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UNICOHERENT PLANE PEANO SETS ARE σ-UNICOHERENT

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A *Peano space* is a connected, locally connected and locally compact metric space. A *region* in a space X is an open and connected subset of X. A space X is *σ-connected* if every sequence $A_1, A_2, \cdots$ of closed, mutually disjoint subsets of X, with at least two of them nonempty, fails to cover X. A connected space X is *unicoherent* (resp., *σ-unicoherent*) if for every pair H, K of closed and connected (resp., and σ-connected) sets with union X, the intersection $H \cap K$ is connected (resp., σ-connected).

THEOREM. Let X be a plane Peano space. Then the following properties are equivalent:

(a) X is unicoherent;

(b) There exists a cover of X formed by unicoherent regions $U_1 \subset U_2 \subset \cdots$ with compact closures;

(c) X is σ-unicoherent, and

(d) If $M_1, M_2, \cdots$ is a sequence of closed, mutually disjoint subsets of X such that $X - M_i$ is connected for every i, then $X - (M_1 \cup M_2 \cup \cdots)$ is connected.

1. Introduction.

It is a well known fact that every unicoherent Peano continuum X satisfies the following property, which we shall call *property* A:

If $M_1, M_2, \cdots$ is a sequence of closed, mutually disjoint subsets of X such that $X - M_i$ is connected for every i, then $X - \bigcup_{i=1}^{\infty} M_i$ is connected.

It has been proved that certain unicoherent, noncompact Peano spaces also satisfy this property. In 1923, Miss A. Mullikin ([7]) proved that the plane has property A. (In 1924, S. Mazurkiewicz ([6]) simplified considerably Miss Mullikin's proof). In 1952, van Est ([10]) proved all Euclidean spaces also have property A. Recently, in 1971, J. H. V. Hunt ([4]) gave an example of a unicoherent, noncompact Peano space (contained in R^3) which does not have this property, and proposed the problem of finding a class of Peano spaces with property A and containing all Euclidean spaces. Finally, in 1973, E. D. Tymchatyn and Hunt himself ([9])[1] discovered such a class, described by the following theorem:

[1] The authors are indebted to Professor Hunt for his many helpful comments.

Every Peano space with the following property (which we shall call *B*):

There exists a cover of the space formed by unicoherent regions $U_1 \subset U_1^- \subset U_2 \subset U_2^- \subset U_3 \subset \cdots$ *with compact closures*[2], *has also property* A.

Analyzing Hunt's example quoted above one wonders if there exists a plane Peano space which is unicoherent but does not have property A. In 3.2 below, we give a negative answer to this question, because for plane Peano spaces, unicoherence, property A, property B and σ-unicoherence[3] are all proved to be equivalent. In the proof we use the theorem of Tymchatyn-Hunt quoted before.

2. **Definitions and preliminary results.** A *Peano space* is a connected, locally connected and locally compact metric space. A *continuum* is a compact and connected space. The space X is a *semicontinuum* if for every pair of points $a, b \in X$ there exists a continuum in X containing a, b. A *region in a space* X is a connected and open subspace of X. A connected space X is *unicoherent* if for every pair H, K of closed connected sets with union X, the intersection $H \cap K$ is connected. A space X is *σ-connected* if every sequence $A_1, A_2, \cdots$ of closed, mutually disjoint subsets of X at least two of which are nonempty, fails to cover X. A space X is *locally σ-connected* if for every $x \in X$ and every neighborhood V of x, there exists a σ-connected neighborhood of x contained in V. A connected space X is *σ-unicoherent* if for each pair H, K of closed σ-connected sets with union X, the intersection $H \cap K$ is σ-connected.

We shall state without proof some results needed in the proof of the main Theorem 3.2. The first of them is obvious. For the others, we give a reference.

2.1. *Let A, X be subsets of R^2 such that $A \subset X$. If $R^2 - X$ has no bounded components, then every bounded component of $R^2 - A$ is contained in X.*

2.2. *Let U be a proper open set in a Hausdorff continuum X and let T be a component of U. Then* $\mathrm{Fr}\, T \cap \mathrm{Fr}\, U \neq \Phi$. (See, for instance, [2], 2.48.)

2.3. *Let $X \subset S^2$ be unicoherent and locally connected. Then $S^2 - X$ is a semicontinuum.* (See [1], page 75, Th. 1.)

[2] Since the regions U_n form a cover and each U_n^- is compact, it is clearly equivalent to assume that $U_1 \subset U_2 \subset \cdots$.

[3] This last concept was introduced by A. Garcia-Máynez in [3].

2.4. *Let X be connected, locally σ-connected and completely normal. If X is σ-unicoherent, then X satisfies property* A. (This can be obtained easily from Theorem 3.2 in [3]).

2.5. *Let X be connected and locally connected. If X satisfies property* A, *then X is unicoherent.*

Proof. By Theorem 3 in [8], it is enough to prove that if R, S are regions in X with union X, then $R \cap S$ is connected. But then $A = X - R$ and $B = X - S$ are disjoint, closed and nonseparating subsets of X. Since X has property A, the set $X - (A \cup B) = R \cap S$ is connected.

3. Main theorem. Before proving the main theorem, we shall prove a result which we have not found in the literature.

3.1. *Let $X \subset R^2$ be connected and locally connected.*

(a) *If X is unicoherent, then $R^2 - X$ has no bounded components.*

(b) *If X is a G_δ and $R^2 - X$ has no bounded components, then X is unicoherent.*

Proof. (a) Identify S^2 with $R^2 \cup \{\infty\}$. According to 2.3, $S^2 - X$ is a semicontinuum. Proceeding by contradiction, assume $R^2 - X$ has a bounded component H. Select a point $q \in H$. Let L be a continuum in $S^2 - X$ containing ∞, q. Let T be the component of $L - \{\infty\}$ containing q. Since $T \subset R^2 - X$ and $T \cap H \neq \Phi$, we must have $T \subset H$. Therefore, T is bounded, that is, $\infty \notin T^-$. But according to 2.2, every component of $L - \{\infty\}$ contains ∞ in its closure, a contradiction.

