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GENERALIZATION OF A THEOREM OF McFADDEN

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McFadden's relation $|N, 1/(n+1)| \subset |C, k| (k>0)$ is strengthened to $|N, p_n| \subset |R, \lambda(w), k| (k>0)$ for suitable $\{p_n\}$ and $\lambda(w)$.

1. Let $\{p_n\}$ be a sequence of complex numbers such that for $n > 0$,

$$P_n = p_0 + p_1 + \cdots + p_n \neq 0 .$$

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with $\{s_n\}$ as its partial sums. We define the (N, p_n)*-transform* $\{t_n(s_n)\}$ of $\{s_n\}$ *generated by the sequence* $\{p_n\}$ *by the formula*

$$t_n(s_n) = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v . \tag{1.1}$$

Similarly, $\{t_n(na_n)\}$ denotes the (N, p_n)-transform of the sequence $\{na_n\}$ generated by the sequence $\{p_n\}$. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|N, p_n|$ if $\{t_n(s_n)\} \in BV$, i.e., $\sum_{n=1}^{\infty} |t_n - t_{n-1}|$ is convergent. (See [7], [5].) In the special case when $p_n = \binom{n+k-1}{k-1}$, $(k > -1)$, summability $|N, p_n|$ is summability $|C, k|$.

Let $\lambda = \lambda(w)$ be a differentiable, monotonically increasing function of w in (A, ∞), where A is a finite positive number; and let $\mu(w)$ be its differential and let $\lambda(w)$ tend to infinity with w. For $k \geqq 0$, we write

$$R_\lambda^k(w) = \sum_{n \leqq w} \{\lambda(w) - \lambda(n)\}^k a_n .$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|R, \lambda, k|$ if

$$\int_A^\infty |d[R_\lambda^k(w)/\lambda^k(w)]| < \infty , \tag{1.2}$$

see [8], [9]. For $k > 0$, $N < w < N+1$ $(N = 1, 2, \cdots)$

$$\frac{d}{dw}[R_\lambda^k(w)/\lambda^k(w)] = \frac{k\mu(w)}{\lambda^{k+1}(w)} \sum_{n \leqq w} \{\lambda(w) - \lambda(n)\}^{k-1} \lambda(n) a_n .$$

Hence, summability $|R, \lambda, k|$ is equivalent to the convergence of the integral

$$\int_A^\infty \frac{\mu(w)}{\lambda^{k+1}(w)} \left| \sum_{n \leqq w} \{\lambda(w) - \lambda(n)\}^{k-1} \lambda(n) a_n \right| dw .$$

If every series summable by the method P is summable by the method Q, we write $P \subseteqq Q$. If $P \subseteqq Q$ and $Q \subseteqq P$, we write $P \sim Q$.

We now define a sequence of constants $\{c_n\}$ by the identity

$$\left[\sum_{n=0}^{\infty} p_n x^n\right]^{-1} = \sum_{n=0}^{\infty} c_n x^n, \quad c_{-1} = 0 .$$

If, for $n = 0, 1, 2, \cdots$

$$p_n > 0, \ \frac{p_{n+1}}{p_n} \leqq \frac{p_{n+2}}{p_{n+1}} \leqq 1 ,$$

we shall write $\{p_n\} \in \mathscr{M}$. We write

$$d_n = c_0 + c_1 + \cdots + c_n ;$$

$$e_n = d_0 + d_1 + \cdots + d_n .$$

We write $P(v)$, $d(v)$, $e(v)$ in place of P_v, d_v, e_v respectively when v is replaced by a more complicated expression. We let $\Delta f_n = f_n - f_{n+1}$, for any sequence $\{f_n\}$.

The following inclusion theorems are known:

$$|C, 0| \subset \left|N, \frac{1}{n+1}\right| \subset |C, k| \sim |R, n, k|, \ (k > 0) .$$

The first one is due to Mears [6], the second one is due to McFadden [4] and the equivalence is due to Hyslop [3].

Our object is to prove that under certain conditions on $\{p_n\}$ and $\lambda(w)$,

$$|N, p_n| \subset |R, \lambda(\omega), k| \ (k > 0) .$$

2. We establish the following.

THEOREM. *Let*

$$\{p_n\} \in \mathscr{M} , \tag{2.1}$$

$$P(v^2) = O(P_v) , \tag{2.2}$$

(2.3) $\lambda(w)$ be an indefinite integral of some function $\mu(w)$,

$$(n+1)\left\{\frac{\lambda(n+1) - \lambda(n)}{\lambda(n+1)}\right\} = O(1) . \tag{2.4}$$

Then $|N, p_n|$ *implies* $|R, \lambda(w), k|$ *for all* $k > 0$.

