii) can be verified as follows: If $\tilde{T} \in CK_X$ is an arbitrary function with corresponding operator $T \in Z(X)$, then we have $\|\tilde{T}\| = \|T\| = \sup\{\|Tx\| \mid \|x\| = 1\} = \sup\{\|\tilde{T}\omega(x)\| \mid \|x\| = 1\} \leqq \sup\{|\tilde{T}(k)| \mid X_k \neq 0\}$. This implies that $\{k \mid X_k \neq 0\}$ is dense in K_X.

THEOREM 2.4. *Let X(resp. Y) be a function module in $\prod_{k\in K}^\infty X_k$ (resp. $\prod_{1\in L}^\infty Y_1$), where K and L are compact Hausdorff spaces. For $\sum_{i=1}^r x_i \otimes y_i \in X \otimes Y$ let $\sum_{i=1}^r x_i \tilde{\otimes} y_i$ be the element*

$$\left(\sum_{i=1}^r x_i(k) \otimes y_i(1)\right)_{(k,1)\in K\times L}$$

of $\prod_{k,1}^\infty X_k \hat{\otimes}_\varepsilon Y_1$. Then

(i) $\|\sum_{i=1}^r x_i \otimes y_i\| = \|\sum_{i=1}^r x_i \tilde{\otimes} y_i\|$ for $\sum_{i=1}^r x_i \otimes y_i \in X \otimes Y$ so that $X \hat{\otimes}_\varepsilon Y$ can be identified with a closed subspace of $\prod_{k,1}^\infty X_k \hat{\otimes}_\varepsilon Y_1$; further, it is not necessary to distinguish between $x \otimes y$ and $x \tilde{\otimes} y$.

(ii) $X \hat{\otimes}_\varepsilon Y$ is a function module in $\prod_{k,1}^\infty X_k \hat{\otimes}_\varepsilon Y_1$.

Proof. (i) We will use the fact that the extreme points of the unit ball $S_1^{X'}$(resp. $S_1^{Y'}$) of X'(resp. Y') are contained in the set of functionals of the form $x \mapsto f(x(k))$(resp. $y \mapsto \tilde{f}(y(1))$) where $k \in K$, $f \in (X_k)'$, $\|f\| \leqq 1$(resp. $1 \in L$, $\tilde{f} \in (Y_1)'$, $\|\tilde{f}\| \leqq 1$); [6].

$$\begin{aligned}
\left\|\sum_{i=1}^r x_i \otimes y_i\right\| &= \sup\{\textstyle\sum F(x_i)\tilde{F}(y_i) \mid F \in X', \|F\| \leqq 1, \tilde{F} \in Y', \|\tilde{F}\| \leqq 1\}\\
&= \sup\{\textstyle\sum F(x_i)\tilde{F}(y_i) \mid F \in \operatorname{ex} S_1^{X'}, \tilde{F} \in \operatorname{ex} S_1^{Y'}\}\\
&= \sup\{\textstyle\sum f(x_i(k))\tilde{f}(y_i(1)) \mid k \in K, f \in (X_k)', \|f\| \leqq 1,\\
&\qquad 1 \in L, \tilde{f} \in (Y_1)', \|\tilde{f}\| \leqq 1\}\\
&= \sup\{\|\textstyle\sum x_i(k) \otimes y_i(1)\| \mid k \in K, 1 \in L\}\\
&= \|\textstyle\sum x_i \tilde{\otimes} y_i\|\,.
\end{aligned}$$

Similarly one can prove that $\|\sum_{i=1}^r x_i(k) \otimes y_i\| = \sup_{1\in L}\|\sum_{i=1}^r x_i(k) \otimes y_i(1)\|$ for $k \in K$ (where the norms are calculated in $X_k \hat{\otimes}_\varepsilon Y$ and $X_k \hat{\otimes}_\varepsilon Y_1$, respectively).

(ii) We only have to show that

(a) $h(\sum x_i \otimes y_i) \in X \hat{\otimes}_\varepsilon Y$ for $h \in C(K \times L)$, $\sum x_i \otimes y_i \in X \otimes Y$.

(b) $(k, 1) \mapsto \|\sum x_i(k) \otimes y_i(1)\|$ is upper semi-continuous for $\sum x_i \otimes y_i \in X \otimes Y$

(c) $X \otimes Y$ is dense in $X \hat{\otimes}_\varepsilon Y$.

