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TOPOLOGICAL MEASURE THEORY FOR COMPLETELY REGULAR SPACES AND THEIR PROJECTIVE COVERS

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This paper investigates the relationships among tight, τ-additive, and σ-additive Baire measures on a completely regular Hausdorff space X and its projective cover $E(X)$. The most interesting questions arise in the σ-additive case, and lead to the following definitions: the space X has the weak (resp. strong) lifting property if for each σ-additive measure on X, some (resp., every) pre-image measure on $E(X)$ is σ-additive. It is shown that every weak *cb* space has the strong lifting property, while the Dieudonné plank fails even the weak lifting property. Also, if X is weak *cb*, then X is measure-compact if and only if $E(X)$ is measure-compact.

Some applications to extensions of measures on lattices and to strict topologies on spaces of continuous functions are given. A relationship between the lifting properties mentioned above and conventional use of the term "lifting" in measure theory is indicated.

A topological space is said to be extremally disconnected if the closure of every open set is again open. Such a property seems remote from the topological settings usually encountered in analysis; for example, a metric space with this property must be discrete. Nonetheless, the property of extremal disconnectedness occurs with surprising frequency in many basic results of modern analysis. Here are some of them:

(1) The lattice $C(X)$ of continuous real-valued functions on a completely regular space X is Dedekind complete if and only if X is extremally disconnected.

(2) A Boolean algebra is complete if and only if its Stone space is extremally disconnected.

(3) If X is a compact Hausdorff space, then $C(X)$ with the supremum norm is isometrically isomorphic to a dual Banach space if and only if X is hyperstonian (i.e., extremally disconnected, and the union of the supports of the normal measures on X is dense in X).

This paper is concerned with Baire measures on completely regular spaces. The critical fact which motivates the work is that for each completely regular Hausdorff space X, there is an extremally disconnected space $E(X)$, called the projective cover or absolute of X, and a perfect irreducible map κ of $E(X)$ onto X. We can observe at once that (1) $f \to f \circ \kappa$ is an isometric embedding of $C^*(X)$ in $C^*(E(X))$; and (2) the adjoint κ^* of this embedding maps $M(E(X))$, the space of finitely-additive Baire measures on $E(X,)$ onto $M(X)$.

A principal focus of topological measure theory is the delineation and study of certain subspaces of $M(X)$ - notably the tight measures (M_t), τ-additive measures (M_τ) and σ-additive measures (M_σ). For example, a space X is said to be measure-compact if $M_\sigma(X) = M_\tau(X)$, and one seeks purely topological conditions for this to occur; it is known that the Lindelöf property is sufficient, and realcompactness is necessary.

The plan of attack in this paper is to relate corresponding spaces of measures on X and its projective cover: for example, $M_\tau(X)$ and $M_\tau(E(X))$. There is a basic reason for doing this: in extremally disconnected spaces, the Baire sets have a relatively simple structure. For example, the zero-sets of continuous real-valued functions are precisely the countable intersections of clopen sets. This suggests that a topological formulation of measure-compactness (and other concepts of similar type) may be easier to obtain for extremally disconnected spaces than in the general setting. Then, via the correspondences mentioned above, the results could be extended to all completely regular spaces.

We summarize the principal results of the paper as follows: for M_t and M_τ there is an exact correspondence between X and $E(X)$, in the sense that $\kappa^{*-1}(M_t^+(X)) = M_t^+(E(X))$, and $\kappa^{*-1}(M_\tau^+(X)) = M_\tau^+(E(X))$. The situation is much more interesting and complicated for M_σ, and we are led to the following definitions:

Definition 1. X has the weak lifting property (WLP) if for each $\mu \in M_\sigma^+(X)$, $\exists \nu \in M_\sigma^+(E(X))$ with $\kappa^*\nu = \mu$.

Definition 2. X has the strong lifting property (SLP) if for each $\mu \in M_\sigma^+(X)$, *every* pre-image of μ in $M^+(E(X))$ is in $M_\sigma^+(E(X))$ (i.e., $\kappa^{*-1}(M_\sigma^+(X)) = M_\sigma^+(E(X))$).

We find examples of (1) a space which fails the WLP and (2) a space which has the WLP but fails the SLP. These examples emphasize the almost realcompact spaces introduced by Frolik [10] and recently studied by Kato [18]. The notion of a weak *cb* space, introduced by Mack and Johnson [25] in their study of the lattice completion of $C(X)$, is characterized in a new way, leading to the principal positive result of the paper: If X is weak *cb*, then X has the SLP. As a corollary, if X is weak *cb*, then X is measure-compact if and only if $E(X)$ is measure-compact.

The major question raised and left unresolved by this paper seems to be: if X is realcompact, must X have the weak (or strong) lifting property? In investigating this problem we find a relationship between the lifting properties defined above and the conventional use of the term "lifting" in measure theory, via the density topology on the real line.

As applications of our results, we analyze a recent paper by Sultan [39] on extensions of measures in our setting, and also examine the relationships among the various strict topologies on $C^*(X)$ and $C^*(E(X))$.

In closing this introductory section, several general remarks seem to be in order:

(1) The relationship between normal measures on X and $E(X)$ has been studied by Lacey and Hebert [22], Flachsmeyer [9], and others, and a very complete and satisfying theory has been obtained.

(2) Rosenthal [32] has obtained strong results on measures on extremally disconnected compact spaces; his work has been extended and simplified by Kupka [21] and others. It was in the course of studying these results that the author came to feel that $E(X)$ should play a significant role in the study of measures on X.

(3) Recent developments in topological measure theory have tended to stress the embedding of X as a subspace of its Stone-Čech compactification βX. In this paper the emphasis is reversed, since X is the range, rather than the domain, of the map κ. Nonetheless, the equality $E(\beta X) = \beta E(X)$ allows us to make good use of the basic X vs. βX theory.

Finally, the author thanks Grant Woods for a number of very helpful conversations about projective covers.

1. Notation and preliminary remarks. A basic reference for topological measure theory is Varadarajan [43]. We shall also refer to more recent work of Knowles [19], Moran [28, 29], Mosiman and Wheeler [30], and Sentilles [35]. Throughout the discussion X denotes a completely regular Hausdorff space, and $C^*(X)$ is the space of bounded continuous real-valued functions on X. If $\mu \in M(X)$, then μ is (a) σ-additive, if $\mu(Z_n) \to 0$ for every decreasing sequence (Z_n) of zero-sets with empty intersection; (b) τ-additive, if the corresponding result holds for nets of zero-sets which decrease (compatibly with the partial order) and have empty intersection; (c) tight, if for every $\varepsilon > 0$ there is a compact subset K of X with $\mu_*(X-K) < \varepsilon$. As usual, $M_\sigma(X)$, $M_\tau(X)$, and $M_t(X)$ denote the collections of σ-additive, τ-additive, and tight measures on X. It is well-known that $M_t \subset M_\tau \subset M_\sigma$.

Each $\mu \in M(X)$ gives rise (via the identification of $C^*(X)$ and $C^*(\beta X)$) to a corresponding regular Borel measure ν on βX. A nonnegative measure μ is σ-additive (resp. τ-additive) if and only if the corresponding ν vanishes on all zero-sets (resp. compact sets) of $\beta X - X$ [19].