For the proof we require the following lammas.

LEMMA 1 [1]. *Let* $\{p_n\} \in \mathscr{M}$. *A necessary and sufficient condi-*

tion for $\Sigma a_n \in |N, p_n|$ *is*

$$\sum_{n=1}^{\infty} \frac{|t_n(na_n)|}{n} < \infty . \tag{2.4'}$$

LEMMA 2 [1]. *Let* $\{p_n\} \in \mathscr{M}$. *Then*

(i) $c_0 > 0$, $c_n \leqq 0$ $(n = 1, 2, 3, \cdots)$

(ii) $\sum_{n=v+1}^{\infty} |c_n| \leqq d_v$

(iii) $d_n \geqq 0$ *and monotonic nonincreasing*

(iv) $P_n d_n \leqq 1$

(v) $P_n e_n \leqq (2n+1)$.

For (i), (ii) *see Hardy* [2] *Theorem* 22, *p.* 68.

LEMMA 3. *Let* $\{p_n\} \in \mathscr{M}$. *Then for any fixed* k *with* $0 < k < 1$, (2.2) *is equivalent to*

$$P_v = O(P(u)), \text{ where } u = [v^k]. \tag{2.5}$$

Proof. If (2.2) holds, by successive application of (2.2) we see that for any fixed integer r,

$$P(v^{2^r}) = O(P_v) . \tag{2.6}$$

Choose r so that $2^r > 1/k$. Then if $u = [v^k]$, $v \leqq (u)^{1/k} < (u)^{2^r}$. So, since P_v is increasing, (2.5) follows from (2.6).

Conversely, suppose that (2.5) holds. Given any positive integer v, define v_r inductively (on r) by taking $v_0 = v$ and defining $v_r (r > 1)$ as the least integer greater than or equal to $v_{r-1}^{1/k}$. Since $\{p_n\} \in \mu$ implies that

$$\frac{p_r}{p_{r-1}} \longrightarrow 1 , \tag{2.7}$$

as $r \to \infty$, we see that (2.5) is equivalent to

$$P(v_1) = O(P_v) . \tag{2.8}$$

By successive application of (2.8) we deduce that, for any fixed r,

$$P(v_r) = O(P_v) . \tag{2.9}$$

Choose r so that $(1/k)^r > 2$. Then $v_r > v^2$ so that (again since P_v is increasing) (2.9) implies (2.2).

For the proof of the theorem we require (2.5). The condition (2.2) is preferable to (2.5) because the former is simpler and independent of k.

LEMMA 4. *If* (2.4) *is satisfied then*

$$\frac{\lambda(n+1)}{\lambda(n)} = O(1), \text{ as } n \longrightarrow \infty .$$

This is obvious.

3. **Proof of the theorem.** It is enough to consider the case $0 < k < 1$. This implies the result for $k \geqq 1$. We can assume without loss of generality that $a_0 = 0$. Then by Lemma 1 and (1.2) it is enough to show that (2.4′) implies

$$\int_1^\infty \left| \frac{d}{dw}(R_\lambda^k(w)/\lambda^k(w)) \right| dw < \infty .$$

Now,

$$na_n = \sum_{v=1}^{n} c_{n-v} P_v t_v(va_v) .$$

Then

$$\begin{aligned}\frac{d}{dw}\frac{R_\lambda^k(w)}{\lambda^k(w)} &= \frac{k\mu(w)}{\lambda^{k+1}(w)}\left[\sum_{n=1}^{[w]} \{\lambda(w) - \lambda(n)\}^{k-1} \frac{\lambda(n)}{n} \sum_{v=1}^{n} c_{n-v} P_v t_v(va_v)\right] \\ &= \frac{k\mu(w)}{\lambda^{k+1}(w)}\left[\sum_{v=1}^{[w]} P_v t_v(va_v) \sum_{n=v}^{[w]} \{\lambda(w) - \lambda(n)\}^{k-1} \frac{\lambda(n)}{n} c_{n-v}\right] .\end{aligned}$$