(a), (b), and (c) easily imply that $(X \otimes_\varepsilon Y)^- = X \hat{\otimes}_\varepsilon Y$ is a function module (cf. the note at the end of 2.1).

(a) Let $h \in C(K\times L)$, $\sum x_i \otimes y_i \in X \otimes Y$. For $\varepsilon > 0$ there are $h_1, \cdots,$

$h_n \in CK$, $g_1, \cdots, g_n \in CL$ such that $\|\sum_{j=1}^{n} h_j \otimes g_j - h\| \leqq \varepsilon$. We thus have

$$\begin{aligned}&\|h \sum x_i \otimes y_i - (\sum h_j \otimes g_j)(\sum x_i \otimes y_i)\| \\ &\quad = \|h \sum x_i \otimes y_i - \sum_{i,j} h_j x_i \otimes g_j y_i\| \leqq \varepsilon \|\sum x_i \otimes y_i\| .\end{aligned}$$

Since $\sum_{i,j} h_j x_i \otimes g_j y_i \in X \otimes Y$ this implies that $h \sum x_i \otimes y_i \in X \hat{\otimes}_\varepsilon Y$.

(b) Let $a \in \boldsymbol{R}$, $(k_0, 1_0) \in K \times L$, $\sum_{i=1}^{r} x_i \otimes y_i \in X \otimes Y$, $\|\sum x_i(k_0) \otimes y_i(1_0)\| < a$. We have to show that there are neighbourhoods U of k_0, V of 1_0 such that $\|\sum x_i(k) \otimes y_i(1)\| < a$ for $k \in U, 1 \in V$.

At first we will prove that there is a neighborhood $\tilde{V}$ of 1_0 such that $\|\sum x_i(k_0) \otimes y_i(1)\| < a - 2\eta$ for $1 \in \tilde{V}$ (where $\eta > 0$ is a number such that $\|\sum x_i(k_0) \otimes y_i(1_0)\| < a - 3\eta$). To this end we choose an (η/R)-net $f_1, \cdots, f_N$ in the dual unit ball of the linear hull of $x_1(k_0), \cdots, x_r(k_0)$ $(R := \sum \|x_i\| \|y_i\| + 1)$. It follows that, for $f \in (X_{k_0})'$, $\|f\| \leqq 1$, there is an $f_j \in \{f_1, \cdots, f_N\}$ such that $\|\sum_i f_j(x_i(k_0)) y_i(1) - \sum_i f(x_i(k_0)) y_i(1)\| \leqq \|f_j - f\| \|\sum x_i(k_0) \otimes y_i(1)\| \leqq \|f_j - f\| R$ (all $1 \in L$), i.e.,

$$\begin{aligned}\|\sum x_i(k_0) \otimes y_i(1)\| &= \sup \{\|\sum f(x_i(k_0)) y_i(1)\| \mid f \in (X_{k_0})', \|f\| \leqq 1\} \\ &\leqq \sup \{\|\sum f_j(x_i(k_0)) y_i(1)\| \mid j = 1, \cdots, N\} + \eta\end{aligned}$$

(all $1 \in L$).

For $j \in \{1, \cdots, N\}$, $\sum_i f_j(x_i(k_0)) y_i$ belongs to Y and $\|\sum_i f_j(x_i(k_0)) y_i(1_0)\| \leqq \|\sum_i x_i(k_0) \otimes y_i(1_0)\| < a - 3\eta$ so that by 2.1(b) there is a neighbourhood $\tilde{V}$ of 1_0 with $\|\sum_i f_j(x_i(k_0)) y_i(1)\| < a - 3\eta$ for $1 \in \tilde{V}$ and $j \in \{1, \cdots, N\}$. For $1 \in \tilde{V}$ we thus have $\|\sum x_i(k_0) \otimes y_i(1)\| < a - 2\eta$.

