NONOPENNESS OF THE SET OF
THOM-BOARDMAN MAPS

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1. **Introduction.** In this paper, we show that the set of all C^∞ Thom-Boardman maps from an n-dimensional manifold is not open iff corank two singularities occur generically. The latter is known to occur iff either $n \leq p$ and $2p \leq 3n - 4$ or $n > p$ and $2p \geq n + 4$. In the course of the proof, we establish a variation of Mather's Multitransversality Theorem: we show that jets have extensions which are multitransverse to given submanifolds of the jet bundle except possibly at the original jet. As an application of this extension theorem, we show that, in Mather's "nice range of dimensions," each jet z has a representative f $(z = j^k f(x))$ such that f is infinitesimally stable on a deleted neighborhood of x.

First we recall some properties of Thom-Boardman singularities; for more details, the reader is referred to [1], [11] and [3]. In this paper N (respectively P) will always be an n (respectively p) dimensional manifold without boundary. There is a finite partition of the jet bundle $J^k(N, P)$ into embedded submanifolds S^I, called *Thom-Boardman Singularities*, each I a nonincreasing sequence of nonnegative integers. Consider f in $C(N, P)$, the set of smooth maps from N to P. Let $S^I(f)$ denote $(j^k f)^{-1} S^I$, $j^k f$ the jet extension of f. Then $j^k f \pitchfork S^i$ implies that $S^i(f)$ is the set of points at which dim ker $Tf = i$; $j^k f \pitchfork S^{i,j}$ implies $j^k f \pitchfork S^i$ and $S^{i,j}(f) = S^j(f \mid S^i(f))$, etc. We call f a *Thom-Boardman map* if $j^k f \pitchfork S^I$ for all I, for all k. By Thom's Transversality Theorem, the set of Thom-Boardman maps is residual (i.e., is a countable intersection of open, dense sets), hence is dense, in the Whitney C^∞ topology.

THEOREM 1.1. *The Thom-Boardman maps form an open subset of $C(N, P)$ iff either $n \leq p$ and $2p > 3n - 4$ or $n > p$ and $2p < n + 4$.*

Let $r = \max(n - p, 0)$, let S_i denote S^{i+r} (jets of corank i), let $S_{i,j}$ denote $S^{i+r,j}$, etc. The condition on n and p in the theorem is precisely the condition that the codimension (abbreviated cod) of S_2 be greater than n, hence that maps cannot take on S_2 singularities transversally.

If cod S_2 is greater than n, then a map is Thom-Boardman iff it is transverse to all Morin singularities (which are the $S_{1;k}$, where $1; k$ means $1, \cdots, 1, 0, 1$ occurring k times). In this case, a map is

Thom-Boardman iff its germ at each point is stable. It follows that the set of Thom-Boardman maps is open in these dimensions. It is helpful in understanding the proof of our theorem to consider a more general method for demonstrating the openness of a set of maps defined by transversality conditions.

A *stratification* of a closed set X in a manifold M is a locally finite partition of X into embedded submanifolds, called strata, satisfying the "condition of the frontier": if S and T are strata and $S \cap \text{Cl}(T) \neq \varnothing$ (Cl denotes closure), then $S \subset \text{Cl}(T)$. Whitney's condition A requires that, for $x \in S$ and $x_n \in T$, if x_n converges to x and $T_{x_n}(= \text{tangent space to } T \text{ at } x_n)$ converges to some L in the Grassman bundle, then $S_x \subset L$.

The following proposition is folklore:

PROPOSITION 1.2. *Suppose X is a closed subset of $J^k(N, P)$ and $\mathscr{S}$ is a stratification of X satisfying Whitney's condition A. Then $\{f: j^k f \pitchfork S \text{ for all } S \in \mathscr{S}\}$ is open in $C(N, P)$ with the Whitney C^∞ topology.*

Using the method for constructing Whitney stratifications given in [13] together with the fact that the Morin singularities $S_{1;k}$ are orbits of a smooth group action on $J^k(N, P)$ (the group being $\mathscr{K}$; see [7] and [8]), one can show that there is a stratification of $J^k(N, P)$ satisfying condition A with the Morin singularities among the strata. In the case $\text{cod } S_2 > n$, the Thom-Boardman maps are precisely those which are transversal to all those strata, hence form an open set by Proposition 1.2.

However, we will give examples in §2 showing that $S_{2,0} \cap \text{Cl}(S_{1;k}) \neq \varnothing$ for all k. Since $\text{cod } S_{1;k} > \text{cod } S_2$ if k is sufficiently large, condition A cannot hold between $S_{1;k}$ and S_2. (In fact, it is shown in [5] that condition A already fails between $S_{1,1}$ and $S_{2,0}$.) Our examples will also show that the condition of the frontier fails.

