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LATTICE VARIETIES COVERING THE SMALLEST NON-MODULAR VARIETY

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There are sixteen varieties of lattices that are known to cover $\boldsymbol{N}$, the variety generated by the five-element non-modular lattice N. Fifteen of these are generated by finite subdirectly irreducible lattices $L_1, L_2, \cdots, L_{15}$, and the sixteenth is jointly generated by N and the diamond M_3. We show that every variety of lattices that properly contains $\boldsymbol{N}$ includes one of the lattices $M_3, L_1, L_2, \cdots, L_{15}$. Of these sixteen lattices, the first six fail to be semidistributive; in fact, every variety of lattices in which the semidistributive law fails contains one of these six.

1. Introduction. By a *variety* of lattices is meant the class of all those lattices satisfying some fixed set of lattice identities. With respect to set inclusion the set of all varieties of lattices itself constitutes a lattice. The least element of this lattice is the class of all one-element lattices and the greatest element is the class of all lattices. Moreover, this lattice is distributive [5] and it has cardinality $2^{\aleph_0}$ [1], [7].

Let K denote a class of lattices and let $\boldsymbol{K}$ denote the variety generated by K. To determine $\boldsymbol{K}$ by finding all of the identities that hold in every lattice in K is often very difficult. On the other hand, there is an alternative approach to the problem of describing $\boldsymbol{K}$ which stems from the well known fact, due to G. Birkhoff, that a variety of lattices is determined by its subdirectly irreducible members. In fact, it is customary, where possible, to identify a given variety of lattices with its subdirectly irreducible members. For instance, in the lattice of varieties of lattices there is a unique atom whose only subdirectly irreducible member is the two-element chain: the variety of all distributive lattices. Covering this variety are precisely two varieties: one is $\boldsymbol{M}_3$, the variety generated by the diamond, M_3 (the five element modular non-distributive lattice); the other is $\boldsymbol{N}$, the variety generated by the pentagon N (the five-element non-modular lattice). While there is a great deal known about varieties of modular lattices (for instance, that the least modular variety $\boldsymbol{M}_3$ is covered by precisely three varieties, each generated by its finite subdirectly irreducible members [6] (cf. [4])) the non-modular case has proved to be more difficult to describe.

In [8] R. McKenzie lists fifteen finite, subdirectly irreducible, non-modular lattices $L_1, L_2, \cdots, L_{15}$ (Fig. 1) each of which generates

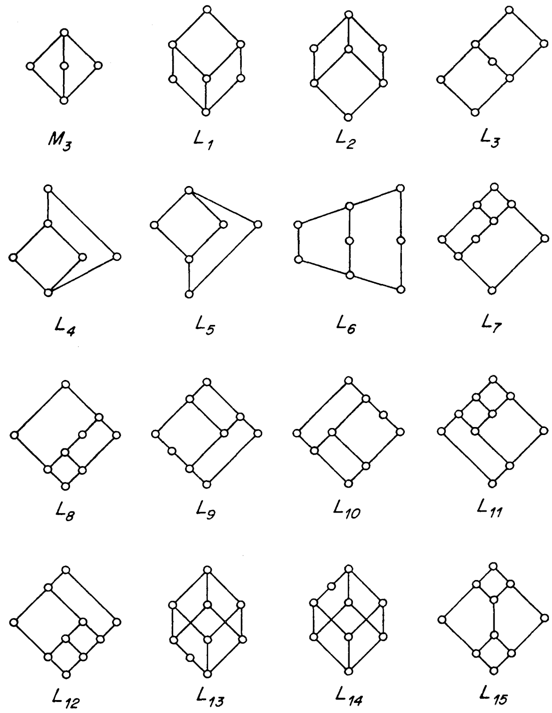


Figure 1

a variety that covers N. A sixteenth cover is jointly generated by N and M_3. Our principal result shows that McKenzie's list is complete.

THEOREM 1.1. *Every variety of lattices that properly contains N includes one of the lattices $M_3, L_1, L_2, \cdots, L_{15}$.*

This theorem was first established by I. Rival [9] under the additional assumption that the variety in question is generated by a lattice in which every chain is finite. Subsequently, B. Jonsson succeeded in removing this condition.

The proof of Theorem 1.1 consists of three main parts corresponding to a cumulative classification of the lattices $M_3, L_1, L_2, \cdots, L_{15}$.

The first part concerns *semidistributivity*. A lattice L is *semidistributive* if, for all $u, v, x, y, z \in L$, $u = x + y = x + z$ implies $u = x + yz$, and dually, $v = xy = xz$ implies $v = x(y + z)$. Call a variety of lattices *semidistributive* if each of its members is semidistributive. The main result of this part of the proof is of some independent interest.

THEOREM 1.2. *A variety of lattices is semidistributive if and* *only if it contains none of the lattices* M_3, L_1, L_2, L_3, L_4, *and* L_5.

This result was first proved by B. A. Davey, W. Poguntke and I. Rival [2] for those varieties generated by a lattice satisfying the double chain condition.