We now choose a function $g \in CL$ such that $\|g\| = 1$, $g(1) = 1$ in a suitable neighborhood V of 1_0 contained in $\tilde{V}$ and $g|_{L \setminus V} = 0$. We then have (cf. the proof of (i)) $\|\sum x_i(k_0) \otimes g y_i\| = \sup_{1 \in L} \|\sum x_i(k_0) \otimes g(1) y_i(1)\| \leqq a - 2\eta$. Similarly to the first step of this proof we select an (η/R)-net $\tilde{f}_1, \cdots, \tilde{f}_M$ in the dual unit ball of the linear hull of $g y_1, \cdots, g y_r$ (it follows that $\|\sum x_i(k) \otimes g y_i\| \leqq \sup \{\|\sum_i \tilde{f}_j(g y_i) x_i(k)\| \mid j = 1, \cdots, M\} + \eta$ for $k \in K$). For $j \in \{1, \cdots, M\}$ we have $\sum_i \tilde{f}_j(g y_i) x_i \in X$ and $\|\sum \tilde{f}_j(g y_i) x_i(k_0)\| < a - \eta$. Therefore there is a neighborhood U of k_0 such that $\|\sum \tilde{f}_j(g y_i) x_i(k)\| < a - \eta$ for $k \in U$, $j = 1, \cdots, M$. This yields

$$\begin{aligned}\sup_{1 \in V} \|\sum x_i(k) \otimes y_i(1)\| &\leqq \sup_{1 \in L} \|\sum x_i(k) \otimes (g y_i)(1)\| \\ &= \|\sum x_i(k) \otimes g y_i\| \\ &\leqq \sup \{\|\sum \tilde{f}_j(g y_i) x_i(k)\| \mid j = 1, \cdots, M\} + \eta \\ &< a \quad \text{for } k \in U .\end{aligned}$$

(c) This is obvious.

REMARK. For the rest of this paper we will assume that X and Y are real Banach spaces which are identified with function modules in $\prod_{k\in K_X}^{\infty} X_k$ resp. $\prod_{l\in K_Y}^{\infty} Y_l$ as described in 2.3. With this identification, $X\tilde{\otimes}_\varepsilon Y$ is a function module in $\prod_{k,l}^{\infty} X_k \hat{\otimes}_\varepsilon Y_l$ by 2.4.

Another way of representing the centralizer as a space of real-valued continuous functions is the Dauns-Hofmann type theorem of Alfsen-Effros ([2], 4.9). The relationship between this and the function module approach (2.3(ii)) is shown by the following proposition.

PROPOSITION 2.5. *Let X, K_X, $(X_k)_{k\in K_X}$ be as above,* $K_X^*: = \{k \mid k\in K_X,\ X_k \neq 0\}$.

(i) *Every $h_0 \in C^b(K_X^*)$ has a unique continuous extension to K_X (so that $K_X = \beta K_X^*$).*

(ii) *Let E_X be the set of extreme points in the unit ball of X'. By [6] we have $E_X = \bigcup_{k\in K_X^*} E_{X_k}$. Let $\pi: E_X \to K_X^*$ be defined by $\pi(p): = k$ for $p \in E_{X_k}$. Then, for every bounded structurally continuous mapping $g: E_X \to \boldsymbol{R}$ there is a function $h \in C^b(K_X^*)$ such that $g = h\circ\pi$. Conversely, for $h \in C^b(K_X^*)$, $h\circ\pi$ is structurally continuous.*

Proof. (i) Let $h_0 \in C^b(K_X^*)$ be given. We define $h: K_X \to \boldsymbol{R}$ by $h(k): = h_0(k)$ for $k \in K_X^*$ and $h(k) = 0$ for $k \in K_X \backslash K_X^*$. Let $x \in X$ be given and $\varepsilon > 0$. h is continuous on the closed set $D: = \{k \mid \|x(k)\| \geqq \varepsilon\} \subset K_X^*$ so that we may choose a continuous function $h_D: K_X \to \boldsymbol{R}$ such that $h|_D = h_D|_D$, $\|h\| = \|h_D\|$. We then have $h_D x \in X$ and $\|h_D x - hx\| \leqq 2\varepsilon\|h\|$ so that we may conclude that $hx \in X^- = X$. 2.2 and 2.3(ii) imply that there is a function $h' \in CK_X$ such that $M_h = M_{h'}$. h' is obviously a continuous extension of h which is uniquely determined by 2.3 (iii).

(ii) Let $g: E_X \to \boldsymbol{R}$ be a bounded structurally continuous function. By [2], 4.9, there is a $T \in Z(X)$ such that $p\circ T = g(p)p$ for every $p \in E_X$. Let $\tilde{T} \in CK_X$ be that function which corresponds to T. We then have $\tilde{T}(k)p = g(p)p$ for p in E_{X_k} so that $\tilde{T}\circ\pi = g$. Conversely, let $\tilde{T} \in CK_X$ be given. For $p \in E_{X_k}$ we have $p\circ T = \tilde{T}(k)p = (\tilde{T}\circ\pi)(p)p$. By [2], 4.9 this implies that $\tilde{T}\circ\pi$ is structurally continuous.

