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GENERAL PEXIDER EQUATIONS (PART II): AN APPLICATION OF THE THEORY OF WEBS

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Given open connected Ω, $\tilde{\Omega}\subseteq R^n$ and continuous T: $\Omega\to R$, F: $\tilde{\Omega}\to R$ both strictly monotonic in each variable separately. The equation $h\{T(x_1, \cdots, x_n)\}=F\{f_1(x_1), \cdots, f_n(x_n)\}$ for the unknowns h: $T(\Omega)\to R$ and π: $(f_1, \cdots, f_n)$: $\Omega\to\tilde{\Omega}$ can be interpreted within the theory of webs (the "Gewebe" of Blaschke-Bol). The web structure is then used to prove: any continuous solution π is uniquely determined on Ω by its value at two points of Ω; if a solution π is not continuous on Ω, then $\pi(\omega)$ is dense in $\tilde{\Omega}$ for every open ω in Ω; if a solution π is continuous at one point of Ω, it is continuous on Ω.

1. Formulation of results.

We consider the equation

$$h\{T(x_1, \cdots, x_n)\} = F\{f_1(x_1), \cdots, f_n(x_n)\}$$

for given T, F, with $h, f_1, \cdots, f_n$ the unknowns. Specifically, we assume given two sets $\Omega, \tilde{\Omega}$ in R^n and two functions $T: \Omega\to R$, F: $\tilde{\Omega}\to R$. Let Ω_i and $\tilde{\Omega}_i$ denote the projections of Ω and $\tilde{\Omega}$ onto the ith coordinate axis; by a *product mapping* $\pi: \Omega\to\tilde{\Omega}$ is understood the restriction to Ω of a mapping $(f_1, \cdots, f_n)$: $X_1^n\Omega_i\to R^n$ defined by the n functions $f_i: \Omega_i\to\tilde{\Omega}_i$. The above equation becomes $h\circ T = F\circ\pi$ with h: $T(\Omega)\to R$ and $\pi: \Omega\to\tilde{\Omega}$ the unknowns.

The present note is self-contained in that Part I [2] served only to indicate that the following hypotheses on T and F are not as restrictive as one might at first suppose. For the moment we assume only that:

(A.1) *T is continuous and strictly monotonic in each variable on Ω,*

(A.2) *F is strictly monotonic in each variable on $\tilde{\Omega}$,*

(A.3) *Ω is open and connected.*

THEOREM 1. *With* (A. 1, 2, 3) *assume that $h\circ T = F\circ\pi$ and $\bar{h}\circ T = F\circ\bar{\pi}$ hold for two product mappiugs $\pi, \bar{\pi}$ on Ω. If π and $\bar{\pi}$ are equal at the end points of some line segment $l\subset\Omega$ parallel to a coordinate axis, then π and $\bar{\pi}$ coincide on a set dense in Ω. Hence two continuous solutions must be identical on Ω if they agree at the end points of such a line segment.*

In the event $\Omega = R^n$ as in the classical case $T(x_1, \cdots, x_n) = x_1+$

$\cdots + x_n$ (and in other cases indicated in the proofs), any two points may be chosen.

COROLLARY 1. *Under the hypotheses of Theorem* 1 *and the further assumption* $\Omega = R^n$, *any two solutions* $\pi, \bar{\pi}$ *which coincide at two distinct points of* R^n *must coincide on a set dense in* R^n. *A continuous solution is uniquely determined by its value at two points in* R^n.

Throughout this note results are stated only in terms of the mapping π since h is then uniquely determined by the equation itself. It should be noted that for the above theorem, F need not be continuous, no assumptions are made on $\tilde{\Omega}$, and Ω need not be a convex region in R^n. By strengthening the conditions on F and $\tilde{\Omega}$ it is possible to generalize the following known result (for example [5], p. 21):

if $f: R \to R$ satisfies Cauchy's equation $f(x+y) = f(x) + f(y)$ then either $f(x) = f(1)\cdot x$ or else $f(\omega)$ is dense in R for every open ω.

Specifically we assume

(B.1) T, F *continuous, strictly monotonic in each variable on* $\Omega, \tilde{\Omega}$ *respectively,*

(B.2) $\Omega, \tilde{\Omega}$ *open and connected subsets of* R^n.

THEOREM 2. *With* (B. 1, 2) *assume* $h \circ T = F \circ \pi$ *satisfied on* Ω *by a product mapping* π. *Then either* π *is continuous on* Ω *or else* $\pi(\omega)$ *is dense in* $\tilde{\Omega}$ *for every open subset* $\omega \subset \Omega$. *Hence continuity of* π *at one point implies the continuity of* π *at every point of* Ω.

In terms of the component functions f_i, Theorem 2 implies the following result.