The second part of the proof concerns the behavior of congruence relations in non-modular lattices. Let a, b, and c be elements of a lattice L which generate a pentagon; that is, $bc < a < c < a + b$. We write $N(a, b, c)$ to indicate that this relation holds. Call a quotient c/a of L an *N-quotient* if $N(a, b, c)$ for some b. Let L be a lattice in a semidistributive variety that contains none of the lattices L_6, L_7, L_8, L_9, L_{10}, L_{11}, and L_{12}. The basic theme of this part of the proof is that projectivities between N-quotients in L behave like projectivities between quotients in a distributive lattice.

The final part concerns *critical edges* of a subdirectly irreducible lattice L. We call a quotient c/a of a subdirectly irreducible lattice L a *critical edge* if every non-trivial congruence relation on L identifies a and c. Let V be a variety that contains none of the lattices $M_3, L_1, L_2, \cdots, L_{12}$ and let $L \in V$ be subdirectly irreducible and non-distributive. We prove that L has a unique critical edge c/a, that c/a is the only N-quotient of L, and that the smallest congruence relation $\mathrm{con}(a, c)$ which identifies a and c identifies no two distinct elements besides a and c. Moreover, $L/\mathrm{con}(a, c)$ is distributive (cf. $L = L_{13}, L_{14}$ or L_{15}).

Therefore L is locally finite, and since every variety is determined by its finitely generated subdirectly irreducible members, we may assume that L is finite. It is now only a matter of straightforward calculations to show that if V does not contain L_{13}, L_{14} or L_{15} then L must be a pentagon.

The final section of this paper is devoted to several results related to Theorem 1.1.

We are indebted to Mr. Wilfried Ruckelshausen, who called our attention to a gap in one of our proofs, and also pointed out simplifications of two other arguments.

2. Semidistributivity. The principal aim of this section is the proof of Theorem 1.2. This generalization of the main result of [2] is realized by focussing attention on the lattices L^σ, of all ideals of L, and L^π, of all dual ideals of L. Of course, each of L, L^σ, and L^π generates the same variety of lattices. Moreover, L is embeddable in both L^σ and L^π. The advantage of L^σ over L lies in the fact that L^σ is compactly generated, whence weakly atomic. For instance, for $a, b \in L$ there exists an element c in L^σ such that $a \leqq c$ and which is covered by $a + b$ ($c \prec a + b$).

Theorem 1.2 is an immediate consequence of the following result.

LEMMA 2.1. *If the lattice L is not semidistributive, then either $L^{\sigma\pi\sigma}$ or $L^{\pi\sigma\pi}$ contains a sublattice isomorphic to one of the lattices M_3, L_1, L_2, L_3, L_4 or L_5.*

Proof. Let us suppose that the semidistributive law fails in L. By duality we may assume that there exist $u, x, y, z \in L$ such that

$$u = x + y = x + z \,, \tag{1}$$

but not $u = x + yz$. We claim that in the larger lattice $L^{\sigma\pi}$ we can find elements u, x, y, z that satisfy not only (1), but also

$$yz \leqq x \prec u, xy \prec y, xz \prec z \,. \tag{2}$$

In fact, given elements $u, x, y, z \in L$ such that (1) holds and $x + yz < u$, we can find $x' \in L^{\sigma}$ such that $x + yz \leqq x' \prec u$, and we therefore have

$$u = x' + y = x' + z, yz \leqq x' \prec u \,.$$

In $L^{\sigma\pi}$ we can then find minimal elements y' and z' subject to the conditions $u = x' + y' = x' + z'$, $y' \leqq y$, $z' \leqq z$. Then $x'y' < y'$. Furthermore, if $x'y' < t \leqq y'$, then $x' < x' + t \leqq u$ and hence $u = x' + t$, so that $t = y'$. Thus, y' covers $x'y'$ and, similarly, z' covers $x'z'$. Therefore (1) and (2) are satisfied if we replace x, y, and z by x', y', and z'.

We now assume that the elements $u, x, y, z \in L^{\sigma\pi}$ satisfy (1) and (2), and begin by looking at the sublattice generated by y, z, xy, and xz. In view of (2) we have

$$y \leqq xy + z \quad \text{or} \quad y(xy + z) = xy \,,$$
$$z \leqq xz + y \quad \text{or} \quad z(xz + y) = xz \,.$$

Of the four cases that arise, three easily yield one of the lattices L_i, $i \leqq 5$, as a sublattice of $L^{\sigma\pi}$. First, let $y \leqq xy + z$ and $z \leqq xz + y$. Let $v = xy + xz$, and observe that $y \nleqq x$ and $z \nleqq x$, hence $y \nleqq v$ and $z \nleqq v$. Consequently, $yv = xy$ and $zv = xz$. Also, $y + z = y + v = z + v$, and, therefore, L_2 is a sublattice of $L^{\sigma\pi}$ (Fig. 2).

Next, let us suppose that $y(xy + z) = xy$ and $z \leqq xz + y$. The lattice generated by y, z, xy, and xz is a homomorphic image of the lattice in (Fig. 3). Let $v = xy + z$. If $x(y + z) + v = y + z$, then y, v, and $x(y + z)$ generate a lattice isomorphic to L_5, or to M_3 if $xv = xy$, while if $x(y + z) + v < y + z$, then x, y, and $x(y + z) + v$ generate a lattice isomorphic to L_3 (Fig. 4). The case in which $y \leqq xy + z$ and $z(xz + y) = xz$ is symmetric to the preceding case, and it remains, therefore, to consider only the case in which $y(xy + z) = xy$ and $z(xz + y) = xz$.