3. **Centralizer-norming systems.** In view of the following considerations we want to single out those Banach spaces for which, in a sense, the centralizer is "not too great".

DEFINITION 3.1. Let X be a real Banach space. A finite family $x_1, \cdots, x_n$ in X is called a *centralizer-norming system* (abbreviated: *cns*) if there is a number $r > 0$ such that $\max\{\|Tx_i\| \mid i = 1, \cdots, n\} \geqq$

$r\|T\|$ for every $T \in Z(X)$. Obviously X has a *cns* iff the norm topology and the strong operator topology coincide on $Z(X)$.

EXAMPLES. (1) Let X be a Banach space for which $Z(X)$ if finite-dimensional (those spaces play an important role in the applications of M-structure to theorems of the Banach-Stone type; cf. [3], [4]). It is clear that X has a *cns* (in fact, X has a *cns* consisting of a single element).

We note that, for example, spaces which are smooth or strictly convex have one-dimensional centralizer and that $Z(X)$ is finite-dimensional for every reflexive space X([4]).

(2) If L is a locally compact Hausdorff space, then C_0L has a *cns* iff L is compact. In this case we may choose $n = 1$ and $x_1 = 1$ (= the constant function assuming the value 1 at every point).

(3) Let A be a C^*-algebra with unit e, X the self-adjoint part of A. Then $\{e\}$ is a cns in X since $Z(X)$ is just the space of multiplication operators corresponding to the self-adjoint elements in the center of A ([2], Cor. 6.17).

(4) One might suggest that for Banach spaces X having a *cns* it is always possible to find a *cns* consisting of a single element. We will use the Borsuk-Ulam theorem from algebraic topology to prove that $\inf\{n \mid n \in \boldsymbol{N}$, there exists a *cns* in X consisting of n elements$\}$ may be an arbitrarily large number:

For $m \in \boldsymbol{N}$ let S^m be the m-dimensional sphere (i.e., the surface of the unit ball in the $(m+1)$-dimensional Hilbert space), $X := \{f \mid f \in C(S^m),\ f(-x) = -f(x)$ for all $x \in S^m\}$. (X is just the space $C_\Sigma(S^m)$, where $\sum: S^m \to S^m$ is the homeomorphism $x \mapsto -x$; cf. [7], Chapter 3, p. 71). A routine computation shows that $T \in Z(X)$ iff there is a continuous function $h: S^m \to \boldsymbol{R}$ such that $h(x) = h(-x)$ for all $x \in S^m$ and $Tf = hf$ for $f \in X$. Therefore a family $f_1, \cdots, f_n$ in X is a *cns* iff $\max\{|f_i(x)| \mid i = 1, \cdots, n\} > 0$ for all $x \in S^m$. X obviously has a *cns* consisting of $m+1$-elements (for example, $f_i(x)$: = the ith component of x, $x \in S^m$, $i = 1, \cdots, m+1$, defines a family of functions with this property). On the other hand, if $g_1, \cdots, g_m$ are arbitrary functions in X, there is an $x_0 \in S^m$ such that $g_1(x_0) = \cdots = g_m(x_0) = 0$, i.e., $g_1, \cdots, g_m$ cannot be a *cns* ([1], p. 485).

We will need the fact that there is a characterization of centralizer-norming systems in terms of the function module representation 2.3:

LEMMA 3.2. *Let X be a real Banach space, X represented as a function module in $\prod_{k \in K_X}^{\infty} X_k$ as described in* §2.

A finite family $x_1, \cdots, x_n$ in X is a *cns* iff $\inf_k \max_i \|x_i(k)\| > 0$.

