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A REMARK ON GENERALIZED HAAR SYSTEMS IN L_p, $1 < p < \infty$

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We show that any chain from a generalized Haar system in L_p is equivalent to the unit vector basis in l_p. The constant of the equivalence depends only on p.

We answer a question raised in [1]. Specifically, we prove

THEOREM. *Let $1 < p < \infty$. There exists a constant K, depending only on p, such that whenever (h_n) is a chain from a generalized Haar system in L_p, (h_n) is K-equivalent to the unit vector basis in l_p.*

Our notation and terminology is standard. If A is a subset of a Banach space, $[A]$ denotes the closed linear span of A. The unit vector basis in l_p is denoted by (e_n), and μ denotes Lebesgue measure on $(0, 1)$.

A generalized Haar system [1] in L_p is a sequence (h_n) defined as follows. Let $\{A_{n,i}: n = 0, 1, \cdots; 0 \leqq i < 2^n\}$ satisfy $A_{0,0} = (0, 1)$; $A_{n+1,2i} \cup A_{n+1,2i+1} = A_{n,i}$; and $A_{n+1,2i} \cap A_{n+1,2i+1} = \phi$. Let

$$H_{n,i} = \frac{1}{\mu(A_{n+1,2i})}\chi_{A_{n+1,2i}} - \frac{1}{\mu(A_{n+1,2i+1})}\chi_{A_{n+1,2i+1}},$$

and define $h_0 \equiv 1$, $h_{2^n+i} = H_{n,i}/\|H_{n,i}\|$.

A chain from (h_n) is a subsequence (h'_n) such that supp $h'_{n+1} \subset$ supp h'_n.

In [1] it is proved that a generalized Haar system is a monotone, unconditional basic sequence in L_p, with unconditional constant λ depending only on p.

The proof of the theorem is based on the following lemma (see [2] and [3]).

LEMMA. *Let $1 \leqq \lambda < \infty$, $\delta > 0$, $1 \leqq p \leqq 2$, and (x_n) be a normalized unconditional basic sequence in L_p with unconditional constant $\leqq \lambda$. Then,*

(a) $\|\sum a_n x_n\| \leqq \lambda(\sum |a_n|^p)^{1/p}$, *and*

(b) *If there exist disjoint sets (B_n) with*

$$\|x_n \mid B_n\| \geqq \delta, \quad \textit{then} \quad \frac{\delta}{\lambda}(\sum |a_j|^p)^{1/p} \leqq \|\sum a_n x_n\|,$$

for any scalar sequence (a_n).

Proof of theorem. We shall denote the chain by (h_n), and let $A_{n,1} = \operatorname{supp} h_n^+$, $A_{n,2} = \operatorname{supp} h_n^-$. We will assume $\operatorname{supp} h_{n+1} \subset A_{n,1}$.

Any chain in L_2 is an orthonormal system, so a chain in L_2 is isometrically equivalent to the unit vectors in l_2.

We consider now the case $1 < p < 2$. Let $N_1 = \{n: \|h_n \mid A_{n,2}\| \geqq 2^{-1/p}\}$, $N_2 = \{n: \|h_n \mid A_{n,1}\| > 2^{-1/p}\}$, and consider first the chain $(h_n)_{n \in N_1}$. Setting $B_n = A_{n,2}$ and $\delta = 2^{-1/p}$, it follows from the lemma that for all sequences (a_j),

$$\frac{2^{-1/p}}{\lambda}\Big(\sum_{j \in N_1} |a_j|^p\Big)^{1/p} \leqq \Big\|\sum_{j \in N_1} a_i h_j\Big\| . \tag{1}$$

As for the chain $(h_n)_{n \in N_2}$, note that for each $n \in N_2$, we have $\mu(A_{n,2}) > \mu(A_{n,1})$. Thus, if j is the successor (in N_2) of n, $\mu(A_{n,1} - A_{j,1}) > (1/2)\mu(A_{n,1})$. Setting $B_n = A_{n,1} - A_{j,1}$ we have $\|h_n \mid B_n\| > 2^{-2/p}$, so that

$$\frac{2^{-2/p}}{\lambda}\Big(\sum_{j \in N_2} |a_j|^p\Big)^{1/p} \leqq \Big\|\sum_{j \in N_2} a_j h_j\Big\| . \tag{2}$$

Using (1), (2), part (a) of the lemma, and the unconditionality of (h_n) we have

$$\begin{aligned} \frac{2^{-2/p}}{\lambda^2}\Big(\sum |a_j|^p\Big)^{1/p} &\leqq \frac{2^{-2/p}}{\lambda^2}\Big(\sum_{j \in N_2} |a_j|^p\Big)^{1/p} + \frac{2^{-1/p}}{\lambda^2}\Big(\sum_{j \in N_1} |a_j|^p\Big)^{1/p} \\ &\leqq \frac{1}{\lambda}\Big\|\sum_{j \in N_2} a_j h_j\Big\| + \frac{1}{\lambda}\Big\|\sum_{j \in N_1} a_j h_j\Big\| \\ &\leqq 2\Big\|\sum a_j h_j\Big\| \leqq 2\lambda\Big(\sum |a_j|^p\Big)^{1/p} , \end{aligned}$$

as desired.

Now suppose (h_n) is a chain from L_p, $2 < p < \infty$. Then $[(h_n)]$ is isometric to l_p, as we may regard $h_n = c_n e_1 + b_n e_n - \sum_{j=n+1}^{\infty} b_j e_j$. The biorthogonal sequence (h_n^*) is a chain from a generalized Haar system in L_q, with $1/q + 1/p = 1$. Since $1 < q < 2$, (h_n^*) is equivalent to e_n^*. Letting $T: l_q \to l_q$ be the isomorphism realizing this equivalence, we have that $T^* e_n = h_n$ and T^* is an isomorphism. Hence (h_n) is equivalent to (e_n).

References

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