Let $y_0 = y$ and $z_0 = z$, and, for $n = 0, 1, \cdots$, let

$$y_{n+1} = y + xz_n,\ z_{n+1} = z + xy_n\ .$$

Then (1) obviously holds with y and z replaced by y_n and z_n. Denote by (2_n) the formula obtained from (2) be replacing y and z by y_n and z_n. Suppose (2_n) holds, and consider (2_{n+1}). We may assume that $y_n z_{n+1} = xy_n$ and $z_n y_{n+1} = xz_n$, for otherwise one of the three cases already considered would apply with y and z replaced by y_n and z_n. As before, we can assume that $y_n(xy_n + z_n) = xy_n$ and $z_n(xz_n + y_n) = xz_n$, for otherwise we are done. We have $z \leqq z_{n+1}$ and $z \not\leqq x$, so $z_{n+1} \not\leqq x$, and hence, $xz_{n+1} < z_{n+1}$. If $xz_{n+1} < t < z_{n+1}$, then the elements, x, z_n and t generate a lattice isomorphic to L_3 (Fig. 5). We may, therefore, assume that $xz_{n+1} \prec z_{n+1}$ and, similarly, $xy_{n+1} \prec y_{n+1}$. We may also assume that $y_{n+1}z_{n+1} \leqq x$, for otherwise y_n, xy_{n+1} and $y_{n+1}z_{n+1}$ generate a lattice isomorphic to L_5. Thus, we may assume that (2_n) holds for all n.

In $L^{\sigma\pi\sigma}$ we now form the join y_∞ of all the elements y_n, and the join z_∞ of all the elements z_n. Obviously

$$u = x + y_\infty = x + z_\infty,\ y_\infty + z_\infty = y + z\ .$$

Furthermore, $x \not\leqq y_n$ for all n and, therefore, $x \not\leqq y_\infty$. Thus, $xy_\infty < y_\infty$, and since $xy_n \prec y_n$ for all n we have in fact that $xy_\infty \prec y_\infty$; similarly $xz_\infty \prec z_\infty$. Finally, from the fact that $xy_n + xz_n \leqq y_{n+1}z_{n+1} \leqq x$ for all n we infer that $xy_\infty = xz_\infty = y_\infty z_\infty$.

Dropping the subscripts in order to simplify the notation, we now have four elements u, x, y, and z in $L^{\sigma\pi\sigma}$ that satisfy (1) and (2) and, in addition, $xy = xz = yz$. Letting $v = x(y + z)$, we consider four cases depending on whether or not the equations $y + z = y + v$ and $y + z = z + v$ hold. If both equations fail, then the elements y, z, $y + v$, and $z + v$ generate a homomorphic image of L_1 (Fig. 6). We may assume that this is a proper homomorphism, so that $v = yz$; then x, y, and z generate a lattice isomorphic to L_4. If just one equation holds, say, $y + z = y + v \succ z + v$, then y, z, and v generate a lattice isomorphic to L_4. Finally, if both equations hold, then y, z, and v generate a diamond.

This completes the proof of Lemma 2.1, and therefore also the proof of Theorem 1.2.

The remainder of this section is concerned with the behavior of congruence relations in a semidistributive lattice. We first dispense with the necessary preliminaries.

Given two quotients p/q and r/s in a lattice L if $r = p + s$ and

$s \geqq q$ then we say that *p/q transposes weakly up* onto r/s and that r/s is a *weak upper transpose* of p/q, —in symbols $p/q \nearrow_w r/s$, —and we refer to the map $t \to t + s$ ($t \in p/q$) as a *weak upper transposition.* Dually, if $qr = s$ and $r \leqq p$ then we say that *p/q transposes weakly down* onto r/s and that r/s is a *weak lower transpose* of p/q, —in symbols $p/q \searrow_w r/s$, —and we refer to the map $t \to tr$ ($t \in p/s$) as a *weak lower transposition* of p/q onto r/s. If there exists a sequence of quotients $x_0/y_0, x_1/y_1, \cdots, x_n/y_n$ with $x_0/y_0 = p/q$ and $x_n/y_n = r/s$ such that, for each $i < n$, x_i/y_i transposes weakly up or down onto x_{i+1}/y_{i+1}, then we say that *p/q projects weakly onto r/s*, and we refer to the composition of the weak transpositions of x_i/y_i onto x_{i+1}/y_{i+1} for $i = 0, 1, \cdots, n-1$ as a *weak projectivity* of p/q onto r/s.

If both $p/q \nearrow_w r/s$ and $r/s \searrow_w p/q$, that is, if $p + s = r$ and $ps = q$, then we say that *p/q transposes up* onto r/s and that *r/s transposes down* onto p/q, —in symbols $p/q \nearrow r/s$ and $r/s \searrow p/q$, —and we say that r/s is an *upper transpose* of p/q and p/q is a *lower transpose* of r/s. In this case the maps $t \to t + s$ ($t \in p/q$) and $t \to tp$ ($t \in r/s$) are referred to as an *upper transposition* of p/q onto r/s and a *lower transposition* of r/s onto p/q, respectively. If there exists a sequence of quotients $x_0/y_0, x_1/y_1, \cdots, x_n/y_n$ with $x_0/y_0 = p/q$ and $x_n/y_n = r/s$ such that, for each $i < n$, x_i/y_i transposes up or down onto x_{i+1}/y_{i+1}, then p/q is said to *project* onto r/s, and the composition of the transportations of x_i/y_i onto x_{i+1}/y_{i+1} for $i < n$ is called a *projectivity* of p/q onto r/s.