(b) We proceed again by contradiction assuming X is not unicoherent. There exists then an essential mapping $f: X \to S^1$. According to (3), page 84 in [1], there exists a simple closed curve $J \subset X$ such that $f | J: J \to S^1$ is essential. Let D be the bounded component of $R^2 - J$. Necessarily, $D - X \neq \Phi$, because $D \subset X$ would imply that $f | J \cup D$ is nonessential (because $J \cup D$ is a disk and hence is contractible) and, therefore, $f | J$ would be also nonessential. There exists, therefore, a component H of $R^2 - X$ intersecting D. Then, $H \subset D$ and H is bounded, a contradiction.

The Example 3 described in [5] is a connected and locally connected subset X of R^2 which is an infinite countable union of mutually disjoint closed segments $X_1, X_2, \cdots$, all lying in a square. A direct analysis shows X is not unicoherent. According to Miss Mullikin's theorem quoted in the introduction, $R^2 - X$ is connected (because each $R^2 - X_i$ is connected). This proves, incidentally, that we cannot eliminate the hypothesis "X is a G_δ" in part (b) of Theorem 3.1.

We are now in a position to prove the main result of this paper.

3.2. *Let X be a Peano subspace of R^2. Then the following propositions are equivalent*:

(a) *X is unicoherent*;

(b) *X has property* B;

(c) *X is σ-unicoherent, and*

(d) *X has property* A.

(We shall give a cyclic proof (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a)).

Proof (a) $\Rightarrow$ (b). For this we shall need the following lemma.

LEMMA. *Let $X \subset R^2$ be Peano and unicoherent; let V be an X-region*[4] *with compact X-closure and let $\{L_j\}_{j \in M}$ be the family of bounded components of $R^2 - V$. Then $U = V \cup (\bigcup_{j \in M} L_j)$ is a unicoherent X-region with compact X-closure.*

Proof. By 3.1, $R^2 - X$ has no bounded components and then, by 2.1, each $L_j \subset X$. Hence, $U \subset X$. Each L_j is also a component of $X - V$ and since X is connected and locally connected, $U = V \cup (\bigcup_{j \in M} L_j)$ is connected (because no L_j can be separated from V).

$R^2 - U$ is the only nonbounded component of $R^2 - V$ (because V is bounded). Therefore, $R^2 - U$ is closed in $R^2 - V$, that is, there exists a closed set K in R^2 such that $R^2 - U = K \cap (R^2 - V)$. Also, $X - U = X \cap (R^2 - U) = X \cap K \cap (R^2 - V) = K \cap (X - V)$. That is, $X - U$ is X-closed and U is X-open.

Since the X-closure of V is compact, it coincides with V^- (the closure of V in R^2) and $V^- \subset X$. Further, by 1.47.2 in [2],

$$\operatorname{Fr}\Big(\bigcup_{j \in M} L_j\Big) \subset \Big(\bigcup_{j \in M} \operatorname{Fr} L_j\Big)^- \subset V^- .$$

Therefore,

$$\begin{aligned} U^- = V^- \cup \Big(\bigcup_{j \in M} L_j\Big)^- &= V^- \cup \Big(\bigcup_{j \in M} L_j\Big) \subset \operatorname{Fr}\Big(\bigcup_{j \in M} L_j\Big) \\ &= V^- \cup \Big(\bigcup_{j \in M} L_j\Big) \subset X . \end{aligned}$$

Hence, the X-closure of U coincides with U^- and, being bounded, it is compact.

Finally, 3.1 implies that U is unicoherent, because U is a G_δ in R^2 (since U is U^--open and U^- is a G_δ in R^2) and $R^2 - U$ has no bounded components (in fact, $R^2 - U$ is connected and unbounded).

[4] We shall use the prefix "$X-$" to indicate that the corresponding concept refers to the relative topology of X.

We come back now to the proof of (a) $\Rightarrow$ (b).

Let $\{V_\beta\}_{\beta \in L}$ be an X-cover of X with X-regions with compact X-closure. Since R^2 is hereditarily Lindelöf, there exists a countable subfamily $\{V_1, V_2, \cdots\}$ of $\{V_\beta\}_{\beta \in L}$ covering X. Let $\{H_\alpha\}_{\alpha \in K}$ be the family of bounded components of $R^2 - V_1$. By previous lemma, $U_1 = V_1 \cup (\bigcup_{\alpha \in K} H_\alpha)$ is a unicoherent X-region with compact X-closure. Let $\{V_{i_1}, V_{i_2}, \cdots, V_{i_m}\}$ be a subfamily of $\{V_1, V_2, \cdots\}$ such that $V'_2 = \bigcup_{j=1}^m V_{i_j}$ is an X-region (with compact X-closure) containing V_2 and U_1^-. Applying the lemma again, we can find a unicoherent X-region U_2 with compact X-closure and such that $U_2 \supset V'_2$. Proceeding inductively, we can get an X-cover $\{U_1, U_2, \cdots\}$ formed by unicoherent X-regions with compact X-closures and such that the X-closure of U_i is contained in U_{i+1} for each $i = 1, 2, \cdots$. This proves X satisfies property B.

(b) $\Rightarrow$ (c) This follows from the Tymchatyn-Hunt theorem in [9].

(c) $\Rightarrow$ (d) This is a corollary of 2.4.

(d) $\Rightarrow$ (a) This follows from 2.5.

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