Let ${}_tS_I$ denote those z in S_I such that, for all f with $j^k f(x) = z$, $j^k f \pitchfork S_I$ at z. These jets are called the *transversal elements* of S_I. Proposition 2.1 implies that ${}_tS_{2,0} \cap \text{Cl}(S_{1;k}) \neq \varnothing$ for all k if $\text{cod } S_2 \leq n$. Consider a z in this intersection. By the Transversal Extension Theorem (3.1) there is a Thom-Boardman map f such that $j^k f(x) = z$ for some x. We pick k large enough that $\text{cod } S_{1;k} > n$. There are $z_i \in S_{1;k}$ with z_i converging to z in $J^k(N, P)$. Lemma 3.5 demonstrates that there are $f_i \in C(N, P)$ such that $j^k f_i(x_i) = z_i$ and $f_i \to f$ in the Whitney C^∞ topology; these f_i can not be Thom-Boardman, hence Theorem 1.1 will be proven.

2. **Contact classes in $S_{2,0}$.** First we describe Mather's algorithm

for determining in which S^I a jet z lies (see [11]). Let A_n be the formal power series algebra in n variables over R. Let m_n denote the unique maximal ideal of A_n (i.e., all formal power series with zero constant term). If f is a map germ at 0 from R^n to R^p, let $I(f)$ denote the ideal in A_n generated by the Taylor series at 0 of the component functions of f. The rank of an ideal I is by definition the dimension of $(m_n^2 + I)/m_n^2$ as a real vector space, and crk $I = n - \text{rank } I$. Suppose rank $I = r$; pick f such that $I = I(f)$. Let ∂I be the ideal generated by I and the $r + 1$ by $r + 1$ subdeterminants of the Jacobian matrix of f; ∂I depends only on I, not on f. The Boardman symbol of I, or of f if $I = I(f)$, is $i_1, i_2, \cdots$ where $i_1 = $ crk I, $i_2 = $ crk ∂I, $i_3 = $ crk $\partial(\partial I)$, etc. Mather shows that, if $z = j^k f(x)$, then z is in S^I, where I consists of the first k terms of the Boardman symbol of any coordinate representation of f near x.

The algebra of a k-jet z is $A_n/(I(g) + m_n^{k+1})$ where g is a coordinate representation of a map whose k-jet is z; this algebra is only determined up to algebra isomorphism. If f is a germ at x, let $V_k(f)$ denote the set of k-jets whose algebras are isomorphic to that of $j^k f(x)$; $V_k(f)$ is called the *contact class* of $j^k f(x)$, and is the orbit of the action of the group $\mathscr{K}$ (see [8]). From the previous paragraph we see that each S_I is a union of contact classes.

Next we show that $S_{2,0} \cap \text{Cl}(S_{1;k}) \neq \varnothing$ for all k. First suppose $n \leq p$. It is easy to show using Mather's algorithm for computing the Boardman sequence that the Morin singularity $S_{1,k}$ in J^l, $l > k$, is equal to $V_l(x^{k+1})$(compare with §3 of Chapter VII of [3]). Also one sees that $S_{2,0} \subset J^l$ contains $V_l(xy)$, $l > 1$. Let

$$F_u: R \times R \times R^{n-2} \longrightarrow R \times R \times R^{p-n} \times R^{n-2}$$

be defined by

$$F_u(x, y, z) = (xy + ux^{k+1}, uy, 0, z) ;$$

$j^l F_u(0) \in V_l(x^{k+1})$ if $u \neq 0$ and $j^l F_0(0) \in V_l(xy)$ for $l > k$. Thus $V_l(xy) \subset \text{Cl}(S_{1;k})$ for all $k < l$.

Now suppose $n > p$. Let

$$F_u: R \times R \times R^{n-p} \times R^{p-2} \longrightarrow R \times R \times R^{p-2}$$

be defined by

$$F_u(x, y, w, z) = \left(xy + ux^{k+1} + \sum_{i=1}^{n-p} w_i^2, uy, z \right) ;$$

then $j^l F_u(0) \in V_l(x^{k+1} + \sum_{i=1}^{n-p} w_i^2) \subset S_{1;k}$ if $u \neq 0$ and

$$j^l F_0(0) \in V_l\left(xy + \sum_{i=1}^{n-p} w_i^2 \right) \subset S_{2,0} \quad \text{for} \quad l > k .$$

Thus

$$V_l\left(xy + \sum_{i=1}^{n-p} w_i^2\right) \subset \mathrm{Cl}(S_{1;k}) \quad \text{for all} \quad k < l \ .$$

We wish to show that $_t S_{2,0} \cap \mathrm{Cl}(S_{1;k}) \neq \varnothing$, for all k, if $\operatorname{cod} S_{2,0} \leqq n$. In fact, this follows by applying the next proposition to the contact class $V_l(xy + \sum_{i=1}^{n-p} w_i^2)$.