COROLLARY 2. *Under the hypotheses of Theorem* 2, *with* $\pi = (f_1, \cdots, f_n)$, *if some* f_i *is continuous at one point then all* f_i *are continuous on their respective domains. In the event* $\tilde{\Omega} = R^n$, *it suffices to assume some* f_i *bounded in a neighborhood of some point in order to conclude all* f_i *continuous.*

Before proceeding to the theory of webs and the proofs of the above assertions, we mention the following additional corollary although its proof depends on a result from Part I; under hypothese weaker than (B.1) (B.2) it was shown that any continuous solution π must be either constant or injective.

COROLLARY 3. *As in Corollary* 2, *if some* f_i *is continuous at* *one point, or in the event* $\tilde{\Omega} = R^n$ *if some* f_i *is bounded in some neighborhood of a point, then either all* f_i *are constant or all* f_i *are continuous and strictly monotonic.*

In the event $F(u_1, \cdots, u_n) = u_1 + \cdots + u_n$ with $\tilde{\Omega} = R^n$, some of the above results are related to those of C. T. Ng [3], [4] since now, if h, π and $\bar{h}, \bar{\pi}$ are two solutions so also will $h + \bar{h}, \pi + \bar{\pi}$ be a solution.

For the most part it will suffice to prove these results in the two dimensional case

$$h\{T(x, y)\} = F\{f(x), g(y)\}$$

with T defined on $\Omega \subseteqq R^2$, F on $\tilde{\Omega} \subseteqq R^2$ and $\pi = (f, g): \Omega \to \tilde{\Omega}$. The hypothesis (A.1), common to both theorems, allows us to interpret Ω as a region covered by a 3-*web* (also called a 3-*net* by some authors) in the sense of Blaschke-Bol [1]. Specifically, through each point (x_0, y_0) of Ω there passes three curves: the coordinate curves $x = x_0$, $y = y_0$ and the contour curve $T(x, y) = T(x_0, y_0)$. Hence Ω is covered by three families of curves and any two curves from distinct families determine at most one curve from the third family since T is strictly monotonic – "at most" since two such curves need not intersect within Ω.

But $x = x_0$, $y = y_0$, and $T(x, y) = T(x_0, y_0)$ are then mapped into the curves $u = f(x_0)$, $v = g(y_0)$ and the set of points satisfying $F(u, v) = h\{T(x_0, y_0)\}$ respectively under the mapping $\pi = (f, g)$. Once hypothesis (B.2) is introduced these latter curves define a 3-web on $\tilde{\Omega}$ and π becomes a web preserving mapping.

We now sketch the derivation of some elementary properties of 3-webs in a form needed in the proofs of our theorems; a more complete discussion appears in [1], pp. 1-15.

2. Elementary 3-web properties.

We consider the two dimensional case and in accordance with hypotheses (A.1) and (A.3) we may assume without loss of generality

Ω an open connected subset of the plan R^2, $T: \Omega \to R$ continuous, strictly monotonic increasing in each variable.

The sketching of a few figures will help clarify the following constructions and propositions.

By a *square* S in Ω is understood four points $(x_i, y_j) \in \Omega$ for $i, j = 1, 2$ where $T(x_1, y_2) = T(x_2, y_1)$; a *path* is a sequence $S_1, \cdots, S_n$ of squares such that each $S_i \wedge S_{i+1}$ has exactly two points; if all the points lie on two horizontal lines, say $S_i = \{(x_{i-1}, y_0), (x_{i-1}, y_1), (x_i, y_0), (x_i, y_1)\}$, the path is denoted by

$$(y_0 < y_1; x_0 < \cdots < x_n) \text{ where } T(x_{i+1}, y_0) = T(x_i, y_1) \tag{2.1}$$

or

$$(y_0 < y_1; x_0 > \cdots > x_n) \text{ where } T(x_{i+1}, y_1) = T(x_i, y_0) \tag{2.2}$$

(and analogously if $y_0 > y_1$) and similarly if the points all lie on two vertical lines. The path (2.1) is *increasing horizontal,* (2.2) is *decreasing horizontal*; both *initiate at* x_0 (more precisely (x_0, y_0) and (x_0, y_1) are the *initial points*), *terminate at* x_n (with (x_n, y_0), (x_n, y_1) as *terminal points*), while $x_0, \cdots, x_n$ are *partition points* of the line segment $[x_0, x_n] \times \{y_0\}$ (or of $[x_n, x_0] \times \{y_0\}$). Similarly, increasing or decreasing *vertical paths* can be defined. All horizontal or vertical paths are called *rectilinear.*

PROPERTY 1. *If $[a, b] \times \{y_0\} \subset \Omega$ then there exists an integer N such that every $n \geqq N$ determines a unique $y_1 > y_0$ and unique partition points $x_0, \cdots, x_n$ such that $(y_0 < y_1; x_0 = a < \cdots < x_n = b)$ is an increasing horizontal path, equivalently $(y_0 < y_1; x_0 = b > \cdots > x_n = a)$ is a decreasing horizontal path. For fixed $y_1 > y_0$, the partition points x_i are continuous, strictly increasing functions of the end points a, b. Similarly for $\{x_0\} \times [c, d] \subset \Omega$ and vertical paths.*