A map $f: X \to Y$ is said to be perfect if f is continuous, closed, and onto, and $f^{-1}(y)$ is compact for all $y \in Y$. Also, f is said to be irreducible if the image of every proper closed subset of X is a proper subset of Y.

The projective cover (or absolute) of X is an extremally disconnected space $E(X)$ and a perfect irreducible map κ of $E(X)$ onto X. The construction of the absolute has been extended to successively larger classes of spaces by Gleason [12], Strauss [38], and the Russian school [31]. Some of the deepest results on extremally disconnected spaces can be found in Efimov [8]. The author has found the most valuable single reference in obtaining a working knowledge of projective covers to be a sequence of papers by Grant Woods and co-authors [46–50]. Relying on these sources for details, we collect here the basic results about $E(X)$ that we need.

A subset F of X is regular closed if $F = \mathrm{cl}_X \, \mathrm{int}_X \, F$. The family $RC(X)$ of regular closed subsets of X is a complete Boolean algebra [36] under the operations: $\bigvee_\alpha F_\alpha = \mathrm{cl}_X (\cup F_\alpha)$, $\bigwedge_\alpha F_\alpha = \mathrm{cl}_X \, \mathrm{int}_X (\bigcap_\alpha F_\alpha)$, $F' = \mathrm{cl}_X (X - F)$. The map $F \to \mathrm{cl}_{\beta X} F$ is a Boolean isomorphism of $RC(X)$ onto $RC(\beta X)$. Let S be the Stone space of $RC(\beta X)$, interpreted as the set of ultrafilters of regular closed subsets of βX. Note that "filter" here is in the Boolean sense: $F, G \in \mathscr{F} \Rightarrow F \wedge G \in \mathscr{F}$. Let $\bar{\lambda}: RC(\beta X) \to \mathrm{clop}\,(S)$ be the canonical correspondence between regular closed subsets of βX and clopen subsets of S. There is a natural map $\bar{\kappa}: S \to \beta X$ which sends each ultrafilter of regular closed sets to its limit in βX. Then S is extremally disconnected and $\bar{\kappa}$ is perfect and irreducible, so $S = E(\beta X)$.

Now let $T = \bar{\kappa}^{-1}(X) \subset S$. Then T is dense in S, so T is extremally disconnected, and $\beta T = S$ [11, 6M]. If $\kappa = \bar{\kappa} \mid T$, then κ is perfect and irreducible, so $T = E(X)$. It follows that $\beta E(X) = E(\beta X)$.

We have then the commutative diagram:

$$\begin{tikzcd}[column sep=huge]
\mathrm{clop}\, E(X) \arrow[r, shift left, "C \to \mathrm{cl}_{E(\beta X)} C"] & \mathrm{clop}\, E(\beta X) \arrow[l, shift left, "D \cap E(X) \leftarrow D"] \\
RC(X) \arrow[u, "\lambda"] \arrow[r, shift left, "F \to \mathrm{cl}_{\beta X} F"] & RC(\beta X) \arrow[u, "\bar{\lambda}"'] \arrow[l, shift left, "H \cap X \leftarrow H"]
\end{tikzcd}$$

where each connecting map is a Boolean isomorphism. Intuitively, $E(X)$ is the set of all ultrafilters of regular closed subsets of X which converge to a point of X; κ sends each such ultrafilter to its limit. If $F_0 \in RC(X)$, then $\lambda(F_0) = \{p \in E(X): F_0 \in p\}$ is clopen in $E(X)$.

Then $\kappa(\lambda(F_0)) = F_0$, but $\lambda(F_0) \subsetneqq \kappa^{-1}(F_0)$ in general. The reason for this is as follows: if $x \in X$, $p \in E(X)$, and $\kappa(p) = x$, then p must contain all regular closed sets F for which $x \in \mathrm{int}_X F$. However, if $x \in F_0 - \mathrm{int}_X F_0$, then $x \in F_0'$. Choosing an ultrafilter p_0 which refines $\{F: x \in \mathrm{int}_X F\} \cup \{F_0'\}$, we have $\kappa(p_0) = x$, so $p_0 \in \kappa^{-1}(F_0)$, but $p_0 \notin \lambda(F_0)$.

The exact relationship here is: $\mathrm{cl}_{E(X)}(\kappa^{-1}(\mathrm{int}_X F)) = \mathrm{int}_{E(X)}\kappa^{-1}(F) = \lambda(F) \subset \kappa^{-1}(F)$ [22]. The inclusion is an equality if and only if F is clopen. The reader may find it instructive to consider the following example: $X = \hat{N}$, the one point compactification of N, $E(X) = \beta N$, $\kappa: E(X) \to X$ sends each integer to itself and $\beta N - N$ to ∞, $F = \{\text{evens}\} \cup \{\infty\} \in RC(X)$. Then $\kappa^{-1}(\mathrm{int}_X F) = \{\text{evens}\} \subsetneqq \lambda(F) = \mathrm{cl}_{\beta N}\{\text{evens}\} \subsetneqq \kappa^{-1}(F) = \{\text{evens}\} \cup \{\beta N - N\}$.

The following result (essentially Lemma 2.4 of [48]) will be very useful: if (C_n) is a decreasing sequence of clopen sets in $E(X)$, then

$$\kappa\Big(\bigcap_1^\infty C_n\Big) = \bigcap_1^\infty \kappa(C_n)\ .$$

κ is not an open map, but it is closed and irreducible, so if U is a nonempty open subset of $E(X)$, then $\{x \in X: \kappa^{-1}(x) \subset U\} = X - \kappa(E(X) - U)$ is a nonempty open subset of X.

2. *X vs. $E(X)$.* There are many properties which X and $E(X)$ always have in common. We list some of them.

THEOREM 1. *For the following topological properties P, X has P if and only if $E(X)$ has P*: (a) *compact* (b) *σ-compact* (c) *Lindelöf* (d) *countably compact* (e) *locally compact* (f) *paracompact* (g) *countably paracompact* (h) *metacompact* (i) *pseudocompact* (j) *k-space* (k) *separable* (l) *countable chain condition* (m) *dense in itself.*

Proof. (a)-(g): [13]; (h): X metacompact $\Rightarrow E(X)$ metacompact is an easy exercise [7, p. 254]; proofs of the converse can be found in [17] and [51]; (i): [48]; (j): [2]; (k): follows from the irreducibility of κ; (l): follows from the last sentence of the previous section; (m): if x_0 is an isolated point of X, then there is a unique ultrafilter p of regular closed sets converging to x_0, and $\kappa^{-1}(x_0) = \{p\}$ is clopen in $E(X)$. Conversely, if p is an isolated point of $E(X)$, with $\kappa(p) = x$, then $\{x\}$ is a regular closed set, so x is an isolated point of X.

Let us remark that (a)-(h) and (j) are valid for any perfect map $f: X \to Y$.

There is one more property which belongs in Theorem 1, and which plays an important role in the sequel. A space is *almost realcompact* [10] if every ultrafilter of regular closed sets such that countable subfamilies have nonempty intersection is fixed. It is known that realcompact implies almost realcompact, but not conversely [18]. Almost realcompactness is preserved by closed subsets and by products, from [14], every X admits an "almost-realcompactification" aX, with $X \subset aX \subset \nu X \subset \beta X$. This construction has been examined in detail by Woods [50]. If $f: X \to Y$ is perfect, then X is almost realcompact if and only if Y is almost realcompact [10]. But for extremally disconnected spaces, it is not hard to show that realcompactness and almost realcompactness are equivalent. Thus *X is almost realcompact if and only if $E(X)$ is realcompact* [5].