Then

$$\begin{aligned}&\int_1^\infty \left| \frac{d}{dw}(R_\lambda^k(w)/\lambda^k(w)) \right| dw \\ &= O(1)\left[\int_1^\infty \frac{\mu(w)}{\lambda^{k+1}(w)} \sum_{v=1}^{[w]} vP_v \frac{|t_v(va_v)|}{v} \left| \sum_{n=v}^{[w]} \{\lambda(w) - \lambda(n)\}^{k-1} \frac{\lambda(n)}{n} c_{n-v} \right| dw\right] \\ &= O(1)\left[\sum_{v=1}^{\infty} vP_v \frac{|t_v(va_v)|}{v} \int_v^\infty \frac{\mu(w)}{\lambda^{k+1}(w)} \left| \sum_{n=v}^{[w]} \{\lambda(w) - \lambda(n)\}^{k-1} \frac{\lambda(n)}{n} c_{n-v} \right| dw\right].\end{aligned}$$

Thus it is enough to prove that uniformly in $v \geqq 1$,

$$\begin{aligned}J(v) &= \int_v^\infty \frac{\mu(w)}{\lambda^{k+1}(w)} \left| \sum_{n=v}^{[w]} \{\lambda(w) - \lambda(n)\}^{k-1} \frac{\lambda(n)}{n} c_{n-v} \right| dw \\ &= O\left(\frac{1}{vP_v}\right) .\end{aligned} \tag{3.1}$$

Write $m = \min([w], v+u)$. Let $a = v + u - 1$, $b = v + u + 1$.

Applying partial summation to the sum over the range $v \leqq n \leqq m$, we see that the expression inside the modulus in (3.1) is equal to

$$\text{(3.2)}\quad \sum_{n=v}^{m-1} \Delta_n\Big[(\lambda(w)-\lambda(n))^{k-1}\frac{\lambda(n)}{n}\Big]d_{n-v}+(\lambda(w)-\lambda(m))^{k-1}\frac{\lambda(m)}{m}d_{m-v}$$
$$+\sum_{n=m+1}^{[w]}(\lambda(w)-\lambda(n))^{k-1}\frac{\lambda(n)}{n}c_{n-v}$$
$$=\sum_{n=v}^{m-1}(\lambda(w)-\lambda(n))^{k-1}\Delta\Big(\frac{\lambda(n)}{n}\Big)d_{n-v}$$
$$+\sum_{n=v}^{m-1}\Delta_n(\lambda(w)-\lambda(n))^{k-1}\frac{\lambda(n+1)}{n+1}d_{n-v}$$
$$+(\lambda(w)-\lambda(m))^{k-1}\frac{\lambda(m)}{m}d_{m-v}+\sum_{n=m+1}^{[w]}(\lambda(w)-\lambda(n))^{k-1}\frac{\lambda(n)}{n}c_{n-v}\,.$$

Here the last term is to be omitted when $m=[w]$, i.e., when $w<b$. Hence

$$J(v)\leqq J_1(v)+J_2(v)+J_3(v)+J_4(v)\ ,$$

where $J_1(v)$, $J_2(v)$, $J_3(v)$, $J_4(v)$ denote the expressions obtained by replacing the expression inside the modulus in (3.1) by each of the four terms on the right of (3.2). First,

$$J_1(v)\leqq\sum_{n=v}^{a}d_{n-v}\Big|\,\Delta\Big(\frac{\lambda(n)}{n}\Big)\Big|\int_{n+1}^{\infty}\frac{(\lambda(w)-\lambda(n))^{k-1}\mu(w)}{\lambda^{k+1}(w)}dw$$
$$=O(1)\sum_{n=v}^{a}d_{n-v}\frac{|\Delta\lambda(n)|}{(n+1)\lambda(n)}+O(1)\sum_{n=v}^{a}d_{n-v}\frac{\lambda(n)}{n(n+1)\lambda(n)}\ .$$

Using (2.4) and Lemma 4,

$$J_1(v)=O(1)\sum_{n=v}^{a}\frac{d_{n-v}}{n^2}$$
$$=O(1)\frac{e(u-1)}{v^2}=O\Big(\frac{1}{vP_v}\Big)\,,$$

by Lemma 2(v) and Lemma 3. Next,

$$J_2(v)\leqq\sum_{n=v}^{a}d_{n-v}\frac{\lambda(n+1)}{n+1}$$
$$\times\int_{n+1}^{\infty}\frac{[(\lambda(w)-\lambda(n+1))^{k-1}-(\lambda(w)-\lambda(n))^{k-1}]\mu(w)}{\lambda^{k+1}(w)}dw\ .$$