Proof. Suppose that $x_1, \cdots, x_n$ is a *cns* in X, i.e., there is a number $r > 0$ such that $\max_i \|Tx_i\| \geqq r\|T\|$ for $T \in Z(X)$. We claim that $\max_i \|x_i(k)\| \geqq r$ for $k \in K_X$. Assume that there is a $k_0 \in K_X$ such that $\|x_i(k_0)\| < r$ for $i = 1, \cdots, n$. Since X is a function module, there is a neighborhood U of k_0 such that $\|x_i(k)\| \leqq r' < r$ for $k \in U$ and $i = 1, \cdots, n$. But then, for a suitable function $h \in CK_X$ (which corresponds to $M_h \in Z(X)$) we get $\max_i \|M_h x_i\| = \max_i \|hx_i\| \leqq r'\|h\| < r\|M_h\|$, a contradiction.

The reverse conclusion is obvious.

In §4 we will also need a related definition, which by 3.2 is a local version of Definition 3.1.

DEFINITION 3.3. (X, K_X as in 3.2). Let k_0 be a point of K_X. A finite family $x_1, \cdots, x_n$ is called a *local centralizer-norming system* (local *cns*) at k_0, if there are a number $r > 0$ and a neighborhood U of k_0 such that $\max_i \|x_i(k)\| \geqq r$ for $k \in U$.

A simple compactness argument guarantees that X has a *cns* iff every point in K_X has a local *cns*.

EXAMPLE. Let L be a locally compact Hausdorff space, $X := C_0 L$. A point k in $K_X = \beta L$ has a local *cns* iff $k \in L$. However, every point k in K_X has a local *cns* provided $X_k \neq 0$. There are known to the author only very complicated examples of Banach spaces not having this property. We will say that X has the *local cns property* if every k with $X_k \neq 0$ has a local *cns*.

4. The structure of $Z(X \hat{\otimes}_\varepsilon Y)$.

Let X, K_X, $(X_k)_{k \in K_X}$, Y, K_Y, $(Y_1)_{1 \in K_Y}$ be as in §2.

DEFINITION 4.1. $M(K_X \times K_Y) := \{\alpha \mid \alpha: K_X \times K_Y \to \boldsymbol{R}$ a bounded function, $\alpha(k, 1) = 0$ whenever $X_k \hat{\otimes}_\varepsilon Y_1 = 0$, $M_\alpha(X \hat{\otimes}_\varepsilon Y) \subset X \hat{\otimes}_\varepsilon Y\}$. It is clear that $M(K_X \times K_Y)$ is Banach algebra (with $\|\alpha\| := \sup\{|\alpha(k, 1)| \mid k \in K_X, 1 \in K_Y\}$).

THEOREM 4.2. (i) *The mapping $\alpha \mapsto M_\alpha$ is an isometric algebra isomorphism from $M(K_X \times K_Y)$ onto $Z(X \hat{\otimes}_\varepsilon Y)$ so that we may identify these two spaces.*

(ii) *Let T be an operator in $Z(X \hat{\otimes}_\varepsilon Y)$. Then $T \in (Z(X) \otimes Z(Y))^-$ iff there is an $\alpha \in C(K_X \times K_Y)$ such that $T = M_\alpha$. It follows that $(Z(X) \otimes Z(Y))^- \cong C(K_X \times K_Y)$.*

Proof. (i) The mapping is well-defined by 2.2. For $(k, 1) \in$

$K_X \times K_Y$ such that $X_k \otimes_\varepsilon Y_l \neq 0$, $\varepsilon > 0$, there exist $x \in X$ and $y \in Y$ such that $\|x(k) \otimes y(l)\| = \|x(k)\| \|y(l)\| \geqq 1 - \varepsilon$, $\|x\| \leqq 1$, $\|y\| \leqq 1$. This follows at once from 2.1(a), (b). Because of this fact we have $\|M_\alpha\| = \|\alpha\|$ for $\alpha \in M(K_X \times K_Y)$. The mapping $\alpha \mapsto M_\alpha$ is obviously an algebra homomorphism, and it remains to show that it is onto.