Our next lemma concerns the possibility of shortening a sequence of weak transpositions. Let us suppose that p/q projects weakly onto r/s in n steps, say

$$p/q = x_0/y_0 \nearrow_w x_1/y_1 \searrow_w x_2/y_2 \nearrow_w \cdots x_n/y_n = r/s .$$

Let $n > 2$. If there exists a quotient u/v such that

$$x_0/y_0 \searrow_w u/v \nearrow_w x_2/y_2 ,$$

then we can shorten the sequence of weak transpositions by replacing the two quotients x_1/y_1 and x_2/y_2 by the single quotient u/v. In a distributive lattice this can always be done, and the non-existence of such a quotient u/v is therefore connected with the presence of a diamond or a pentagon as a sublattice of L. If L is semidistributive, then this sublattice must of course be a pentagon. The aim of the lemma is to describe the location of the pentagon relative to the quotients x_i/y_i.

LEMMA 2.2. *Let L be a semidistributive lattice, and let x_0/y_0, x_1/y_1, and x_2/y_2 be quotients in L such that $x_0/y_0 \nearrow_w x_1/y_1 \searrow_w x_2/y_2$. Then either there exists a subquotient p/q of x_0/y_0 such that, for*

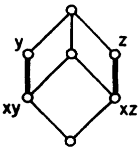


Fig. 2

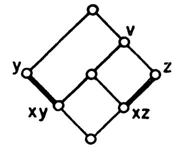


Fig. 3

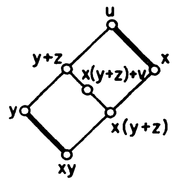
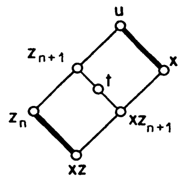


Fig. 4



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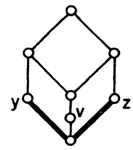
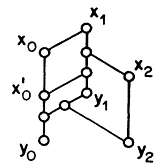


Fig. 6



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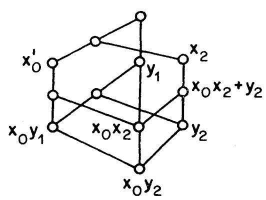

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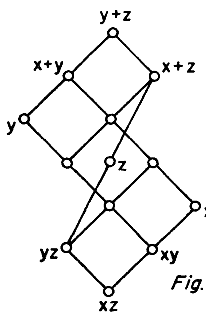


Fig. 9

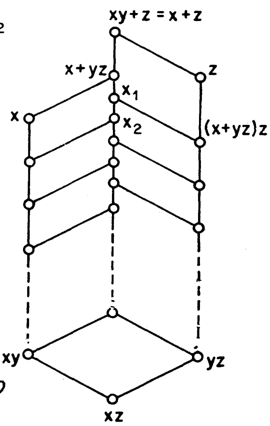


Fig. 10

some quotient u/v, $p/q \searrow u/v \nearrow x_2/y_2$, or else there exist $a, b, c \in L$ with $N(a, b, c)$ such that either b/bc is a subquotient of x_0/y_0, or else $(a+b)/b$ transposes down onto a subquotient of x_0/y_0.

Proof. Let $x'_0 = x_0(y_1 + x_2)$. If $x'_0 + y_1 < x_2 + y_1$, then the elements $a = x'_0 + y_1$, $b = x_0$ and $c = y_1 + x_2$ satisfy $N(a, b, c)$, and $b/bc = x_0/x'_0$ is a subquotient of x_0/y_0 (Fig. 7).

Let $x'_0 + y_1 = x_2 + y_1$. By the semidistributivity of L, $x_2 + y_1 = x'_0x_2 + y_1 = x_0x_2 + y_1$. If $x_0x_2 + y_2 < x_2$, then the elements $a = x_0x_2 + y_2$, $b = y_1$ and $c = x_2$ satisfy $N(a, b, c)$, and $(a + b)/b$ transposes down onto the subquotient x'_0/x_0y_1 of x_0/y_0 (Fig. 8).

Finally, if $x_0x_2 + y_2 = x_2$, then the subquotient $(x_0y_1 + x_0x_2)/x_0y_1$ of x_0y_0 transposes down onto the quotient x_0x_2/x_0y_2, which transposes up onto x_2/y_2.

3. Projectivities between N-quotients. Consider a variety V that contains none of the lattices $M_3, L_1, L_2, \cdots, L_{12}$ and a lattice $L \in V$. Our aim in this section is to show that projectivities between N-quotients in L behave like projectivities between quotients in a distributive lattice.

To this end we require a preliminary result concerning lattices determined by defining relations (relative to the variety of all lattices). The result is most easily formulated by means of a diagram; indeed, the proof itself becomes quite transparent when presented pictorially.