PROPOSITION 2.1. *Suppose* $\operatorname{cod} S_{2,0}$ *is no greater than* n. *Then each contact class in* $S_{2,0} \subset J^l$ *contains an element of* $_t S_{2,0}$.

Proof. Suppose $j^l f(x) = z \in S_2 = S^{r+2}$. The intrinsic second derivative of f at x (see §3 and §4 of Chapter VI of [3] and §2 of [10]) is a bilinear map $D^2 f_x \colon N_x \times K \to C$, where $K = \ker Df_x$ and $C = \operatorname{cok} Df_x$. Local coordinates can be chosen centered about x and $f(x)$ so that f has the form $f(x_1, \cdots, x_n) = (x_1, \cdots, x_s, g)$, where $s = n - r - 2$ is the rank of f, and

$$(2.2) \qquad g(x_1, \cdots, x_n) = \sum_{i,j=1}^{s} a_{ij} x_i x_j + \sum_{j=1}^{s} \sum_{k=s+1}^{n} b_{kj} x_k x_j + \sum_{k,l=s+1}^{n} c_{kl} x_k x_l$$

where the coefficients are in $C(\boldsymbol{R}^n, \boldsymbol{R}^{p-S})$. If we evaluate the coefficients at 0, then (2.2) reduces to a vector-valued quadratic form. Viewed in these coordinates, $D^2 f_x$ is the last two terms of this quadratic form; $d^2 f_x = D^2 f_x | K \times K$ is the last term and $D^2 f_x | K^\perp \times K$ is the second term ($K^\perp$ is the set $x_{s+1} = \cdots = x_n = 0$).

$D^2 f_x$ can also be viewed as a map from N_x to $\operatorname{Hom}(K, C)$, and $\operatorname{Hom}(K, C)$ is canonically isomorphic to the normal space to S_2. Thus z is in $_t S_2$ iff $D^2 f_x \colon N_x \to \operatorname{Hom}(K, C)$ is surjective. Furthermore, z is in $S_{2,0}$ iff $d^2 f_x \colon K \to \operatorname{Hom}(K, C)$ is injective.

Assume z is in $S_{2,0}$. Then the image of $d^2 f_x \colon K \to \operatorname{Hom}(K, C)$ has dimension $n - s$. Since

$$\dim \operatorname{Hom}(K, C) = \operatorname{cod} S_2 \leqq n, \dim(\operatorname{Hom}(K, C)/d^2 f_x(K))$$

is less than or equal to $s = \dim K^\perp$. Since we can choose the coefficient functions in the second term of (2.2) arbitrarily without affecting the ideal, hence the contact class, of f, we can choose an f' contact equivalent to f so that $d^2 f'_x = d^2 f_x$ and $D^2 f'_x | K^\perp$ spans $\operatorname{Hom}(K, C) \bmod d^2 f'_x(K)$. Hence $D^2 f'_x$ is surjective; hence $j^l f'(x)$ is in $_t S_2$.

While it is not relevant to this paper, we would like to mention that Mather has proved the following (Theorem 6.1 of [9]): if U is a C^2 submanifold of $J^k(N, P)$ and $\operatorname{cod} U \leqq n$, then the set of transversal points (i.e., $_t U$) is dense in U.

3. Transversal extensions of jets. In this section we establish a variation of the Multijet Transversality Theorem (due to Mather [9]; see also [3]) and give some applications. Then we complete the proof of Theorem 1.1.

Suppose $X \subset N$ and $h: X \to J^k(N, P)$ is a section. Let $A = \{f \in C(N, P): j^k f \mid X = h\}$; A is a closed subset of $C(N, P)$ in the C^k, hence in the C^∞, topology. Furthermore, A is a Baire space (see Theorem 4.4b of Chapter 2 of [4]), i.e., every residual set (=countable intersection of open, dense sets) is dense.