Let T be an M-bounded operator on $X \hat{\otimes}_\varepsilon Y$. By [2], 4.8, every element of $E_{X \hat{\otimes}_\varepsilon Y}$ is an eigenvector for T'. It can be shown that this is also true for every $p \otimes q$, where $(p, q) \in E_X \times E_Y$. The proof of this fact depends on elementary properties of tensor products and weak*-topologies. We refer the reader to [8], p. 506. Therefore there is a function $a: E_X \times E_Y \to \boldsymbol{R}$ such that $(p \otimes q) \circ T = a(p, q)(p \otimes q)$ for $(p, q) \in E_X \times E_Y$. We claim that a is separately continuous. Let $p \in E_X$ be fixed and x a vector in X such that $p(x) = 1$. For $y \in Y$, the mapping $Y' \ni y' \mapsto (p \otimes y')(T(x \otimes y))$ is linear and weak*-continuous (by the Krein-Smulian theorem we have only to prove continuity on bounded sets, and this is obvious). So there is a vector $T_p y$ such that $y'(T_p y) = (p \otimes y')(T(x \otimes y))$ for every $y' \in Y'$. It is easy to see that $y \mapsto T_p y$ is linear and continuous. In fact we have $T_p \in Z(Y)$ since every $q \in E_Y$ is an eigenvector for T_p' (cf. [2], 4.8): $q \circ T_p(y) = (p \otimes q)(T(x \otimes y)) = a(p, q)(p \otimes q)(x \otimes y) = a(p, q)q(y)$. It follows that the corresponding eigenvalue for $q \in E_Y$ is $a(p, q)$ so that, by [2], 4.9, $q \mapsto a(p, q)$ must be structurally continuous. By symmetry, $p \mapsto a(p, q)$ has the same property for every $q \in E_Y$. By 2.4(ii) a induces a mapping $\alpha_0: K_X^* \times K_Y^* \to \boldsymbol{R}$ which is separately continuous: $\alpha_0(k, l) := a(p, q)$ for $p \in E_{X_k}$, $q \in E_{X_l}$, $k \in K_X^*$, $l \in K_Y^*$ (note that $E_{X \hat{\otimes} X} \subset \{p \otimes q \mid (p, q) \in E_X \times E_Y\}$; [8], p. 506). We thus have proved that $T = M_\alpha$, where $\alpha: K_X \times K_Y \to \boldsymbol{R}$ is defined by $\alpha(k, l) = \alpha_0(k, l)$ for $(k, l) \in K_X^* \times K_Y^*$ and $\alpha(k, l) = 0$ otherwise.

(ii) The operators in $Z(X) \otimes Z(Y)$ are by definition exactly the operators M_α, $\alpha \in CK_X \otimes CK_Y (CK_X \otimes CK_Y$ regarded as a subspace of $C(K_X \times K_Y))$. For $\alpha \in C(K_X \times K_Y)$ we have $\|M_\alpha\| = \|\alpha\|$ (this follows at once from 2.3(iii); cf. also the proof of (i)) so that $(Z(X) \otimes Z(Y))^- = \{M_\alpha \mid \alpha \in (CK_X \otimes CK_Y)^-\} = \{M_\alpha \mid \alpha \in C(K_X \times K_Y)\} \cong C(K_X \times K_Y)$.

COROLLARY 4.3 (Wickstead). *$Z(X \hat{\otimes}_\varepsilon Y)$ is the closure with respect to the strong operator topology of $Z(X) \otimes Z(Y)$.*

Proof. This is a consequence of 4.2 and 2.2.

Because of 4.2 it is clear that in order to describe the relations between $Z(X) \otimes Z(Y)$ and $Z(X \hat{\otimes}_\varepsilon Y)$ when considering the norm topology we have to investigate the continuity properties of the functions $\alpha \in M(K_X \times K_Y)$. The following theorem asserts local continuity if there are local centralizer-norming systems:—

THEOREM 4.4. *Let $k_0 \in K_X$, $1_0 \in K_Y$. If k_0 has a local cns $x_1, \cdots, x_n$ in X and 1_0 has a local* cns *$y_1, \cdots, y_m$ in Y, then all $\alpha \in M(K_X \times K_Y)$ are continuous at $(k_0, 1_0)$.*

Proof. Let U(resp. V) be a neighborhood of k_0(resp. 1_0) such that $\max\{||x_i(k)|| \mid i = 1, \cdots, n\} \geqq r$ for $k \in U$(resp. $\max\{||y_j(1)|| \mid j = 1, \cdots, m\} \geqq \tilde{r}$ for $1 \in V$) where $r \in \boldsymbol{R}$, $r > 0$(resp. $\tilde{r} \in \boldsymbol{R}$, $\tilde{r} > 0$) is a suitable chosen number.