This is important for the following reason: suppose X is almost realcompact but not realcompact, e.g., the Dieudonné plank [37, p. 108; 18]. Thus $X \subsetneqq \nu X$, so $E(X) = \nu E(X) \subsetneqq E(\nu X)$. Thus the "$\nu$" analogue of the "$\beta$" identity $\beta E(X) = E(\beta X)$ fails. Also, if $p \in \nu X - X$, then p gives rise to a σ-additive measure $\delta(p)$ on X (explicitly, $\delta(p)(Z) = 1$ or 0 according as $p \in \mathrm{cl}_{\beta X} Z$ or not), but there is no obvious candidate for a σ-additive measure on $E(X)$ which is a pre-image of $\delta(p)$. This is the first hint that interesting measure-theoretic pathology may arise when we consider the relation between X and $E(X)$.

Now we turn to one-way implications between X and $E(X)$.

THEOREM 2. *If X has P, then $E(X)$ has P, but not conversely*: (a) *realcompact* (b) *topologically complete* (*complete in the finest compatible uniform structure*) (c) *measure-compact* ($M_\sigma = M_\tau$) (d) *strongly measure-compact* ($M_\sigma = M_t$).

Proof. All of these follow from a lemma of Herrlich and van der Slot [14]: If $f: S \to T$ is continuous and onto, and $A \subset S$, $B \subset T$ with $f^{-1}(B) = A$, then A is homeomorphic to a closed subspace of $S \times B$. Merely let $S = E(\beta X)$, $T = \beta X$, $f = \bar{\kappa}$, $B = X$, $A = E(X)$. Now use the fact that each of the four properties is preserved by closed subsets and products with compact spaces (see [29] and [30]).

The Dieudonné plank is a counter-example to all four converses (see §3). However, we will show (Theorem 6, Corollaries) that if X is weak *cb*, then properties (c) and (d) can safely be transferred to Theorem 1; it is well-known [5] that this is true for (a).

THEOREM 3. *If $E(X)$ is normal or collectionwise normal, then so is X.*

Proof. This is not difficult to show. The space $[0, \omega_1)$ is a counter-example to both converses [45].

3. Spaces of measures. The maps $\kappa\colon E(X) \to X$ and $\bar{\kappa}\colon \beta E(X) \to \beta X$ induce linear maps $\kappa^*\colon M(E(X)) \to M(X)$ and $\bar{\kappa}^*\colon M(\beta E(X)) \to M(\beta X)$ defined by, for example, $\kappa^*\mu(f) = \mu(f \circ \kappa)$, for each $f \in C^*(X)$. In view of the natural identification between $M(X)$ and $M(\beta X)$, and $M(E(X))$ and $M(\beta E(X))$, κ^* and $\bar{\kappa}^*$ are essentially the same map.

A word of caution is in order concerning the interpretation of functionals as measures (see §3 of [30]). If $\mu \in M_\sigma(E(X))$, then $\kappa^*\mu(B) = \mu(\kappa^{-1}(B))$, for each Baire set B in X. Unfortunately, this desirable result may fail if μ is only finitely-additive (see §4). Also, we shall think of members of $M(\beta E(X))$ and $M(\beta X)$ as compact-regular Borel measures, and then $\bar{\kappa}^*\mu(B) = \mu(\bar{\kappa}^{-1}(B))$ for each Borel set B in βX.

Now let M_z denote M_t, M_τ, or M_σ. It is known [30, Th. 3.1] that $\kappa^*(M_z(E(X))) \subset M_z(X)$, and we are concerned with the question: Is $\kappa^{*-1}(M_z(X)) = M_z(E(X))$? Unfortunately, the answer is almost always no, and for a very trivial reason: if, say, $x \in \beta X - X$, and p, q are distinct members of $\bar{\kappa}^{-1}(x) \subset \beta E(X) - E(X)$, then $\mu = \delta(p) - \delta(q)$ (a difference of point functionals) is not τ-additive, yet $\kappa^*\mu = 0$. To avoid this difficulty, *we shall be concerned only with nonnegative measures, and regard κ^* as a map from $M^+(E(X))$ to $M^+(X)$.*

LEMMA. *κ^* is a surjection.*

Proof. Since $f \to f \circ \kappa$ is an isometric embedding of $C^*(X)$ into $C^*(E(X))$, the Hahn-Banach theorem tells us that if $\mu \in M^+(X)$, $\exists \lambda \in M(E(X))$ with $||\lambda|| = ||\mu||$ and $\lambda(f \circ \kappa) = \mu(f) \forall f \in C^*(X)$. A standard argument shows that λ is nonnegative, so $\lambda \in M^+(E(X))$ and $\kappa^*\lambda = \mu$.

THEOREM 4. (a) $\kappa^{*-1}(M_t^+(X)) = M_t^+(E(X))$; (b) $\kappa^{*-1}(M_\tau^+(X)) = M_\tau^+(E(X))$.

Proof. Let $\mu \in M_z^+(X)$, where $z = t$ or τ. We may think of μ as a regular Borel measure on βX. Let ν be a regular Borel measure on $\beta E(X)$ with $\bar{\kappa}^*\nu = \mu$.

(a) $z = t$: There is a sequence (K_n) of compact subsets of X with $\mu(\beta X - K_n) < 1/n \forall n$. Then each $\bar{\kappa}^{-1}(K_n)$ is a compact subset of $E(X)$, and $\nu(\beta E(X) - \bar{\kappa}^{-1}(K_n)) < 1/n$, so ν corresponds to a tight measure on $E(X)$.

(b) $z = \tau$: We have $\mu(L) = 0$ for all compact $L \subset \beta X - X$ [19, Th. 2.4]. Then if Q is a compact subset of $\beta E(X) - E(X)$, $\bar{\kappa}(Q)$ is a compact subset of $\beta X - X$, so $\nu(Q) \leqq \nu(\bar{\kappa}^{-1}\bar{\kappa}Q) = \mu(\bar{\kappa}Q) = 0$. Hence, by an appeal to the same reference, ν corresponds to a τ-additive measure on $\beta E(X)$.

COROLLARY. *X satisfies the condition $M_t = M_\tau$ if and only if $E(X)$ does.*

Other proofs of these results can be found, at least implicitly, in the work of Bauer [3, 4] and Topsoe [41]; these proofs rely on nothing more than the fact that κ is a perfect map. See also [20].

It would be nice if the argument of Theorem 4(b) also worked for M_σ. Unfortunately it does not: there is no reason to suppose that if Z is a zero-set in $\beta E(X) - E(X)$, then $\bar{\kappa}(Z)$ is a zero-set in $\beta X - X$. However, we can make the following observation: Suppose X has the property

(*) If Z is a zero-set of $\beta E(X)$, with $Z \subset \beta E(X) - E(X)$, then there is a zero-set W of βX with $\bar{\kappa}(Z) \subset W \subset \beta X - X$.