In the language of webs both $(y_0 < y_1; x_0 = a < \cdots < x_n = b)$ and $(y_0 < y_1; x_0 = b > \cdots > x_n = a)$ define (in general equivalent) n-partitionings of $[a, b] \times \{y_0\}$; the corresponding (x_i, y_0)'s are rational points of $[a, b] \times \{y_0\}$. For given $n \geqq N$ the corresponding $y_1 > y_0$ is unique but a path also exists for $y_1 < y_0$ and these paths may be distinct (unless closure conditions are assumed). Intuitively, as $n \to \infty$ so also the corresponding $y_1 \to y_0$ and $|x_{i+1} - x_i| \to 0$; hence *the rational points on $[a, b] \times \{y_0\}$ are dense.* For sufficiently large n such rectilinear paths can be extended at both ends since $[a, b] \times \{y_0\} \subset \Omega$ open. More specifically.

PROPERTY 2. *Given $a < c < b$ with $[a, b] \times \{y_0\} \subset \Omega$. Then for arbitrary α, β satisfying $a \leqq \alpha < \beta \leqq c$ and for sufficiently large n, an increasing horizontal path exists of the form $(y_0 < y_1; x_0 = \alpha < \cdots < x_n = \beta < \cdots < x_m)$ with both x_{m-1} and x_m within the open interval $]c, b[$. Similarly for decreasing horizontal and for vertical paths.*

Rather than choosing large n, y_1 can be chosen sufficiently close to y_0, thereby obtaining horizontal paths with $|x_{i+1} - x_i|$ arbitrarily small. However in this case the path cannot be assumed to partition a given interval unless y_1 is judiciously chosen. In this case Property 2 can be modified as follows.

PROPERTY 3. *Given* $a < c < b$ *with* $[a, b] \times \{y_0\} \subset \Omega$. *Then there exists a* $\delta > 0$ *such that for arbitrary* $y_1 \in]y_0, y_0 + \delta[$ *and for arbitrary* $\alpha \in [a, c]$, *an increasing horizontal path exists of the form* $(y_0 < y_1; x_0 = \alpha < \cdots < x_m)$ *with both* x_{m-1} *and* x_m *within* $]c, b[$. *Similarly for decreasing horizontal and for vertical paths.*

3. Proof of Theorem 1. For the moment consider the two dimensional case $\Omega, \tilde{\Omega} \subset R^2$ with $T: \Omega \to R$, $F: \tilde{\Omega} \to R$. Let $h \circ T = F \circ \pi$ and $\bar{h} \circ T = F \circ \bar{\pi}$, two solutions with $\pi, \bar{\pi}$ product functions on Ω. The hypotheses imply without loss of generality:

(A-1) *T continuous, strictly monotonic increasing in each variable,*

(A-2) *F strictly monotonic increasing in each variable,*

(A-3) *Ω open and connected.*

LEMMA 1. *If $\pi, \bar{\pi}$ coincide at two points of a square S, then $\pi, \bar{\pi}$ coincide at all points of any path formed with S as member.*

Proof. Since $S_i \wedge S_{i+1}$ always has two points it will suffice to prove that if $\pi = (f, g), \bar{\pi} = (\bar{f}, \bar{g})$ are equal at two points of $S = \{(x_i, y_j) | i, j = 1, 2\}$, then so also at the other two points. But if $\pi(x_i, y_j) = \bar{\pi}(x_i, y_j)$ for $(i, j) = (1, 2)$ and $(2, 1)$, or for $(i, j) = (1, 1)$ and $(2, 2)$, then trivially $f(x_i) = \bar{f}(x_i)$ and $g(y_i) = \bar{g}(y_i)$ for $i = 1, 2$. If for $(i, j) = (1, 1)$ and $(1, 2)$ (the other cases are similar) then $f(x_1) = \bar{f}(x_1)$, $g(y_i) = \bar{g}(y_i)$ for $i = 1, 2$ and hence only $f(x_2) = \bar{f}(x_2)$ remains. But $T(x_1, y_2) = T(x_2, y_1)$ implies

$$F\{f(x_1), g(y_2)\} = F\{f(x_2), g(y_1)\} \text{ and } F\{\bar{f}(x_1), \bar{g}(y_2)\} = F\{\bar{f}(x_2), \bar{g}(y_1)\}$$

and since the left hand sides are equal, $g(y_1) = \bar{g}(y_1)$ implies $f(x_2) = \bar{f}(x_2)$ by (A-2).

In Lemma 1, equality is assumed at two points of one square; for rectilinear paths a stronger result holds.