Now let α be a function in $M(K_X \times K_Y)$, $\varepsilon > 0$ arbitrary. For $i \in \{1, \cdots, n\}$, $j \in \{1, \cdots, m\}$ the function $z_{ij} := \alpha(x_i \otimes y_j) - \alpha(k_0, 1_0)(x_i \otimes y_j)$ is in $X \hat{\otimes}_\varepsilon Y$ and vanishes at $(k_0, 1_0)$. Since the norm of the elements of $X \hat{\otimes}_\varepsilon Y$ is upper semi-continuous (2.4(ii)) there are neighborhoods U' of k_0 and V' of 1_0 such that

$$||z_{ij}(k, 1)|| = |\alpha(k, 1) - \alpha(k_0, 1_0)| \, ||x_i(k)|| \, ||y_j(1)|| \leqq \varepsilon r r'$$

for $k \in U'$, $1 \in V'$, $i = 1, \cdots, n$, $j = 1, \cdots, m$. It follows that $|\alpha(k, 1) - \alpha(k_0, 1_0)| \leqq \varepsilon$ for $(k, 1) \in (U \cap U') \times (V \cap V')$.

THEOREM 4.5. *Let X* and *Y be real Banach spaces such that the norm topology and the strong operator topology are equivalent on $Z(X)$ and $Z(Y)$* (i.e., *X and Y have a* cns). *We will identify $Z(X) \otimes Z(Y)$ with a subspace of $Z(X \hat{\otimes}_\varepsilon Y)$. Then the following assertions are valid*:

(i) $(Z(X) \otimes Z(Y))^- = Z(X \hat{\otimes}_\varepsilon Y)$

(ii) $Z(X) \hat{\otimes}_\varepsilon Z(Y) = Z(X \hat{\otimes}_\varepsilon Y)$

(iii) $K_{X \hat{\otimes}_\varepsilon Y} = K_X \times K_Y$ *(up to homeomorphism)*

(iv) $X \hat{\otimes}_\varepsilon Y$ *has a* cns

(*more precisely*: *if $x_1, \cdots, x_n$ is a* cns *in X and $y_1, \cdots, y_m$ is a* cns *in Y, then $\{x_i \otimes y_j \mid i = 1, \cdots, n, j = 1, \cdots, m\}$ is a cns in $X \hat{\otimes}_\varepsilon Y$*).

Proof. (i) This is a consequence of 4.2(ii) and 4.4.

(ii) This follows from (i) since the norm of the operators in $Z(X) \otimes Z(Y)$ is their tensor product norm.

(iii) $C(K_{X \hat{\otimes}_\varepsilon Y}) \cong Z(X \hat{\otimes}_\varepsilon Y) \cong Z(X) \hat{\otimes}_\varepsilon Z(Y) \cong C(K_X) \hat{\otimes}_\varepsilon C(K_Y) \cong C(K_X \times K_Y)$. It follows that $K_{X \hat{\otimes}_\varepsilon Y} = K_X \times K_Y$ up to homeomorphism.

(iv) It is clear that $\inf\{\max_{i,j} ||x_i(k) \otimes y_j(1)|| \mid (k, 1) \in K_X \times K_Y\} > 0$. As in 3.2 it follows that $\{x_i \otimes y_j \mid i = 1, \cdots, n, j = 1, \cdots, m\}$ is a *cns* in $X \hat{\otimes}_\varepsilon Y$.

Finally, we want to point out that for Banach spaces which are not too pathological the difference between $Z(X \hat{\otimes}_\varepsilon Y)$ and $Z(X) \hat{\otimes}_\varepsilon Z(Y)$ is just the difference between $\beta(K_X^* \times K_Y^*)$ and $\beta K_X^* \times \beta K_Y^*$:—

THEOREM 4.6. *Let X and Y be Banach spaces having the local* cns *property. Then* $K_{X\hat{\otimes}_\varepsilon Y} = \beta(K_X^* \times K_Y^*)$.

Proof. By 4.2 and 4.4, $C(K_{X\hat{\otimes}_\varepsilon Y}) \cong Z(X \hat{\otimes}_\varepsilon Y) \cong C^b(K_X^* \times K_Y^*) \cong C(\beta(K_X^* \times K_Y^*))$. The Banach-Stone theorem implies that $K_{X\hat{\otimes}_\varepsilon Y} = \beta(K_X^* \times K_Y^*)$.

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I. MATHEMATISCHES INSTITUT
DER FREIEN UNIVERSITÄT
HÜTTENWEG 9
D 1000
BERLIN 33