Then it will be true that $\kappa^{*-1}(M_\sigma^+(X)) = M_\sigma^+(E(X))$. The technique of Theorem 4(b), combined with Theorem 2.1 of [19], yields the result, since $\nu(Z) \leqq \nu(\bar{\kappa}^{-1}\bar{\kappa}Z) \leqq \nu(\bar{\kappa}^{-1}(W)) = \mu(W) = 0$.

We will show that X has (*) if and only if X is a *weak cb space*. This concept was introduced by Mack and Johnson [25]; additional details can be found in the book by Alo and Shapiro [1]. For convenience we list several characterizations here.

LEMMA [25]. *X is a weak* cb *space if and only if* (1) *every lacally bounded lower semicontinuous function is bounded above by a continuous function.* (2) *If (F_n) is a decreasing sequence of regular closed subsets of X with empty intersection, there is a decreasing sequence (Z_n) of zero-sets with $Z_n \supset F_n \forall n$ and $\bigcap_1^\infty Z_n = \varnothing$.* (3) *Every countable increasing regular open cover of X has a locally finite partition of unity subordinate to it.*

There is a related concept, due to Horne [15], of a *cb-space*. Characterizations of *cb*-spaces may be obtained in the preceding lemma by deleting "lower semicontinuous" from (1) and "regular" from (2) and (3); see [24]. A space is *cb* is and only if it is weak *cb* and countably paracompact. Other relationships with standard topological properties are summarized in the following diagram:

$$\begin{tikzcd}
& \text{normal and countably paracompact} \arrow[d] & \\
\text{countably compact} \arrow[r] \arrow[d] & cb \arrow[r] \arrow[d] & \text{countably paracompact} \\
\text{pseudocompact} \arrow[r] & \text{weak } cb & \\
\text{extremally disconnected} \arrow[ur] & &
\end{tikzcd}$$

A realcompact, even a measure-compact space need not be weak *cb* (Michael's product space [27]). Conversely, a weak *cb* space need not be realcompact (any pseudocompact, noncompact space). The space R^c is weak *cb* and realcompact, but not measure-compact.

THEOREM 5. *X has* (*) *if and only if X is a weak* cb *space.*

Proof. (a) Suppose X has (*). We prove (3) of the preceding lemma. Let $(U_n)_1^\infty$ be an increasing regular open cover of X. Let $F_n = X - U_n$, and let $L = \bigcap_1^\infty \mathrm{cl}_{\beta X} F_n$. Then L is a compact subset of $\beta X - X$. Each F_n is regular closed in X, so $\mathrm{cl}_{\beta X} F_n$ is regular closed in βX, and $\bar{\lambda}(\mathrm{cl}_{\beta X} F_n) = C_n$ is a clopen subset of $E(\beta X)$.

Let $Z = \bigcap_1^\infty C_n$. Then Z is a zero-set in $E(\beta X)$, and certainly $\bar{\kappa}(Z) \subset L$; indeed, using the technique of [48, Lemma 2.4] one can show that $\bar{\kappa}(Z) = L$. Then $Z \subset \bar{\kappa}^{-1}(L) \subset \beta E(X) - E(X)$, so, from (*), there is a zero-set W of βX with $\bar{\kappa}(Z) = L \subset W \subset \beta X - X$. Then $T = \beta X - W$ is σ-compact locally compact, and $\bigcap_1^\infty \mathrm{cl}_T F_n = \emptyset$, so $\{T - \mathrm{cl}_T F_n\}$ is an open cover of T. Choose a locally finite (in T) partition of unity (f_α) subordinate to this cover. Then if $g_\alpha = f_\alpha | X$, (g_α) is a locally finite partition of unity in X, subordinate to $\{(T - \mathrm{cl}_T F_n) \cap X\} = \{U_n\}$. Thus X is a weak *cb* space.

(b) Assume X is a weak *cb* space, and let Z be a zero-set of $\beta E(X)$, with $Z \subset \beta E(X) - E(X)$. There is a decreasing sequence (C_n) of clopen subsets of $\beta E(X)$ with $\bigcap_1^\infty C_n = Z$. Let $\bar{\kappa}(C_n) = D_n$ and $D = \bigcap_1^\infty D_n$. Then each D_n is regular closed in βX, and (as in part (a)) $\bar{\kappa}(Z) = D \subset \beta X - X$.

Now let $F_n = D_n \cap X$; then (F_n) is a decreasing sequence in $RC(X)$ with empty intersection. From (2) of the lemma, there is a decreasing sequence (Z_n) of zero-sets with $Z_n \supset F_n \forall n$ and $\bigcap_1^\infty Z_n = \emptyset$. Let $Z_n = f_n^{-1}(0)$, $f_n \in C^*(X)$. If $\bar{f}_n$ is the extension to βX, certainly $\mathrm{cl}_{\beta X} Z_n \subset W_n = \bar{f}_n^{-1}(0)$. Let $W = \bigcap_1^\infty W_n$. Then W is a zero-set of βX, and $W \subset \beta X - X$. Hence $\bar{\kappa}(Z) = D = \bigcap \mathrm{cl}_{\beta X} F_n \subset \bigcap \mathrm{cl}_{\beta X} Z_n \subset \bigcap_1^\infty W_n = W \subset \beta X - X$. This shows that (*) holds.

Recall the definitions of the strong and weak lifting properties given in the introduction.

THEOREM 6. *If X is measure-compact or weak* cb, *then X has the* SLP.

Proof. If X is measure-compact, the result is immediate from Theorem 4b. If X is weak *cb*, then Theorem 5 and the remarks following the definition of (*) yield the result.

COROLLARY 1. *If X is weak* cb, *then X is measure-compact if and only if $E(X)$ is measure-compact.*

Proof. Theorems 2c and 6.

COROLLARY 2. *If X is weak* cb, *then X is strongly measure-compact if and only if $E(X)$ is strongly measure-compact.*

Proof. Theorem 2d, Theorem 4 (Corollary), and Theorem 6.

Several remarks are in order here.

(1) If X is weak *cb*, so is νX [25]; thus νX is measure-compact if and only if $E(\nu X)$ is measure-compact.

(2) Corollary 1 could be phrased: If X has the WLP, then X is measure-compact if and only if $E(X)$ is measure-compact. The following is also valid: if $E(X)$ is measure-compact, then X has the WLP if and only if X has the SLP. For if $\mu \in M_\sigma^+(X)$ has a single σ-additive pre-image ν, then ν is τ-additive, hence so is μ, and therefore every pre-image of μ is τ-additive, by Theorem 4b. As we shall see (Example 1 below), it is possible for $E(X)$ to be (strongly) measure-compact without X possessing the WLP.

(3) Let X be the Sorgenfrey plane. Then X is realcompact and weak *cb* [25], but not measure-compact [28]. Hence $E(X)$ is *a realcompact extremally disconnected space which is not measure-compact.*

EXAMPLE 1. A space which fails the WLP.

Let D be the Dieudonné plank. Thus $D = [0, \omega_1] \times [0, \omega_0] - \{(\omega_1, \omega_0)\}$. Here $[0, \omega_0]$ is topologized as usual (order topology), while $[0, \omega_1]$ is given the topology in which each $\alpha < \omega_1$ is isolated, and $\{(\beta, \omega_1]\}_{\beta<\omega_1}$ is a base of neighborhoods of ω_1. D has the subspace topology arising from the product topology. Let $p = (\omega_1, \omega_0)$.