LEMMA 3.1. *Let L be a semidistributive lattice generated by three elements x, y, and z, with $x \leqq xy + z$ and $xz \leqq y$. If L does not have a sublattice isomorphic to L_7, L_8 or L_{12}, then L is a homomorphic image of the lattice in Fig. 9.*

Proof. It is easy to check that Fig. 9 represents the lattice with the defining relations $x \leqq xy+z$, $xz \leqq y$, $(x+y)z=yz$, $(x+yz)y= xy + yz$, and $x + y(x + z) = (x + y)(x + z)$. It therefore suffices to show that under the hypotheses of the lemma the last three of these relations hold.

The lattice determined by x, z, xy, and yz and the defining relations $x \leqq xy + z$ and $xz \leqq y$ (relative to the variety of all lattices) is pictured in Fig. 10. In order to avoid L_{12} we must have $x_1 = x_2$, where $x_1 = xy + (x + yz)z$ and $x_2 = yz + xx_1$. Since $(xy + yz) + x_2x= x_2 = x_1 = (xy + yz) + (x+yz)z$, semidistributivity yields $x_2=(xy+yz)+ x_2x(x + yz)z = xy + yz$. As $z(xy + yz) = yz$, we conclude that $(x + yz)z = (x + yz)zx_1 = (x + yz)z(xy + yz) = yz$. Hence, by the semidistributivity of L, $(x + y)z = yz$. Next, check that the elements x, z, xy, yz and $(x + yz)y$ generate a homomorphic image of the lattice in Fig. 11. To avoid L_8 we must therefore have $(x + yz)y = xy + yz$. Finally, observe that the elements y, z, $x + y$, $x + z$ and $x + y(x + z)$ generate a homomorphic image of the lattice in Fig. 12. To avoid L_7 we must therefore have $x + y(x + z)= (x + y)(x + z)$.

For the remainer of this section let L be a lattice in a variety that contains none of the lattices $M_3, L_1, L_2, \cdots, L_{12}$.

LEMMA 3.2. *If $a, b, c, a', c' \in L$, $N(a, b, c)$, and $c/a \nearrow c'/a'$, then $N(a', b, c')$ and, for all $t \in c/a$ and $t' \in c'/a'$, $(t + a')c = t$ and $t'c + a' = t'$.*

Proof. We have $ca' = a$ and $c + a' = c'$. Taking $x = c$, $y = a'$,

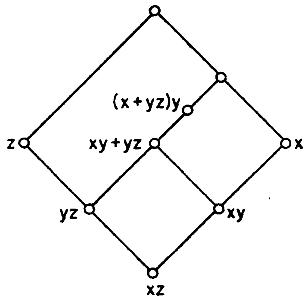


Fig. 11

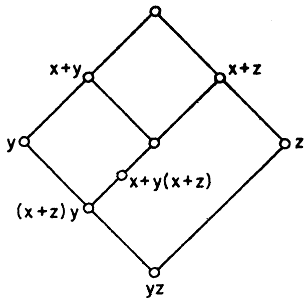


Fig. 12

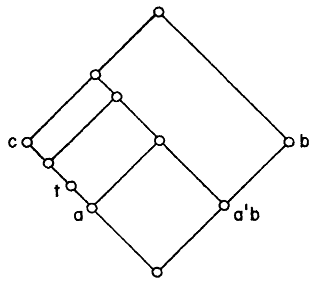


Fig. 13

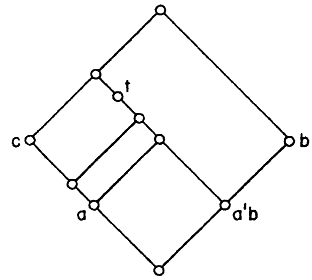
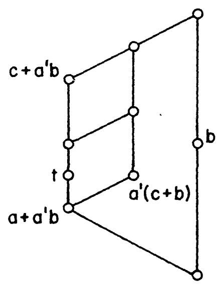


Fig. 14



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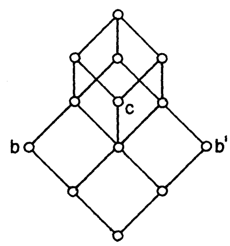

Fig. 16

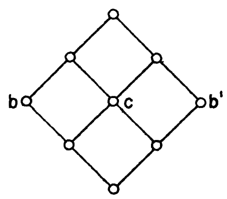

Fig. 17

and $z = b$ in Lemma 3.1, we see that $N(a', b, c')$. For $t \in c/a$ we must have $(t + a'b)c = t$, for otherwise the lattice generated by a', b, c, and t has a sublattice isomorphic to L_{10} (Fig. 13). Similarly, for $t' \in (c + a'b)/(a + a'b)$ we must have $ct' + a'b = t'$ to avoid L_8 (Fig. 14). Dually, for $t' \in c'/a'$ and $t \in c'(c + b)/a'(c + b)$ we must have $t'(c + b) + a' = t'$ and $(t + a')(c + b) = t$. Finally, for $t \in (c + a'b)/(a + a'b)$ and $t' \in c'(c + b)/a'(c + b)$ we must have $(t + a'(c + b))(c + a'b) = t$ and $t'(c + a'b) + a'(c + b) = t'$ in order to avoid L_6 (Fig. 15).