LEMMA 2. *If π and $\bar{\pi}$ coincide at two points of a rectilinear path, they coincide at all points of this path.*

Proof. Given $S_i = \{(x_i, y_1), (x_i, y_2), (x_{i+1}, y_2), (x_{i+1}, y_1)\}$ with $T(x_i, y_2) = T(x_{i+1}, y_1)$, set $\pi(x_i, y_j) = (u_i, v_j)$ and $\bar{\pi}(x_i, y_j) = (\bar{u}_i, \bar{v}_j)$. Then as above,

(1_i) $\quad F(u_i, v_2) = F(u_{i+1}, v_1)$ and (1_1) $F(u_1, v_2) = F(u_2, v_1)$,

(2_i) $\quad F(\bar{u}_i, \bar{v}_2) = F({}_i\bar{u}_{+1}, \bar{v}_1)$ and (2_1) $F(\bar{u}_1, \bar{v}_2) = F(\bar{u}_2, \bar{v}_1)$.

Suppose first that $\pi, \bar{\pi}$ coincide at an initial point, say $u_1 = \bar{u}_1$ and $v_1 = \bar{v}_1$. Monotoneity and $v_2 \lesseqgtr \bar{v}_2$ implies $(1_1) \lesseqgtr (2_1)$ and hence $u_2 \lesseqgtr \bar{u}_2$; but then $(1_2) \gtreqless (2_2)$ and hence $u_3 \lesseqgtr \bar{u}_3$ and by induction $u_i \lesseqgtr \bar{u}_i$. In this case π and $\bar{\pi}$ could never coincide again on the path. Similarly, $u_1 = \bar{u}_1$ with $v_2 = \bar{v}_2$ implies $(1_1) = (2_1)$; hence $v_1 \lesseqgtr \bar{v}_1$, implies $u_2 \gtreqless \bar{u}_2$ and in turn $(1_2) \lesseqgtr (2_2)$ and inductively $u_i \gtreqless \bar{u}_i$. Again $\pi, \bar{\pi}$ cannot be equal at any other point. Hence equality at two points of this path implies equality at two points of one square, and by Lemma 1, at all points of the path. A similar argument applies to vertical paths.

LEMMA 3. *If $\pi, \bar{\pi}$ coincide at two points of a line segment $[a, b] \times \{y_0\} \subset \Omega$, then $\pi, \bar{\pi}$ coincide on a set dense in this line segment. Similarly for $\{x_0\} \times [c, d] \subset \Omega$.*

Proof. Lemma 2 implies that if $\pi, \bar{\pi}$ coincide at two points of an n-partition so also at all points of this partition; if (x_i, y_0) and (x_{i+1}, y_0) are two points of this partition, then $\pi, \bar{\pi}$ will coincide at all rational points of $[x_i, x_{i+1}] \times \{y_0\}$. Hence if $\pi, \bar{\pi}$ coincide at two partition points of a line segment, they coincide on a set dense on this line segment. Suppose now that $\pi, \bar{\pi}$ are equal at (a', y_0), (b', y_0) for $a \leqq a' < b' \leqq b$, and consider $[a', B] \times \{y_0\}$ for $b - \delta < B \leqq b$. For sufficiently large n the n-partitioning of $[a', B] \times \{y_0\}$ will satisfy $|x_{i+1} - x_i| < \delta/2$ and by the continuity in Property 1 some B exists in $[b - \delta, b]$ such that (b', y_0) is a partition point of $[a', B] \times \{y_0\}$, implying that $\pi, \bar{\pi}$ coincide on a set dense in $[a', B] \times \{y_0\}$. Since $\delta > 0$ is arbitrary, equality holds on a set dense in $[a', b] \times \{y_0\}$. Similarly for $[a, b'] \times \{y_0\}$.

To prove the Theorem 1, note that Ω open and connected implies that any two points of Ω can be joined by a polygon with sides parallel to the axes. If $\pi, \bar{\pi}$ coincide at two points of some side, then so also on a set dense on this side; however the adjacent sides may not intersect at one of these points of equality, and hence a "neighboring" polygon is required, as constructed below.

LEMMA 4. *Given two perpendicular line segments $l_1, l_2 \subset \Omega$, parallel to the axes and intersecting at a partition point of l_1. If $\pi, \bar{\pi}$ are equal at the end points of l_1, then they are equal on a set dense in $l_1 \cup l_2$.*

Proof. By Lemmas 2 and 3, the π and $\bar{\pi}$ are equal on any n-partitioning of l_1 and since any partition point of l_1 belongs to a square on which $\pi, \bar{\pi}$ are equal, they will be equal at two points of l_2 provided the partition is sufficiently fine; this last restriction can be satisfied by choosing $l_1 \wedge l_2$ together with any other sufficiently close partition point of l_1.

In R^2, the theorem follows by considering polygons formed by line segments $l_1, l_2, \cdots, l_n$ parallel to the axes and replacing l_2 by a neighboring $\tilde{l}_2$ through a rational point of l_1, l_3 by $\tilde{l}_3$ through a rational point of $\tilde{l}_2$, etc. Any two partition points of l_1 can be joined to within ε of any point in Ω by such a polygon—a path in Ω. By considering any two variables in R^n, the theorem now follows fairly easily by decomposing Ω into overlaping convex sets, in particular n-dimensional rectangles; consider the given line segment as one dimensional edge of an n-dimensional rectangle within $\Omega \subset R^n$, sides parallel to the coordinate hyperplanes. The above argument implies $\pi, \bar{\pi}$ equal on a set dense in each two-dimensional "face" containing this edge. For product mappings, equality follows on a set dense in this n-dimensional rectangle, etc.