Kato [18] has shown that (1) D is almost realcompact; (2) D is not realcompact and (3) $\nu D = D \cup \{p\}$ is Lindelöf.

We will show that (A) $\mu = \delta(p)$ is a σ-additive measure on D with no pre-image in $M_\sigma^+(E(D))$ and (B) $E(D)$ is strongly measure-compact, although D is not even realcompact.

In order to do this, we follow Kato in introducing an auxiliary space $\tilde{D}$. Let $G = [0, \omega_1) \times [0, \omega_0] \subset D$, and let $\tilde{G}$ be a copy of G. Then the embedding $i: \tilde{G} \to G \subset D$ extends to a map $\psi: \beta\tilde{G} \to \beta D$. Since G is dense in D, ψ is onto. By [11, Lemma 6.11], ψ maps $\beta\tilde{G} - \tilde{G}$ onto $\beta D - G$. Now let $\tilde{D} = \psi^{-1}(D)$ and $\Phi = \psi | \tilde{D}$. Then $\tilde{G} \subset \tilde{D}$, and Φ maps $\tilde{D} - \tilde{G}$ onto $D - G$ (the right edge of D).

Now Φ is a perfect map, since it is the restriction of ψ to a complete inverse image; since $\tilde{D}$ contains a dense set of isolated points, Φ is also irreducible. Consider the diagram

$$E(\tilde{D}) \xrightarrow{\tilde{\kappa}} \tilde{D} \xrightarrow{\Phi} D\,.$$

Since $\tilde{\kappa}$ and Φ are perfect and irreducible, so is $\Phi \circ \tilde{\kappa}$; we deduce that $E(D) = E(\tilde{D})$ and $\kappa = \Phi \circ \tilde{\kappa}$.

Proof of (A). Suppose $\exists \lambda \in M_\sigma^+(E(D))$ with $\kappa^*\lambda = \mu$. Let $\nu = \tilde{\kappa}^*(\lambda) \in M_\sigma^+(\tilde{D})$; then $\Phi^*\nu = \mu$. Thus to establish (A), it is enough to show that no such ν can be found.

Suppose there were such a ν. Let $T = \{(\alpha, \omega_0): \alpha < \omega_1\}$ be the top edge of D, and let $\tilde{T} = \Phi^{-1}(T) \subset \tilde{D}$. Since T is a zero-set of D, with $\mu(T) = \mu(D) = 1$, we have $\tilde{T}$ a zero-set of $\tilde{D}$, with $\nu(\tilde{T}) = \nu(\tilde{D}) = 1$. Since $T \subset G$, $\tilde{T}$ and T are homeomorphic under Φ; each is a discrete space of cardinal $\aleph_1$. But there is a critical distinction between them: T is not C^*-embedded in D, but $\tilde{T}$ is C^*-embedded in $\tilde{D}$ (because it is C^*-embedded in $\tilde{G}$, hence in $\beta\tilde{G}$). It follows that every subset of $\tilde{T}$ is a zero-set of $\tilde{D}$. Thus ν is defined on all subsets of a set of cardinal $\aleph_1$, has total mass 1, and assigns measure 0 to all singletons. This contradicts a well-known result of Ulam [42]. Hence no such ν can exist.

Proof of (B). The only τ-additive measures on D are of the form $\sum_1^\infty c_n\delta(x_n)$, $(x_n) \in D$, $\sum_1^\infty |c_n| < \infty$; hence $M_\tau(D) = M_t(D)$, and so, by the Corollary to Theorem 4, $E(D)$ has the same property. It remains to show that $E(D)$ is measure-compact, and, using the same technique as in the proof of Theorem 2, it is enough to show that $\tilde{D}$ is measure-compact. We use the criterion of Moran [28, Th. 2.1].

Suppose $\mu \in M_\sigma^+(\tilde{D})$ and μ has empty support. Let $L_n = \{(\alpha, n): \alpha \leqq \omega_1\}$ for each n. Then L_n is a Lindelöf, clopen subset of D, and so (Theorem 1(c)) $\Phi^{-1}(L_n)$ is a Lindelöf clopen subset of $\tilde{D}$. Thus $\mu|\Phi^{-1}(L_n) \equiv 0 \forall n$. Also, as in (A), $\mu|\tilde{T}$ is defined on all subsets and vanishes on all singletons. Thus μ is the 0 measure, to complete the proof.

REMARK. Let X be any almost realcompact, nonrealcompact space. Then X fails the SLP. Indeed choose $x_0 \in \nu X - X$, and note that, while x_0 has pre-images in $\beta E(X)$, none can lie in $\nu E(X) = E(X)$. Thus $\mu = \delta(x_0) \in M_\sigma^+(X)$, but none of the point-mass pre-images of μ is σ-additive, so X does not have the SLP.

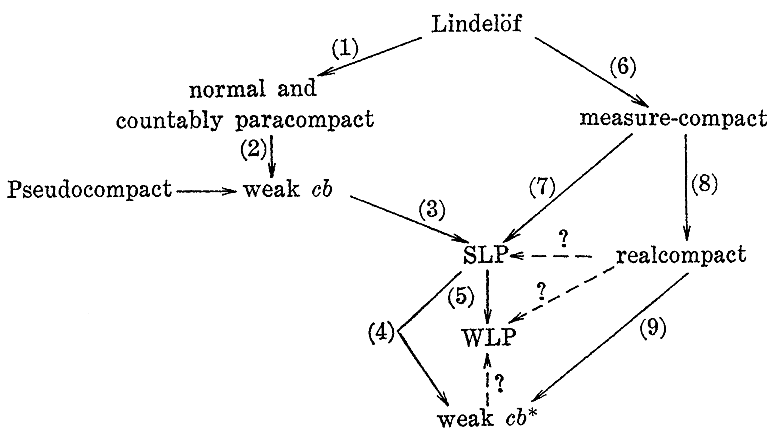
We conjecture that such a μ has no σ-additive pre-image of any kind, so that such an X actually fails the WLP, as we have just seen for the special case $X = D$.

EXAMPLE 2. A space which has the WLP but fails the SLP.

Let X be the space obtained by joining the Dieudonné plank D and the Tychonoff plank T along the edge $\{(\omega_1, n): n \in N\}$. Then X is completely regular Hausdorff, and D and T are regular closed subsets of X, each the Boolean complement of the other. The ultrafilters of regular closed sets thus divide nicely into two classes, those which contain D and those which contain T, and in fact $E(X)$ is the topological sum of $E(D)$ and $E(T)$ (cf. [48, Lemma 2.6]).

Let $p = (\omega_1, \omega_0)$; then $X \cup \{p\}$ is Lindelöf, and contains X as a dense, C-embedded subspace, so $\nu X = X \cup \{p\}$. Then $\mu = \delta(p) \in M_\sigma^+(X)$, and it is easy to see that scalar multiples of μ are the only purely σ-additive measures on X. Thus it is enough to show that $\kappa^{*-1}(\mu)$ contains both σ-additive and nonσ-additive measures. Now D and T are both μ-thick ($\mu^* D = \mu^* T = 1$) and C^*-embedded in X, so μ induces measures $\mu_1 \in M_\sigma^+(D)$ and $\mu_2 \in M_\sigma^+(T)$; each is simply the "point mass at p" in its own setting. By the lemma at the beginning of this section, $\exists \lambda_1 \in M^+(E(D))$ with $\kappa_D^* \lambda_1 = \mu_1$, and $\exists \lambda_2 \in M^+(E(T))$ with $\kappa_T^* \lambda_2 = \mu_2$. By Example 1 λ_1 cannot be σ-additive. Since T is pseudocompact, so is $E(T)$, and therefore λ_2 must be σ-additive [43, p. 172]. Since each can be viewed as a pre-image of μ in $M^+(E(T))$, the proof is complete.