We conclude that the transpositions $t \to t + a'$ and $t' \to tc$ are isomorphisms between the quotients c/a and c'/a', and are inverses of each other, as was to be shown.

COROLLARY 3.3. *If the N-quotient c/a in L projects onto a quotient u/v then u/v is an N-quotient, and the projectivity is an isomorphism.*

COROLLARY 3.4. *If an N-quotient c/a in L projects weakly onto a quotient u/v, then a subquotient of c/a projects onto u/v.*

LEMMA 3.5. *If c_i/a_i, $i = 0, 1, 2$, are N-quotients in L with $c_0/a_0 \nearrow c_1/a_1 \searrow c_2/a_2$ then $c_0/a_0 \searrow c_0c_2/a_0a_2 \nearrow c_2/a_2$.*

Proof. We have $c_1 = c_0 + a_1 = a_1 + c_2$, hence by the semidistributivity of L, $c_1 = a_1 + c_0c_2$. It follows by Lemma 3.2 that

$$a_0 + c_0c_2 = (a_0 + c_0c_2 + a_1)c_0 = c_1c_0 = c_0$$

and, similarly, $a_2 + c_0c_2 = c_2$. Also, $a_0(c_0c_2) = c_0a_1c_2 = a_0a_2$ and $a_2(c_0c_2) = a_0a_2$.

COROLLARY 3.6. *If the N-quotient c/a in L projects onto a quotient u/v, then $c/a \nearrow x/y \searrow u/v$ for some quotient x/y.*

Proof. Apply Corollary 3.3 and the dual of Lemma 3.5.

COROLLARY 3.7. *If c/a is an N-quotient in L, then* con(a, c) *does not collapse any nontrivial quotient u/v with $u \leqq a$ or $c \leqq v$.*

4. **Critial edges.** Let V be a variety that contains none of the lattices $M_3, L_1, L_2, \cdots, L_{12}$ and let $L \in V$ be a subdirectly irreducible, non-distributive lattice. Our aim in this section is to show that L has a unique critical edge c/a and that c/a is also the only N-quotient of L. It follows that $L/\text{con}(a, c)$ is distributive and that L is locally finite.

LEMMA 4.1. *If c/a is a critical edge of L, then c covers a, and c/a is an N-quotient.*

Proof. Since L is non distributive and semidistributive, it has an N-quotient u/v. Since con(u, v) identifies a and c, there exist elements $x_0, x_1, \cdots, x_n \in L$ with $c = x_0 > x_1 > \cdots > x_n = a$ such that u/v projects weakly onto each of the quotients x_i/x_{i+1}. By Corollaries 3.3 and 3.4, all the quotients x_i/x_{i+1} are N-quotients, and, of course, they are all critical. Hence, all the congruence relations con(x_i, x_{i+1}) are equal, and by Corollary 3.7 this implies that $n = 1$. Thus, c/a is an N-quotient. To show that c covers a we again appeal to Corollary 3.7.

LEMMA 4.2. *All the N-quotients in L are critical edges of L, and they are all projective to each other.*

Proof. Choose a critical edge c/a of L. By the preceding lemma, $a \prec c$, and c/a is an N-quotient. By Corollary 3.4, every N-quotient u/v has a subquotient u'/v' that is projective to c/a, and is therefore a critical edge of L. Furthermore, u'/v' cannot be a proper subquotient of u/v, for if, say, $u' < u$, then con(u, u') collapses u'/v', contrary to Corollary 3.7. Thus $u/v = u'/v'$ is a critical edge of L projective to c/a.

LEMMA 4.3. *Let θ be the smallest non-trivial congruence relation on L. Then L/θ is distributive and, for all $u, v \in L$ with $u > v$, θ identifies u and v if and only if u/v is an N-quotient.*

Proof. By the preceding lemma, θ collapses all the N-quotients of L, whence it follows that L/θ cannot contain a pentagon. Since L/θ belongs to V, it does not contain a diamond either, and it must therefore be distributive. The second part of the lemma follows from the fact that, by Lemmas 4.1 and 4.2, the N-quotients in L are precisely the critical edges.

The next step is to prove that con(a, c) idenitfies no two distinct elements other than a and c.

LEMMA 4.4. *If c/a is a critical edge of L, then a is meet irreducible and c is join irreducible.*

Proof. By Lemma 4.1, $a \prec c$ and c/a is an N-quotient. Let us assume that a is meet reducible; that is, $a = cd$ for some $d > a$. Then con(a, d) identifies a and c, and, hence, there exist quotients x_i/y_i, $i = 0, 1, \cdots, n$, with $x_0/y_0 = d/a$, $y_n = a$ and $x_n \geqq c$, such that, for $i < n$, x_i/y_i transposes weakly up or down onto x_{i+1}/y_{i+1}. We assume that n has been chosen as small as possible. Clearly, $n \geqq 2$.

The first two weak transpositions go one up and the other down, and the order cannot be reversed by replacing x_1/y_1 by another quotient. This is obvious when $n > 2$, for if the order could be changed, then the sequence of quotients could be shortened by replacing x_1/y_1 and x_2/y_2 by a single quotient. Regarding the case $n = 2$, we need only observe that we cannot have $d/a \searrow_w u/v \nearrow_w s/a$ with $s \geqq c$, for then $c \leqq u + a \leqq d$.