For Corollary 1 note that if $\pi(a_1, \cdots, a_n) = \bar{\pi}(a_1, \cdots, a_n)$ and $\pi(b_1, \cdots, b_n) = \bar{\pi}(b_1, \cdots, b_n)$ for product mappings $\pi = (f_1, \cdots, f_n)$ and $\bar{\pi} = (\bar{f}_1, \cdots, \bar{f}_n)$ then $f_i(a_i) = \bar{f}_i(a_i)$ and $f_i(b_i) = \bar{f}_i(b_i)$; hence one need only change one a_i to a b_i in $\pi(a_1, \cdots, a_n) = \bar{\pi}(a_1, \cdots, a_n)$ to obtain equality at the end points of a line segment parallel to a coordinate axis.

4. Preliminary lemmas for Theorem 2.

We again consider first the two dimensional case

$$h\{T(x, y)\} = F\{f(x), g(y)\} \tag{4.1}$$

with $T\colon \Omega \to R$, $F\colon \tilde{\Omega} \to R$ and $\pi = (f, g)\colon \Omega \to \tilde{\Omega}$. The hypotheses now imply without loss of generality

(B.1) *T, F continuous and strictly monotonic increasing in each variable,*

(B.2) *Ω and $\tilde{\Omega}$ open connected subsets of the plane R^2.*

Let $\Omega_x, \tilde{\Omega}_x$ and $\Omega_y, \tilde{\Omega}_y$ denote the projections of $\Omega, \tilde{\Omega}$ onto the x and y axes respectively. We again consider rectilinear paths

$$(y_0 < y_1; x_0 < \cdots < x_n) \text{ where } T(x_{i+1}, y_0) = T(x_i, y_1) \tag{4.2}$$

or

$$(y_0 < y_1; x_0 > \cdots > x_n) \text{ where } T(x_{i+1}, y_1) = T(x_i, y_0)\,. \tag{4.3}$$

Since now a 3-web is also defined on $\tilde{\Omega}$, using the contour curves of F, the concepts presented in § 2 also apply to paths in $\tilde{\Omega}$. The significance of such paths lies in the following observation. For the path (4.2), the equation $h \circ T = F \circ \pi$ implies $F\{f(x_{i+1}), g(y_0)\} = F\{f(x_i), g(y_1)\}$ and hence by (B.1), $g(y_0) \gtrless g(y_1)$ implies $f(x_{i+1}) \lessgtr f(x_i)$ for all i while $g(y_0) = g(y_1)$ implies $f(x_{i+1}) = f(x_i)$ for all i. Similarly for the path (4.3). In effect each path $(y_0 < y_1; x_0 < \cdots < x_n)$ in Ω defines a path in $\tilde{\Omega}$, with T replaced by F, of the form $(g(y_0) \lessgtr g(y_1); f(x_0) \gtrless \cdots \gtrless f(x_n))$, assuming f not constant on $x_0, \cdots, x_n$. The following lemmas are then clear.

LEMMA 1. *Given* $[a, b] \times \{y_0\} \subset \Omega$ *with* $f(a) < f(b)$. *Then* π *maps* $[a, b] \times \{y_0\}$ *onto a set dense in* $[f(a), f(b)] \times \{g(y_0)\}$. *Similarly if* $f(b) < f(a)$ *and for vertical paths* $\{x_0\} \times [c, d] \subset \Omega$.

This lemma states that every n-partition of $[a, b] \times \{y_0\}$ has as image an n-partition of $[f(a), f(b)] \times \{g(y_0)\}$, or in the language of webs, rational points on $[a, b] \times \{y_0\}$ must map to rational points of $[f(a), f(b)] \times \{g(y_0)\}$, and the set of rational points on a line segment is dense in that line segment.

LEMMA 2. *For any solution* $\pi = (f, g)$, f *is either constant or strictly monotone on the partition points of any horizontal path. Similarly for* g *on vertical paths.*

The following lemmas are less trivial.

LEMMA 3. *If either* f *or* g *is not monotone on* Ω_x, Ω_y *respectively then both* f *and* g *are not monotone on any open interval in their respective domains.*