Isawata has defined a space X to be a weak cb^* space if whenever (F_n) is a sequence of regular closed sets in X with $\bigcap_1^\infty F_n = \varnothing$ then $\bigcap_1^\infty \mathrm{cl}_{\nu X} F_n = \varnothing$. Hardy and Woods [49] showed that X is weak cb^* if and only if $E(\nu X) = \nu E(X)$. An almost realcompact, nonrealcompact space is never weak cb^*. At this point we summarize our findings in a diagram:



REMARKS. (1), (2), (6), and (8) are well-known, and each converse is false. (3) and (7) are proved in Theorem 6; the disjoint union of Michael's product space and the Sorgenfrey plane has the SLP, but is neither weak *cb* nor measure-compact. (5) is trivial; Example 2 shows that the converse fails. (9) is equally trivial, and the converse fails (any pseudocompact, noncompact space). (4) can be shown as follows: it is always true that $\nu E(X) \subset E(\nu X)$. If $p \in E(\nu X)$, then $\bar{\kappa}(p) \in \nu X$, and so $\delta(\bar{\kappa}(p)) = \kappa^*(\delta(p)) \in M_\sigma^+(X)$. Hence (from the SLP) $\delta(p) \in M_\sigma^+(E(X))$, i.e., $p \in \nu E(X)$.

The SLP (and hence the WLP) do not imply realcompactness (any pseudocompact, noncompact space). The space of Example 2 has the WLP, but is not weak *cb**. To see this, note that $E(\nu X)$ is Lindelöf, since νX has that property. However, $\nu E(X)$ is the topological sum of $\nu E(D) = E(D)$ and $\nu E(T) = \beta E(T)$. Since D is not Lindelöf, neither is $E(D)$, and so $\nu E(X) \subsetneqq E(\nu X)$.

Since a zero-set of $\beta X - X$ must actually lie in $\beta X - \nu X$, there is a natural identification of $M_\sigma(X)$ and $M_\sigma(\nu X)$. This yields the following proposition; we omit the proof.

THEOREM 7(a). *If X has the* WLP (*or* SLP), *so does νX.* (b) *If X is weak* cb*, *and νX has the* WLP (*or* SLP), *then so does X.*

In view of these results, the principal open question appears to be: *Does realcompactness imply the* WLP *or the* SLP? We know that the WLP fails for the Dieudonné plank, but this is basically a topological pathology: there is a point of $\nu X - X$ with no pre-image in $\nu E(X) - E(X)$, simply because the latter set is empty. Failure of the WLP for a realcompact space would be a true measure-theoretic pathology. Such an example would be quite interesting; we mention a possible candidate below.

EXAMPLE 3. Let $X = [0, 1]$, endowed with the density topology $\mathscr{T}_d$. A subset E of $[0, 1]$ is open in this topology if and only if E is Lebesgue measurable and has density one at each of its points; see [34] and [40] for details. We shall need the the following facts: (1) $\mathscr{T}_d$ is finer than the usual topology $\mathscr{T}_0$ on X; (2) $(X, \mathscr{T}_d)$ is realcompact; (3) Every $\mathscr{T}_d$-continuous function is Baire class 1; hence the $\mathscr{T}_d$-Baire sets are precisely the $\mathscr{T}_0$-Borel sets; (4) Every set of Lebesgue measure 0 is $\mathscr{T}_d$-closed and discrete; (5) Let $\mathscr{L}$ be the σ-algebra of Lebesgue measurable sets, m = Lebesgue measure. Then $\beta E(X)$ is the Stone space of the reduced measure algebra $\mathscr{L}/m^{-1}(0)$ (henceforth $E(X)$ stands for $E(X, \mathscr{T}_d)$).

Now let C be the Cantor set, and, thinking of C as 2^N, let μ denote Haar measure on the Borel sets of C. From (3), we may think of μ as a member of $M_\sigma^+(X, \mathscr{T}_d)$. But it is easy to see, using

(4), that spt $\mu = \{x \in X$: if $x \in U(\mathscr{T}_d$-open), then $\mu U > 0\}$ is empty, so μ is not τ-additive; hence $(X, \mathscr{T}_d)$ is not measure-compact.

We now ask: does there exist $\lambda \in M_\sigma^+(E(X))$ with $\kappa^*(\lambda) = \mu$? Intuitively it seems unlikely, since the measure algebra construction suppresses sets like C. However, we now show that if $\mathscr{L}$ admits a Borel lifting of a certain type, then $(X, \mathscr{T}_d)$ has the WLP, so that such a λ does exist.

For a discussion of the theory of lifting, see [16] or [26]. If $A \in \mathscr{L}$, let $\theta(A) = \{x \in X$: the density of A at x is $1\}$.

THEOREM 8. *Suppose there is a lifting* $\mathfrak{l}: \mathscr{L} \rightarrow \mathscr{L}$ *such that* (a) $\theta(A) \subset \mathfrak{l}(A) \forall A \in \mathscr{L}$ *and* (b) *each set* $\mathfrak{l}(A)$ *is a* $\mathscr{T}_0$*-Borel set. Then* $(X, \mathscr{T}_d)$ *has the* WLP.

Proof. The lifting $\mathfrak{l}$ gives rise to a lifting topology $\mathscr{T}_\mathfrak{l}$ on X for which $\{\mathfrak{l}(A): A \in \mathscr{L}\}$ is a base. Condition (a) implies that $\mathscr{T}_0 \subset \mathscr{T}_d \subset \mathscr{T}_\mathfrak{l}$ on X, and condition (b) implies that each $\mathscr{T}_\mathfrak{l}$-continuous function is $\mathscr{T}_0$-Borel measurable; hence the $\mathscr{T}_\mathfrak{l}$-Baire sets coincide with the $\mathscr{T}_0$-Borel sets and the $\mathscr{T}_d$-Baire sets.

From [16, pp. 58–61] we can deduce that there is a dense subspace Y of $E(X)$ which is homeomorphic to $(X, \mathscr{T}_\mathfrak{l})$. Y consists of exactly one point from each set $\kappa^{-1}(x)$; explicitly, x corresponds to the ultrafilter $p = \{F \in RC(X, \mathscr{T}_d): x \in \mathfrak{l}(F)\}$.

Now let $\mu \in M_\sigma^+(X, \mathscr{T}_d)$. There is a corresponding Baire measure $\hat{\mu}$ on $Y \subset E(X)$. Define γ on the Baire sets of $E(X)$ by $\gamma(B) = \hat{\mu}(B \cap Y)$. Then $\gamma \in M_\sigma^+(E(X))$, and if D is a Baire set in $(X, \mathscr{T}_d)$, $\gamma(\kappa^{-1}(D)) = \hat{\mu}(\kappa^{-1}(D) \cap Y) = \mu(D)$. Hence $\kappa^*\gamma = \mu$, and so $(X, \mathscr{T}_d)$ has the WLP.