First, let us suppose that $d/a \nearrow_w x_1/y_1 \searrow_w x_2/y_2$. By Lemma 2.2, there exist $a', b, c' \in L$ with $N(a', b, c')$ such that either b/bc' is a subquotient of d/a, or else $(a' + b)/b$ transposes down onto a subquotient of d/a. In either case, $a \leqq b$. By Lemma 4.2 c/a and c'/a' are projective, whence it follows by Lemma 3.2 that $N(a, b, c)$. However, this is impossible since $a \leqq b$.

Next, let $d/a \searrow_w x_1/y_1 \nearrow_w x_2/y_2$. By the dual of Lemma 2.2 there exist $a', b, c' \in L$ with $N(a', b, c')$ such that either $(a' + b)/b$ is a subquotient of d/a, or else b/bc' transposes up onto a subquotient of d/a. As before, $N(a, b, c)$, thus $a \not\leqq b$, and $(a' + b)/b$ cannot be a subquotient of d/a. Also, b/bc' cannot transpose up onto a subquotient of d/a, for this would imply that $a + b \leqq d$; hence, $c \leqq d$.

COROLLARY 4.5. *L has only one critical edge c/a, and* $\operatorname{con}(a, c)$ *identifies no two distinct elements of L other than a and c.*

Proof. By Lemmas 4.1 and 4.2, all the critical edges of L are projective to each other, but by Lemma 4.4, a critical edge cannot be projective to any quotient distinct from itself. Hence, L has only one critical edge. The second statement of the lemma follows by Lemma 4.3.

COROLLARY 4.6. *L is locally finite.*

Proof. If $\phi(n)$ is the order of a free distributive lattice with n generators, then an n-generated sublattice of L can have at most $\phi(n) + 1$ elements.

5. Proof of Theorem 1.1. Let V be a variety that contains none of the lattices $M_3, L_1, L_2, \cdots, L_{15}$ and let $L \in V$ be a subdirectly irreducible, non-distributive lattice. Since any variety is determined by its finitely generated subdirectly irreducible members we may take L to be finitely generated; whence, by Corollary 4.6, L is, in fact, finite. Let c/a be the unique critical edge of L. To complete the proof of Theorem 1.1 it would suffice to show that L must be a pentagon. This is the objective of this section.

LEMMA 5.1. *There exists $b \in L$ such that $N(a, b, c), bc \prec a$ and $c \prec a + b$.*

Proof. Choose $b \in L$ with $N(a, b, c)$ so that the quotient $(a + b)/bc$ is minimal. Given $c < t \leqq a + b$, we cannot have $bt = bc$, for then t/c would be an N-quotient, contrary to the fact that c/a is the only N-quotient in L. Letting $b' = bt$, we therefore have $a < a + b'$, and hence, $c \leqq a + b'$ by the meet irreducibility of a. Thus $N(a, b', c)$, and in view of the choice of b this yields $a + b' = a + b$; hence, $t = a + b$. Thus, $c \prec a + b$ and, by duality, $bc \prec b$.

LEMMA 5.2. *The elements a and c are doubly irreducible.*

Proof. By the preceding lemma we can choose $b \in L$ with

$N(a, b, c)$, $bc \prec a$ and $c \prec a + b$. According to Lemma 4.4, a is meet irreducible and c is join irreducible, so that by duality it suffices to show that c is meet irreducible. If this is not the case, then there exists $d \in L$ with $c = (a + b)d$ and $c < d$. As in the proof of Lemma 4.4, we see that there exists a quotient u/v such that one and only one of the following two statements holds:

(i) d/c transpose weakly up onto a quotient that transposes weakly down onto u/v;

(ii) d/c transposes weakly down onto a quotient that transposes weakly up onto u/v.

We shall show that either case leads to a contradiction.

Case (i). By Lemma 2.2 and the fact that c/a is the only N-quotient in L, there exists $b' \in L$ with $N(a, b', c)$ such that either $b'/b'c$ is a subquotient of d/c, or else $(a + b')/b'$ transposes weakly down onto a subquotient of d/c. Regardless of which alternative applies, we have $c \leqq b'$, contrary to the fact that $N(a, b', c)$.

Case (ii). Using the dual of Lemma 2.2, we obtain $b' \in L$ with $N(a, b', c)$ such that either $(a + b')/b'$ is a subquotient of d/c or else $b'/b'c$ transposes up onto a subquotient of d/c. The former case is excluded by the fact that $c \nleqq b'$. In the latter case $b' \leqq d$, and hence, $(a + b)(a + b') = c$. The elements a, b, and b' generate a sublattice K of L with the property that the congruence relation $\theta = \mathrm{con}(a, c)$ identifies no two distinct elements of K except a and c, and that K/θ is distributive. Since θ identifies the elements $(a + b)(a + b') = c$ and a, K/θ is a homomorphic image of the lattice in Fig. 16.

Let $a > ab' + ab$. Then θ does not identify c and $ab' + ab$. Also, θ does not identify c with either $a + b$ or $a + b'$. Consequently, $a + b, a + b'$, and $b + b'$ generate, in this case, an eight element Boolean algebra. Then $a + b$, $a + b'$, $b + b'$, and a generate a lattice isomorphic to L_{13}.