Proof. Suppose f monotone on some interval I and let $]a_0, c[$ be the union of all open intervals, containing I, on which f is monotone. We prove $]a_0, c[= \Omega_x$. If $c \in \Omega_x$ then $(c, y_0) \in \Omega$ for some y_0 and so also $[a, b] \times \{y_0\} \subset \Omega$ for some $a < c < b$. Choose γ, β with $c \leqq \gamma < \beta \leqq b$; then by Lemma 2, f will be monotone on any path $(y_0 < y_1; \beta > \cdots > \gamma = x_n > \cdots > x_m)$ where, by Property 2, the x_{m-1} and x_m can be made to lie within $]a, c[$. Monotoneity is thus extended to $[c, b]$ contradicting the fact that $]a_0, c[$ was maximal. Hence $c \notin \Omega_x$ and similarly for a_0. With f not monotone it remains to prove g not monotone on Ω_y. Choose $a < b$ with $[a, b] \times \{y_0\} \subset \Omega$ for some y_0. Since f is not monotone on any interval, choose this a, b such that for some c with $a < c < b$ we have say $f(a) < f(b) < f(c)$ (the other cases are similar). By partitioning $[a, c] \times \{y_0\}$ and $[c, b] \times \{y_0\}$, Property 1 guarantees the existence of y_1, y_2 for the paths $(y_0 < y_1; x_0 = a < \cdots < x_n = c)$ and $(y_0 < y_2; \bar{x}_0 = c < \cdots < \bar{x}_n = b)$ where $T(x_i, y_1) = T(x_{i+1}, y_0)$ and $T(\bar{x}_i, y_2) = T(\bar{x}_{i+1}, y_0)$ respectively. But then $h \circ T = F \circ \pi$ implies $F\{f(x_i), g(y_1)\} = F\{f(x_{i+1}), g(y_0)\}$ and $F\{(f(\bar{x}_i), g(y_2)\} = F(f(\bar{x}_{i+1}), g(y_0)\}$ respectively. But since $f(a) < f(c)$, Lemma 2 implies $f(x_i) < f(x_{i+1})$ and by (B.1), $g(y_1) > g(y_0)$. Similarly $f(b) < f(c)$ implies $f(\bar{x}_i) > f(\bar{x}_{i+1})$ and hence $g(y_2) < g(y_0)$, that is, $g(y_2) < g(y_0) < g(y_1)$. But since $x_i < x_{i+1}$ so $y_1 > y_0$ and since $\bar{x}_i < \bar{x}_{i+1}$ so also $y_2 > y_0$ implying g not montone.

LEMMA 4. *Given any compact subset* $K \subset \tilde{\Omega}$ *and any* $v_1 \neq v_0$. *Then there exists an* $\varepsilon > 0$ *such that*

$$F(u_1, v_0) = F(u_0, v_1) \text{ implies } |u_1 - u_0| > \varepsilon$$

whenever (u_1, v_0) *and* (u_0, v_1) *are in* K.

Proof. If not, then convergent sequences $\{(u_{1n}, v_0)\}$, $\{(u_{0n}, v_1)\}$ exist in K with $|u_{1n} - u_{0n}| < 1/n$, and with $F\{u_{1n}, v_0\} = F\{u_{0n}, v_1\}$. The limit points (u, v_0) and (u, v_1) then satisfy $F\{u, v_0\} = F\{u, v_1\}$, a contradiction.

Given any $[a, b] \times \{y_0\} \subset \Omega$, if f is not monotone on its domain Ω_x then by Lemma 3 it is not monotone on any sub-interval of $[a, b]$. Hence it may be assumed that a and b were chosen such that for some c

$$(4.4) \quad \begin{cases} a < c < b \text{ and either (i) } f(a) < f(b) < f(c) \text{ or} \\ \text{(ii) } f(c) < f(a) < f(b) \text{ or (iii) } f(c) < f(b) < f(a) \\ \text{or finally (iv) } f(b) < f(a) < f(c)\,. \end{cases}$$

LEMMA 5. *Assume* (4.4) *valid for some* $[a, b] \times \{y_0\} \subset \Omega$ *and let* $v_0 = g(y_0)$. *Let* A *denote the greatest lower bound (including* $-\infty$) *and* B *the least upper bound (including* $+\infty$) *of* f *on* $[a, b]$. *Then the following assertions hold*:

(4.5) *the image of* $[a, b] \times \{y_0\}$ *is dense in* $[A, B] \times \{v_0\}$, *and*

(4.6) *neither* (A, v_0) *nor* (B, v_0) *can belong to* $\tilde{\Omega}$.

Proof. The case (4.4i) is proved in detail, (4.4iii) is outlined and the other cases are similar. Recall that for any increasing path $(y_0 < y_1; x_0 < \cdots < x_n)$, the relation $T(x_i, y_1) = T(x_{i+1}, y_0)$ and hence $F\{f(x_i), g(y_1)\} = F\{f(x_{i+1}), g(y_0)\}$ holds; however for decreasing paths $(y_0 < y_2; x_0 > \cdots > x_n)$ these relations become $T(x_i, y_0) = T(x_{i+1}, y_2)$ and hence $F\{f(x_i), g(y_0)\} = F\{f(x_{i+1}), g(y_2)\}$.

Case (4.4i) $f(a) < f(b) < f(c)$.