Thanks are due to Dennis Sentilles for some very helpful ideas relative to this proof.

Assuming the continuum hypothesis, Siegfried Graf has shown [52, Th. 9.2] that a lifting satisfying the conditions of Theorem 8 exists (see also [23] and [44]). Without the continuum hypothesis, however, the question of existence of such a lifting seems to be open.

4. Extension of measures. Up to this point we have treated $M^+(E(X))$ and $M^+(X)$ as distinct collections of measures connected by the map κ^*. Here is another point of view: if $\mu \in M^+(X)$, then μ induces a set function λ on $\{\kappa^{-1}(B)$: B a Baire set in $X\}$ in the obvious way: $\lambda(\kappa^{-1}(B)) = \mu(B)$. The question then becomes: does λ have an extension to a zero-set regular measure ν defined on all Baire sets of $E(X)$? Call such a ν a *measure extension* of μ.

We consider this problem in the context of a recent paper on extensions of measures by Sultan [39]. We record some relevant terminology. A lattice $\mathscr{L}$ of subsets of a set X is called a paving. $\mathscr{L}$ is a delta paving if $\mathscr{L}$ is closed under countable intersections; $\mathscr{L}$ is a normal paving if disjoint members of $\mathscr{L}$ are contained in disjoint complements of members of $\mathscr{L}$. If $\mathscr{L}_1 \subset \mathscr{L}_2$ are two lattices of subsets of X, then $\mathscr{L}_1$ semiseparates $\mathscr{L}_2$ if whenever $A \in \mathscr{L}_1$, $B \in \mathscr{L}_2$, and $A \cap B = \varnothing$, there exists a $C \in \mathscr{L}_1$ such that $B \subset C$ and $A \cap C = \varnothing$. Finally, $\mathscr{L}_2$ is $\mathscr{L}_1$ *c.b.* if $B_n \downarrow \varnothing$ in $\mathscr{L}_2$ implies there exist (A_n) in $\mathscr{L}_1$ with $B_n \subset A_n \downarrow \varnothing$.

Now let $\mathscr{L}_1 = \{\kappa^{-1}(Z)\colon Z \text{ a zero-set of } X\}$, and let $\mathscr{L}_2 = \{W\colon W$ a zero-set of $E(X)$. Then $\mathscr{L}_1 \subset \mathscr{L}_2$, and each is a delta-normal paving of subsets of $E(X)$. $M^+(X)$ corresponds to $MR(\mathscr{L}_1)$, and $M^+(E(X))$ corresponds to $MR(\mathscr{L}_2)$.

We now give topological characterizations of the semiseparation and *c.b.* properties. Mack [54] has termed a space *weakly δ-normally separated* if each regular closed set and zero-set disjoint from it are completely separated. He has shown that every weak *cb* space has this property.

LEMMA. *Suppose X is any completely regular space, and F, H are disjoint closed subsets of X, with F a zero-set and H the intersection of a decreasing sequence of regular closed sets (an RC_δ). Then there is a regular closed set D with $H \subset D$ and $F \cap D = \varnothing$.*

Proof. Let $H = \bigcap_1^\infty H_n$, where each H_n is regular closed and $H_n \supset H_{n+1} \forall n$. Let $A = \kappa^{-1}(F)$, $B = \bigcap_1^\infty \lambda(H_n)$. Then A and B are disjoint zero-sets in $E(X)$, and so there is a clopen set C with $B \subset C$ and $A \cap C = \varnothing$. Then $D = \kappa(C)$ is the desired regular closed set.

THEOREM 9. *$\mathscr{L}_1$ semiseparates $\mathscr{L}_2$* [39] *if and only if X is weakly δ-normally separated.*

Proof. Assume the topological condition, and let $A \in \mathscr{L}_1$, $B \in \mathscr{L}_2$ with $A \cap B = \varnothing$. Let $A = \kappa^{-1}(F)$, where F is a zero-set of X. Now B is a countable intersection of clopen subsets of $E(X)$, and so $\kappa(B) = H$ is an RC_δ in X. From the hypothesis and the lemma, there is an $f \in C^*(X)$, $f|F \equiv 1$, $f|H \equiv 0$. Then if $C = \kappa^{-1}(Z(f))$, we have $B \subset C$ and $A \cap C = \varnothing$.

Conversely, suppose $\mathscr{L}_1$ semiseparates $\mathscr{L}_2$. Then for disjoint F and H in X, F a zero-set and H regular closed, $\kappa^{-1}(F) \in \mathscr{L}_1$ and $\lambda(H) \in \mathscr{L}_2$ are disjoint, hence there is a zero-set Z of X with $\lambda(H) \in \kappa^{-1}(Z)$ and $\kappa^{-1}(Z) \cap \kappa^{-1}(F) = \varnothing$. It follows that F and H are completely separated.

COROLLARY. *The following are equivalent:*

(1) *X is weakly δ-normally separated.*

(2) *if $\mu \in M^+(X)$, and ν is any measure in $M^+(E(X))$ with $\kappa^*\nu = \mu$, then ν is a measure extension of μ.*

(3) *if $p \in \beta X$, and q is any point of $\beta E(X)$ with $\bar{\kappa}(q) = p$, then $\delta(q)$ is a measure extension of $\delta(p)$.*

Proof. (1) $\Rightarrow$ (2): Theorem 9, and Lemma 4.4 of [39]. (2) $\Rightarrow$ (3): trivial, since (3) is a special case of (2). (3) $\Rightarrow$ (1): Woods [55] has shown that X is weakly δ-normally separated if and only if $\mathrm{cl}_{\beta E(X)} \kappa^{-1}(Z) = \bar{\kappa}^{-1}(\mathrm{cl}_{\beta X} Z)$ for each zero-set Z of X. If (1) fails, choose a Z_0 for which $\mathrm{cl}_{\beta E(X)} \kappa^{-1}(Z_0) \subsetneqq \bar{\kappa}^{-1}(\mathrm{cl}_{\beta X} Z_0)$, and find a point q in the second set, but not the first. Then, with $\bar{\kappa}(q) = p$, we have $\delta(p)(Z_0) = 1$, since $p \in \mathrm{cl}_{\beta X} Z_0$, but $\delta(q)(\kappa^{-1}(Z_0)) = 0$. Thus $\delta(q)$ is not a measure extension of $\delta(p)$.

EXAMPLE. Referring to Example 2, let Y be the space obtained by deleting from X the top edge of the Tychonoff plank T, and let U be the portion of T which remains. Then Y is not weakly δ-normally separated (consider Z, the top edge of D, and U). As in Example 2, $\nu Y = Y \cup \{p\}$, so $\delta(p) \in M_\sigma^+(Y)$. $E(Y)$ is the topological sum of $E(D)$ and $E(U)$; choose $q \in \beta E(U)$ with $\bar{\kappa}(q) = p$. Then $\delta(p)(Z) = 1$, but $\delta(q)(\kappa^{-1}(Z)) \leqq \delta(q)(E(D)) = 0$. Thus $\delta(p)$ is a σ-additive measure with a functional extension $\delta(q)$ which is not a measure extension; $\delta(q)$ is (necessarily) only finitely-additive.