Now suppose X is closed in N. Let $\pi: {}_sJ^k(N, P) \to N^{(s)} \times P^s$ denote the s-fold k-jet bundle and let $X^{(s)}$ be the set of s-tuples with at least one component in X; $X^{(s)}$ is closed in $N^{(s)}$. Suppose W is an immersed submanifold of $\pi_1^{-1}(N^{(s)} - X^{(s)})$, where π_1 is the composition of π with the projection of $N^{(s)} \times P^s$ on its first factor. Let $A^t = \{f \in A: {}_sj^k f \mid (N^{(s)} - X^{(s)}) \pitchfork W\}$.

PROPOSITION 3.1. (*Multijet Transversal Extension Theorem.*) A^t *is residual in* A.

Proof. W can be covered by a countable collection of compact, codimension 0 submanifolds M_i (with boundary); furthermore, the M_i may be chosen so that, for each i, there exist relatively compact, mutually disjoint coordinate patches $U_{i,1}, \cdots, U_{i,s}$ whose closures are contained in $N - X$ and $V_{i,1}, \cdots, V_{i,s}$ in P such that $\pi(M_i) \subset U_{i,1} \times \cdots \times U_{i,s} \times V_{i,1} \times \cdots \times V_{i,s}$.

Let $B_i = \{f \in C(N, P): {}_sj^k \pitchfork W \text{ on } M_i\}$ and let $A_i = B_i \cap A$; A_i is open in A since B_i is open (see II. 4.14 of [3]). Pick any neighborhood U of f. Exactly as in the proof of the Multijet Transversality Theorem ([9] or [3]), there is a g in $B_i \cap U$ which agrees with f outside $U_{i,1} \cup \cdots \cup U_{i,s}$; thus g is in $A_i \cap U$. Hence A_i is dense in A; $A^t = \cap A_i$ is residual.

COROLLARY 3.2. *If* $\{W_i\}$ *is a countable collection of immersed submanifolds of* $\pi_1^{-1}(N^{(s)} - X^{(s)})$, *then* $\{f \in A: j^k f \mid (N^{(s)} - X^{(s)}) \pitchfork W_i \text{ for all } i\}$ *is residual.*

In Mather's nice range of dimensions ([9] and [10]), ${}_sJ^k(N, P)$ is the disjoint union of finitely many contact classes C_i and finitely many other manifolds M_i of codimension greater than n. Thus, in these dimensions, a map is multitransverse to all contact classes iff it is multitransverse to the C_i's and M_i's.

Mather shows in [9] that if f is multitransverse to all contact classes and if $f \mid C(f)$ ($C(f)$ is the set of points in N at which the rank of f is less than p) is proper, then f is infinitesimally stable;

if in addition f is proper, then f is stable.

COROLLARY 3.3. *In the nice range of dimensions, for every k-jet z there is a map f defined on a neighborhood U of x in N such that $j^k f(x) = z$ and $f \,|\, (U - \{x\})$ is infinitesimally stable; if $n \leq p$, then $f \,|\, (U - \{x\})$ is stable.*

Proof. First, we can find an f on a U so that $j^k f(x) = z$ and f is finite (see Example 5.2 of Chapter VIII of [12] and §2 of Chapter VII of [3]). Since we are in the nice range of dimensions and finite maps form an open set, we can assume that f is multitransverse to all contact classes on $U - \{x\}$. Since f is finite, shrinking U if necessary, $f \,|\, (U - \{x\})$ will be proper on $C(f)$; if $n \leq p$, f will be proper on all of $U - \{x\}$.

Note that, by Lemma 4.4 of [9] and Theorem 1.13 of [7], if $j^p f$ is transverse to the contact class of $j^p f(x)$, then the germ of f at x is stable.

A germ f_x is finitely-determined if, for some l, every germ g_x having the same l-jet as f_x at x can be transformed into f_x by local coordinate changes. If f_x is finitely-determined, then for every representative f of f_x there is a neighborhood U of x such that $f \,|\, (U - \{x\})$ is multitransverse to all contact classes (see [2]); the converse is true if f is complex analytic. This suggests the conjecture that, in the nice range of dimensions, every jet has a finitely-determined representative. C. T. C. Wall has recently confirmed that this is true in a range of dimensions slightly larger than the nice range.

We return to the proof of Theorem 1.1.

COROLLARY 3.4. *If $\mathrm{cod}\, S_2 \leq n$, then for each $x \in N$ and all $k \geq 1$ there is a Thom-Boardman map f such that $j^{k+1} f(x) \in S_{2,0} \cap \mathrm{Cl}(S_{1;k})$.*

Proof. We know that $V_{k+1}(xy + \sum_{i=1}^{n-p} w_i^2)$ is contained in $S_{2,0} \cap \mathrm{Cl}(S_{1;k})$. Choose in this contact class a transversal element z of S_2 so that $\pi_1(z) = x$. Let $A = \{f \in C(N, P) : j^{k+1} f(x) = z\}$. Applying Corollary 3.2 to the case $s = 1$ and $\{W_i\}$ the Thom-Boardman singularities, we see that there is an f such that $j^{k+1} f(x) = z$ and away from x $j^{k+1} f$ is transverse to all Thom-Boardman singularities. Thus f is a Thom-Boardman map.