By Property 1 the paths $(y_0 < y_1; x_0 = a < \cdots < x_n = c)$ and $(y_0 < y_2; x_0 = b > \cdots > x_n = c)$, for all $n \geqq N$, uniquely determine y_1 and y_2. For n sufficiently large the corresponding y_1 and y_2 will satisfy $|y_i - y_0| < \delta$ where $\delta > 0$ is as in Property 3. Repeated applications of Property 3 permit the construction of sequences $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ as follows (where m may denote different integers each time it appears):

(1_1) $(y_0 < y_1; x_0 = a < \cdots c < \cdots < x_m = b_1)$ for some $b_1 \in]c, b[$,

(2_1) $(y_0 < y_2; x_0 = b_1 > \cdots > x_m = a_1)$ for some $a_1 \in]a, c[$,

(1_k) $(y_0 < y_1; a_{k-1} < \cdots < x_m = b_k)$ for some $b_k \in]c, b[$,

(2_k) $(y_0 < y_2; b_k > \cdots > x_m = a_k)$ for some $a_k \in]a, c[$.

By (1_1), $F\{f(x_i), g(y_1)\} = F\{f(x_{i+1}), g(y_0)\}$ and since $f(a) < f(c)$ by hypothesis, Lemma 2 implies all $f(x_i) < f(x_{i+1})$, and in turn $g(y_1) > g(y_0)$. But then the same argument applied to (1_k) yields $f(x_i) < f(x_{i+1})$ and in turn $f(a_{k-1}) < f(b_k)$ for $k = 1, 2, \cdots$. Since y_2 is defined by $(y_0 < y_2; x_0 = b > \cdots > x_n = c)$, implying $F\{f(x_i), g(y_0)\} = F\{f(x_{i+1}), g(y_2)\}$, the hypothesis $f(b) < f(c)$ together with Lemma 2 yields $f(x_i) < f(x_{i+1})$, that is $g(y_0) > g(y_2)$. But then a similar argument applied to (2_k) also yields $f(x_i) < f(x_{i+1})$, that is, $f(b_k) < f(a_k)$. Hence $f(a_{k-1}) < f(b_k) < f(a_k)$ for $k = 1, 2, \cdots$; in particular $f(a_k)$ is strictly monotone increasing with k, and in fact $f(a_k) > f(a)$ in view of (1_1). By Lemma 1 the image of $[a, b] \times \{y_0\}$ must be dense in $[f(a), B[\times \{v_0\}$ where $B = l.u.b.\ f(a_k)$. If now $(B, v_0) \in \tilde{\Omega}$ set $K = [\alpha, \beta] \times [v_{-1}, v_1]$ for some $\alpha < B < \beta$ and $v_{-1} < v_0 < v_1$ such that $K \subset \tilde{\Omega}$. Consider (1_k) with $u_1 = f(x_{i+1})$ and choose $v_1 < g(y_1)$. By Lemma 4 there exists an $\varepsilon > 0$ such that within K, $F\{u_0, v_1\} = F\{f(x_{i+1}), v_0\}$ implies $f(x_{i+1}) > u_0 + \varepsilon$. If v_1 is increased, u_0 must decrease and hence $F\{f(x_i), g(y_1)\} = F\{f(x_{i+1}), v_0\}$ implies $f(x_{i+1}) > f(x_i) + \varepsilon$. With $f(a_{k-1}) \in [\alpha, B]$, by (1_k) follows $f(b_k) > f(a_{k-1}) + m\varepsilon$ and analogously (choose $v_{-1} > g(y_2)$), $f(a_k) < f(b_k) + m\varepsilon'$ for some $\varepsilon' > 0$; clearly $(b, v_0) \notin \tilde{\Omega}$. To prove the corresponding result for the $g.l.b.\ A$, use the same y_1 and y_2 as above (with $g(y_1) > g(y_0)$ and $g(y_0) > g(y_2)$) but replace the previous paths with the following:

(1_1) $(y_0 < y_1; x_0 = c > \cdots > x_m = a_1)$ for some $a_1 \in]a, c[$,

(2_1) $(y_0 < y_2; x_0 = a_1 < \cdots < x_m = b_1)$ for some $b_1 \in]c, b[$,

(1_k) $(y_0 < y_1; x_0 = b_{k-1} > \cdots > x_m = a_k)$ for some $a_k \in]a, c[$,

(2_k) $(y_0 < y_2; x_0 = a_k < \cdots < x_m = b_k)$ for some $b_k \in]c, b[$.

For (1_k) as before, $F\{f(x_i), g(y_0)\} = F\{f(x_{i+1}), g(y_1)\}$ and hence $f(x_i) > f(x_{i+1})$. Hence $f(c) > f(a_1)$ and $f(b_{k-1}) > f(a_k)$. For (2_k) follows $F\{f(x_i), g(y_2)\} = F\{f(x_{i+1}), g(y_0)\}$ implying $f(x_i) > f(x_{i+1})$ and hence

$f(a_k) > f(b_k)$. Consequently $f(c) > f(a_1) > \cdots > f(b_{k-1}) > f(a_k) > f(b_k)$ implying that now $f(a_k)$ is a monotone decreasing sequence. As before the image of $[a, b] \times \{y_0\}$ will be dense in $]A, f(c)] \times \{v_0\}$ where $A = g.l.b.\ f(a_k)$. Again $(A, y_0) \notin \tilde{\Omega}$.