REMARK. For any X and any $p \in \beta X$, there is always at least one $q \in \beta E(X)$ with $\bar{\kappa}(q) = p$ and $\delta(q)$ a measure extension of $\delta(p)$. Indeed $\{\kappa^{-1}(Z) \colon p \in \mathrm{cl}_{\beta X} Z\}$ is a z-filterbase on $E(X)$; any z-ultrafilter which extends it corresponds to such a q.

Question. Is it true that for any X and $\mu \in M^+(X)$, there is always at least one $\nu \in M^+(E(X))$ with $\kappa^*(\nu) = \mu$ and ν a measure extension of μ?

THEOREM 10. *$\mathscr{L}_2$ is $\mathscr{L}_1$ c.b. [39] if and only if X is a weak* cb *space.*

Proof. Assume X is weak cb, and let $B_n \downarrow \varnothing$ in $\mathscr{L}_2$. Let $B_n = \bigcap_i C_{n,i}$, where each $C_{n,i}$ is clopen, and let $D_n = \bigcap_{m,i=1}^n C_{m,i}$. Then $D_n \downarrow \varnothing$, and so $\kappa(D_n) \downarrow \varnothing$ in $RC(X)$. Choose a sequence (Z_n) of zero-sets of X with $Z_n \supset \kappa(D_n)$ and $Z_n \downarrow \varnothing$. Then with $A_n = \kappa^{-1}(Z_n)$, we have $B_n \subset A_n$ and $A_n \downarrow \varnothing$.

The proof of the converse is left to the reader.

In view of Theorems 9 and 10 and Mack's result that every weak cb space is weakly δ-normally separated, Theorem 5.1 of [39] yields another proof that every weak cb space has the strong lifting property (Theorem 6).

5. **Spaces of functions.** As we have remarked, the map Φ: $C^*(X) \to C^*(E(X))$ defined by $\Phi(f) = f \circ \kappa$ is an isometric embedding of $C^*(X)$ as a norm-closed subspace of $C^*(E(X))$. Since κ is closed and onto, it is a quotient map; hence $g \in C^*(E(X))$ is a member of $\Phi(C^*(X))$ if and only if $g | \kappa^{-1}(x)$ is constant for each $x \in X$. It follows that $\Phi(C^*(X))$ is actually a pointwise-closed subspace of $C^*(E(X))$.

We now investigate whether Φ is a topological embedding with respect to the strict topologies β_0, β, and β_1 [35] which can be placed on $C^*(X)$ and $C^*(E(X))$. As might be expected from §3, the situation is as nice as possible for β_0 and β, and somewhat complicated for β_1.

THEOREM 11. *Φ is a topological isomorphism of $(C^*(X), \mathscr{T})$ onto a closed subspace of $(C^*(E(X)), \mathscr{T})$ for $\mathscr{T} = \beta_0$ or β.*

Proof. It is known [30] that Φ is $\mathscr{T}$-continuous in either case. ($\mathscr{T} = \beta_0$) Let U be a β_0-neighborhood of 0 in $C^*(X)$. We may assume [35] that $U = \bigcap_1^\infty \{f \mid \|f\|_{H_i} \leqq a_i\}$, where each H_i is a compact subset of X and $0 < a_i \uparrow \infty$. Let $L_i = \kappa^{-1}(H_i)$. Then each L_i is compact, and $\Phi(U) \supset \Phi(C^*(X)) \cap \bigcap_1^\infty \{g \mid \|g\|_{L_i} \leqq a_i\}$, so Φ is open onto its range.

($\mathscr{T} = \beta$) Let U be a β-neighborhood of 0 in $C^*(X)$. We may assume [35] that $U = H^\circ$, where H is weak*-compact subset of $M_\tau^+(X)$. Now $\kappa^{*-1}(H) = Q$ is weak*-compact in $M^+(E(X))$, by Alaoglus, theorem, and so, by Theorem 4b, Q is a weak*-compact subset of $M_\tau^+(E(X))$, hence β-equicontinuous. Thus $\Phi(U) \supset \Phi(C^*(X)) \cap Q^\circ$, so Φ is open onto its range.

We remark that (1) the $\mathscr{T} = \beta_0$ proof adapts easily to show that Φ is an embedding of $C(X)$ in $C(E(X))$ for the compact-open topology; (2) If $C^*(E(X))$ is β_0 or β-complete, so is $C^*(X)$. In the β_0-case this says, topologically, that if $E(X)$ is a k_R-space, (i.e., every real-valued function which is continuous on compact subsets is continuous), then so is X. Of course, this is trivial to prove directly. The converse (X a k_R-space $\Rightarrow E(X)$ a k_R-space) seems to be open.

DEFINITION 3. X is β_1-stable if Φ is a topological isomorphism of $(C^*(X), \beta_1)$ onto a closed subspace of $(C^*(E(X)), \beta_1)$.

LEMMA [53, p. 156]. *If E is a locally solid vector lattice, F is a linear subspace and sublattice of E, and μ is a positive continuous linear functional on F, then there is a positive continuous linear functional λ on E with $\lambda | F = \mu$.*

THEOREM 12. SLP $\Rightarrow$ β_1-*stable* $\Rightarrow$ WLP.

Proof. That the SLP $\Rightarrow \beta_1$-stable can be proved just as in the $\mathscr{T} = \beta$ case of the previous theorem. Now suppose that X is β_1-stable, and let $\mu \in M_\sigma^+(X)$. Then μ can be thought of as a β_1-continuous linear functional on $\Phi(C^*(X))$, so by the Hahn-Banach theorem, $\exists \lambda \in M_\sigma(E(X))$ with $\kappa^* \lambda = \mu$. Since β_1 is locally solid, the lemma shows that we can choose λ to be nonnegative.

The first implication, at least, of Theorem 12 cannot be reversed.

EXAMPLE 2 (continued). The join X of the Dieudonne plank D and the Tychonoff plank T is β_1-stable, but fails the SLP.

Proof. We only sketch the argument. It is enough to show that if H is a weak*-compact subset of $M_\sigma^+(X)$, then there is a weak*-compact subset Q of $M_\sigma^+(E(X))$ with $\kappa^*(Q) = H$. It is convenient to identify $M_\sigma^+(X)$ and $M_\sigma^+(\nu X)$, because νX is a strongly measure-compact Prohorov (or T-) space [30]; every σ-additive Baire measure on νX is of the form $\sum_1^\infty c_n \delta(x_n)$, where $x_n \in \nu X \forall n$ and $\sum_1^\infty |c_n| < \infty$. Then, by looking at "horizontal lines" in νX and using [30, Th. 4.4] one can show that H lives on $A \cup T \cup \{p\} = A \cup \beta T$, where A is a subset of D of the form $[0, \alpha_0] \times [0, \omega_0]$, $\alpha_0 < \omega_1$. Then $H | A$ is a weak*-compact subset of $M_\tau^+(D)$, and $H | \beta T$ is a weak*-compact subset of $M^+(\beta T)$, so each can be pulled back to a weak*-compact subset of $M_\sigma^+(E(X))$. The sum of these two pre-images contains the desired pre-image of H.

We do not know an example of a space with the WLP which is not β_1-stable.

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