The inner integral can be evaluated and is equal to

$$\frac{1}{k}\Big[\frac{1}{\lambda(n+1)}-\frac{1}{\lambda(n)}\Big\{1-\Big(\frac{\lambda(n+1)-\lambda(n)}{\lambda(n+1)}\Big)^k\Big\}\Big]$$
$$=\frac{1}{k\lambda(n)}\Big[\Big(\frac{\lambda(n+1)-\lambda(n)}{\lambda(n+1)}\Big)^k-\Big(\frac{\lambda(n+1)-\lambda(n)}{\lambda(n+1)}\Big)\Big]$$
$$=O\Big(\frac{1}{n^k\lambda(n)}\Big)\,,$$

by (2.4). Hence, by Lemma 4,

$$J_2(v) = O(1)\sum_{n=v}^{a}\frac{d_{n-v}}{n^{k+1}} = O(1)\frac{e(u-1)}{v^{k+1}}$$
$$= O\left(\frac{1}{vP_v}\right),$$

by Lemma 2(v) and Lemma 3.

Suppose $N \leqq w < N+1$. Then,

$$J_3(v) \leqq \int_v^{v+u}(\lambda(w)-\lambda(N))^{k-1}\frac{\lambda(N)}{N}d_{N-v}\frac{\mu(w)}{\lambda^{k+1}(w)}dw$$
$$+\frac{\lambda(v+u)}{v+u}d(u)\int_{v+u}^{\infty}(\lambda(w)-\lambda(v+u))^{k-1}\frac{\mu(w)}{\lambda^{k+1}(w)}dw$$
$$= J_{31}(v) + J_{32}(v)\,.$$

Since

$$\int_{v+u}^{\infty}(\lambda(w)-\lambda(v+u))^{k-1}\frac{\mu(w)}{\lambda^{k+1}(w)}dw = \frac{1}{k\lambda(v+u)}\,,$$

clearly

$$J_{32}(v) = O\left(\frac{d(u)}{v+u}\right) = O\left(\frac{1}{vP_v}\right),$$

by Lemma 2(iv) and Lemma 3.

Now,

$$J_{31}(v) \leqq \sum_{\sigma=0}^{u-1}\int_{v+\sigma}^{v+\sigma+1}(\lambda(w)-\lambda(N))^{k-1}\frac{\lambda(N)}{N}d_{N-v}\frac{\mu(w)}{\lambda^{k+1}(w)}dw$$
$$\leqq \sum_{\sigma=0}^{u-1}\frac{d_\sigma}{(v+\sigma)\lambda^k(v+\sigma)}\int_{v+\sigma}^{v+\sigma+1}(\lambda(w)-\lambda(N))^{k-1}\mu(w)dw$$
$$\leqq \sum_{\sigma=0}^{u-1}\frac{d_\sigma}{(v+\sigma)\lambda^k(v+\sigma)}\int_{v+\sigma}^{v+\sigma+1}(\lambda(w)-\lambda(N))^{k-1}\mu(w)dw$$
$$= O(1)\sum_{\sigma=0}^{u-1}\frac{d_\sigma}{(v+\sigma)\lambda^k(v+\sigma)}\,\frac{\lambda^k(v+\sigma+1)}{(v+\sigma+1)^k}$$
$$= O\left(\frac{1}{vP_v}\right),$$

by Lemma 4, Lemma 2(v), and Lemma 3. Hence

$$J_3(v) = O(1/vP_v)\,.$$

Lastly,

$$
\begin{aligned}
J_4(v) &\leqq \int_b^\infty \frac{\mu(w)}{\lambda^{k+1}(w)} \left[\sum_{n=m+1}^{[w]} (\lambda(w) - \lambda(n))^{k-1} \frac{\lambda(n)}{n} | c_{n-v} | \right] dw \\
&= \sum_{n=b}^{\infty} | c_{n-v} | \frac{\lambda(n)}{n} \int_n^\infty (\lambda(w) - \lambda(n))^{k-1} \frac{\mu(w)}{\lambda^{k+1}(w)} dw \\
&= \frac{1}{k} \sum_{n=b}^{\infty} \frac{| c_{n-v} |}{n} \leqq \frac{1}{k} \frac{d(u)}{b} \\
&= O(1/vP_v) ,
\end{aligned}
$$

by Lemma 2 (ii), (iv), and Lemma 3. Hence (3.1) is proved.

This completes the proof of the theorem.

By putting $p_n = 1/(n+1)$, $\lambda(w) = w$ (integer) we get the inclusion $|N, 1/(n+1)| \subset |R, n, k|$, $k > 0$ due to McFadden [4].

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