Case (4.4iii) $f(c) < f(b) < f(a)$.

Now define y_1 and y_2 by $(y_0 < y_1; x_0 = c < \cdots < x_n = b)$ and $(y_0 < y_2; x_0 = c > \cdots > x_n = a)$. For y_1 follows $F\{f(x_i), g(y_1)\} = F\{f(x_{i+1}), g(y_0)\}$ and since $f(c) < f(a)$ so $f(x_i) < f(x_{i+1})$ and $g(y_1) > g(y_0)$. For y_2 follows $F\{f(x_i), g(y_0)\} = F\{f(x_{i+1}), g(y_2)\}$ and since $f(c) < f(a)$, so now $g(y_0) > g(y_2)$. Consider

(1_1) $(y_0 < y_1; x_0 = c > \cdots > x_m = a_1)$ for some $a_1 \in]a, c[$,

(2_1) $(y_0 < y_2; x_0 = a_1 < \cdots < x_m = b_1)$ for some $b_1 \in]c, b[$,

(1_k) $(y_0 < y_1; x_0 = b_{k-1} > \cdots > x_m = a_k)$ for some $a_k \in]a, c[$,

(2_k) $(y_0 < y_2; x_0 = a_k < \cdots < x_m = b_k)$ for fome $b_k \in]c, b[$.

For (1_k) follows $F\{f(x_i), g(y_0)\} = F\{f(x_{i+1}), g(y_1)\}$ and hence $f(x_i) > f(x_{i+1})$ and $f(b_{k-1}) > f(a_k)$. For (2_k) follows $F\{f(x_i), g(y_2)\} = F\{(x_{i+1}), g(y_0)\}$, hence $f(x_i) > f(x_{i+1})$ and $f(a_k) > f(b_k)$. Hence $f(c) > f(a_1) > \cdots > f(b_{k-1}) > f(a_k) > f(b_k)$. In particular the sequence $f(a_k)$ is monotone decreasing. To obtain a monotone increasing sequence use

(1_k) $(y_0 < y_1; a_{k-1} < \cdots < b_k)$

and

(2_k) $(y_0 < y_2; b_k > \cdots > a_k)$

with $a_0 = c$.

5. Proof of Theorem 2. For the moment continue with the two dimensional case. With $[a, b] \times \{y_0\} \subset \Omega$ let $v_0 = g(y_0)$; in $\tilde{\Omega}$ the image $(f(a), v_0)$ of $(a; y_0)$ lies on some maximal open horizontal line segment $]A, B[\times \{v_0\} \subset \tilde{\Omega}$. The image $(f(b), v_0)$ of (b, v_0) may or may not lie within this line segment; conceivably $\tilde{\Omega}$ could contain $]A, B[\times \{v_0\}$ and $]B, C[\times \{v_0\}$ with $(B, v_0) \notin \tilde{\Omega}$ while $f(a) \in]A, B[$ and $f(b) \in]B, C[$. Nevertheless if f is not continuous then by Lemmas 3 and 5, the image of $[a, b] \times \{y_0\}$ must be dense in $]A, C[\times \{v_0\}$, and in particular dense in $]A, B[\times \{v_0\}$. Similarly for g. Hence for any point $(x_0, y_0) \in \omega \subset \Omega$ with ω open, the image $\pi(\omega)$ must be dense in any horizontal or vertical line segment contained in $\tilde{\Omega}$ and containing $(f(x_0), g(y_0))$.

By considering any two variables in R^n, a similar result holds: for any point $p_0 \in \omega \subset \Omega \subset R^n$ with ω open, the image $\pi(\omega)$ must be dense in any line segment $\gamma \subset \tilde{\Omega}$ with γ parallel to a coordinate axis and with $q_0 = \pi(p_0) \in \gamma$. Since $\tilde{\Omega}$ is open and connected this q_0 can be joined to any other point $q_n \in \tilde{\Omega}$ by a polygon consisting of line segments $\gamma_0, \cdots, \gamma_n$ parallel to the axes with $\gamma_i \subset \tilde{\Omega}$, $q_0 \in \gamma_0$ and $q_n \in \gamma_n$. Since $\pi(\omega)$ is dense in γ_0, γ_1 can be replaced by γ_1^* arbitrarily close to γ_1 and with $\gamma_0 \wedge \gamma_1^* \in \pi(\omega)$, and in general, with $\pi(\omega)$ dense in γ_i^*, γ_{i+1} can be replaced by γ_{i+1}^* arbitrarily close to γ_{i+1} but with $\gamma_i^* \wedge \gamma_{i+1}^* \in \pi(\omega)$. The polygon $\gamma_0, \gamma_1^*, \cdots, \gamma_n^*$ then joins q_0 to q_n^* with q_n^* arbitrarily close to q_n and since q_n was arbitrary, $\pi(\omega)$ is dense in $\tilde{\Omega}$.

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