LEMMA 3.5. *Suppose $z_m \to j^k f(x) \in J^k(N, P)$. Then there are maps f_m and points x_m such that $j^k f_m(x_m) = z_m$ and $f_m \to f$ in the Whitney C^∞ topology.*

Proof. Necessarily $\pi(z_m) = (x_m, y_m)$ converges to $\pi(z) = (x, y)$. Choose coordinate charts (U, r) about x and (V, l) about y such that $r(U) = \boldsymbol{R}^n$, $r(x) = 0$, $r(x_m) = u_m$ and $l(V) = \boldsymbol{R}^p$. Choose any maps g_m such that $j^k g_m(x_m) = z_m$. Let $g'_m = l \circ g_m \circ r^{-1}$ and $f' = l \circ f \circ r^{-1}$. Let p_m be a polynomial of degree k expanded about u_m such that $j^k p_m(u_m) = j^k(g'_m - f')(u_m)$. Since $j^k p_m(u_m)$ converges to the zero jet at 0, the coefficients of the p_m's go to 0. For any compact set $K \subset \boldsymbol{R}^n$, there is a C such that $|u - u_m|^i < C$ for all $u \in K$, $m > 0$ and $0 < i \le k$. Thus $p_m \to 0$ uniformly on K, as does each of its derivatives. Pick $a \in C(\boldsymbol{R}^n, \boldsymbol{R})$ such that $a = 1$ on a neighborhood of 0 and $a = 0$ off a compact set. Let $f_m | U = l^{-1}(a p_m + f') \circ r$ and $f_m | (N - U) = f$. Then $f_m \to f$ in the Whitney C^∞ topology and $j^k f_m(x_m) = z_m$.

Choose k sufficiently large that $\operatorname{cod} S_{1;k} > n$. Choose f as in Corollary 3.4. By the preceding lemma, f is the limit of maps f_m having $S_{1;k}$ singularities. Necessarily f_m is not a Thom-Boardman map. Thus the proof of Theorem 1.1 is complete.

EXAMPLE 3.6. Let $f(x, y, s, t) = (g(x, y) + sx + ty, xy, s, t)$, where g is in m_2^2; then $j^2 f(0)$ is in $_t S_{2,0}$. If we take $g(x, y) = x^3 + y^3$, then f is a Thom-Boardman map and is the limit of maps which, having $S_{1;5}$ singularities, are not Thom-Boardman. The computational details involved in this example are worked out in [6].

REFERENCES

1. J. M. Boardman. *Singularities of differentiable maps*, Publ. Math. I.H.E.S., **33** (1967). 21–57.
2. T. Gaffney, *Properties of Finitely Determined Germs*, Thesis, Brandeis Univ., 1975.
3. M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, Springer-Verlag, New York, 1973.
4. M. Hirsch, *Differential Topology*, Springer-Verlag, New York, 1976.
5. L. Lander, *Stratification by orbits, with an application to $J^2(2,2)$*, J. London Math. Soc., (2), **8** (1974), 443–450.
6. ———, Thesis, Universität Regensburg, 1972.
7. J. Mather, *Stability of C^∞ mappings, III: finitely determined map germs*, Publ. Math. I.H.E.S., **35** (1968), 127–156.
8. ———, *Stability of C^∞ mappings, IV: classification of stable germs by R-algebras*, Publ. Math. I.H.E.S., **37** (1969), 223–248.
9. ———, *Stability of C^∞ mappings, V: transversality*, Advances in Math., **4** (1970), 301–336.
10. ———, *Stability of C^∞ mappings, VI: the nice dimensions*, Proceedings of the Liverpool Singularities Symposium I, Lecture Notes in Math. No. 192, Springer-Verlag, New York, 1971.
11. ———, *On Thom-Boardman Singularities*, Dynamical Systems, Academic Press, New York, 1973.
12. J. C. Tougeron, *Idéaux de Fonctions Différentiables*, Ergebnisse Band 71, Springer-Verlag. New York. 1972.

13. C. T. C. Wall, *Regular Stratifications, Dynamial Systems-Warwick* 1974, Lecture Notes in Math. No. 468, Springer-Verlag, New York, 1975.

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