Thus, we must have $a = ab + ab'$, and K/θ must be a homomorphic image of the lattice in Fig. 17. Actually, this homomorphism must be an isomorphism, since no two of the elements b, b', and c are comparable modulo θ. However, this implies that K is isomorphic to L_{15}, so this too leads to a contradiction.

LEMMA 5.3. *L is a pentagon.*

Proof. By Lemma 5.1 we can choose $b \in L$ so that $N(a, b, c)$, $bc \prec a$ and $c \prec a + b$. Let $u = a + b$ and $v = bc$.

We claim that $u(s+t) = us + ut$ for all $s, t \in L$. By Lemma 4.3 and Corollary 4.5, this holds modulo con(a, c), and the only way the equation can fail is if $u(s+t) = c$ and $us + ut = a$. Since c is doubly irreducible $c = s + t$; hence, $s = c$ or $t = c$, so that $us + ut = c > a$.

Defining $s\phi t$ by $us = ut$, we infer that ϕ is a congruence relation on L. Since ϕ does not identify a and c, ϕ must be trivial. From this we infer that $t \leqq u$ for all $t \in L$, since ϕ always identifies $u + t$ with u. Similarly, $t \geqq v$ for all $t \in L$.

No element other than a, c, u, and v is comparable with either a or c, for if $t \leqq a$, then $t = a$ or $t = v$, while if $a < t$, then $c \leqq t$ by the meet irreducibility of a, and therefore, $t = c$ or $t = u$. If t is not comparable with a or c, then $a + t = u$ and $ct = v$, so that $N(a, t, c)$. From this, we infer that $v \prec b \prec u$, for if $b < t < u$, say, then $N(b, c, t)$, contrary to the fact that c/a is the only N-quotient of L. Thus if $t \in L$ is distinct from a, b, c, u, and v, then $b + t = c + t = u$ and $bt = ct = v$, so that b, c, and t generate a diamond.

6. Related results.

While semidistributivity as applied to varieties of lattices, rather than individual lattices, is not equivalent to a conjunction of identities the next result shows that semidistributivity is equivalent to the disjunction of countably many identities.

THEOREM 6.1. *Let* $y_0 = y, z_0 = z$ *and, for* $n = 0, 1, 2, \cdots$, *let* $y_{n+1} = y + xz_n, z_{n+1} = z + xy_n$. *Then a variety* V *is semidistributive if and only if, for some* $n = 0, 1, 2, \cdots, x(y+z) = xy_n = xz_n$ *and its dual hold in* V.

Proof. If $L \in V$ is not semidistributive then there are elements x, y, z in L such that $xy = xz < x(y+z)$ say. Then, for all $n = 0, 1, 2, \cdots, y_n = y$ and $z_n = z$ whence $xy_n = xz_n < x(y+z)$.

Conversely, let us suppose that V is a semidistributive variety. It suffices to show that, for some n, $x(y+z) = xy_n = xz_n$ in the free lattice $F_V(3)$ of V generated by x, y, and z. In $F_V(3)^\sigma$ let y_∞ be the join of the elements y_n and let z_∞ be the join of the elements z_n. Then $xy_n \leqq xz_{n+1}$ and $xz_n \leqq xy_{n+1}$ so that $xy_\infty = xz_\infty$. Now, $y_n + z_n = y + z$, and semidistributivity implies that $x(y+z) = xy_\infty = xz_\infty$. It follows that, for some n, $x(y+z) = xy_n = xz_n$.

The proof of Theorem 6.1 yields the next result.

COROLLARY 6.2. *A variety* V *of lattice is semidistributive if* *and only if the lattices $F_V(3)^\sigma$ and $F_V(3)^\pi$ are semidistributive.*

As we mentioned at the outset the problem of finding a set of identities which describes a given variety is usually quite difficult. This task was accomplished by R. McKenzie [8] in the case of the smallest non-modular variety $\boldsymbol{N}$. Once these identities are exhibited, however, the matter of verifying that they describe precisely $\boldsymbol{N}$ is, in view of Theorem 1.1, a simple computation.

THEOREM 6.3. *$\boldsymbol{N}$ is precisely the class of all lattices satisfying the two identities.*

$$x(y+z)(y+w) \leqq x(y+zw)+xz+xw$$

and

$$x(y+z(x+w)) = x(y+xz)+x(xy+zw)\ .$$

A lattice L is said to satisfy (W) if, for all $x, y, u, v \in L$, $xy \leqq u+v$ implies that either $xy \leqq u$ or $xy \leqq v$ or $x \leqq u+v$ or $y \leqq u+v$. It is easy to verify that each of the lattices $M_3, L_1, L_2, \cdots, L_{15}$ satisfies (W). According to a result of B.A. Davey and B. Sands [3], every finite lattice satisfying (W) is a retract of any finite lattice of which it is a homomorphic image. On the other hand, each subdirectly irreducible member of a variety $\boldsymbol{L}$ generated by a finite lattice L is a homomorphic image of a sublattice of L [5]. Combining these observations with Theorem 1.1 yields our final result.

THEOREM 6.4. *Let L be a finite non-modular lattice. If L is not a member of the smallest non-modular variety then L contains a sublattice isomorphic to one of $M_3, L_1, L_2, \cdots, L_{15}$.*

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