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TWO QUESTIONS ON WALLMAN RINGS

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In this paper we give an example of a Wallman ring $\mathscr{A}$ on a topological space X such that the associated compactification $\omega(X, Z(\mathscr{A}))$ is disconnected and $\mathscr{A}$ is not a direct sum of any two proper ideals, herewith solving a question raised by H. L. Bentley and B. J. Taylor. Also, an example of a uniformly closed Wallman ring which is not a sublattice is given.

I. Introduction. Biles [2] has called a subring $\mathscr{A}$ of the ring $C(X)$, of all real-valued continuous functions on a topological space X, a Wallman ring on X whenever $Z(\mathscr{A})$, the zero-sets of functions belonging to $\mathscr{A}$, forms a normal base on X in the sense of Frink.

H. L. Bentley and B. J. Taylor [1] studied relationships between algebraic properties of a Wallman ring $\mathscr{A}$ and topological properties of the compactification $\omega(X, Z(\mathscr{A}))$ of X. They proved that if $\mathscr{A}$ is a Wallman ring on X such that $\mathscr{A} = \mathscr{B} \oplus \mathscr{C}$ where $\mathscr{B}$ and $\mathscr{C}$ are proper ideals of $\mathscr{A}$, then $\omega(X, Z(\mathscr{A}))$ is disconnected. We shall prove that the converse of this result is not valid. But, when $\omega(X, Z(\mathscr{A}))$ is disconnected we find a Wallman ring $\mathscr{A}^\circ$, equivalent to $\mathscr{A}$, which is a direct sum of any two proper ideals.

It is well-known that every closed subring of $C^*(X)$, the ring of all bounded functions in $C(X)$, that contains all the rational constants is a lattice. But this is not true for arbitrary closed subrings of $C(X)$. We give an example of a uniformly closed Wallman ring on a space Y which is not a sublattice of $C(Y)$. This corrects an assertion stated in ([1], p. 27).

II. Definitions and basic results. All topological spaces under consideration will be completely regular and Hausdorff. A nonempty collection $\mathscr{F}$ of subsets of a nonempty set X is said to be a ring of sets if it is closed under the formation of finite unions and finite intersections. The collection $\mathscr{F}$ is said to be disjunctive if for each closed set G in X and point $x \in X \sim G$ there is a set $F \in \mathscr{F}$ satisfying $x \in F$ and $F \cap G = \emptyset$. It is said to be normal if for F_1 and F_2 in $\mathscr{F}$ with empty intersection there exist G_1 and G_2 which are complements of members of $\mathscr{F}$ satisfying $F_1 \subset G_1$, $F_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$. The collection $\mathscr{F}$ is a normal base for the topological space X in case it is a normal, disjunctive, ring of sets that is a base for the closed sets of X.

Throughout this section $\mathscr{D}$ will denote a disjunctive ring of closed sets in a topological space X that is a base for the closed sets of X. Let $\omega(X, \mathscr{D})$ denote the collection of all $\mathscr{D}$-ultrafilters, and topologize them with a topology having as a base for the closed sets, sets of the form $D^* = \{\mathscr{U} \in \omega(X, \mathscr{D}) : D \in \mathscr{U}\}$ where $D \in \mathscr{D}$. Then X can be embedded in $\omega(X, \mathscr{D})$ as a dense subspace when it carries the relative topology. The embedding map takes each $x \in X$ into the unique $\mathscr{D}$-ultrafilter of supersets of x in $\mathscr{D}$. The space $\omega(X, \mathscr{D})$ is a T_1-compactification of X ([3], p. 122).

We now state some facts concerning the space $\omega(X, \mathscr{D})$ which will be needed. For a proof see ([3], p. 119, p. 123).

PROPOSITION 2.1. *The space $\omega(X, \mathscr{D})$ is Hausdorff if and only if $\mathscr{D}$ is a normal base on X.*

The following result is an interesting characterization of $\omega(X, \mathscr{D})$ due to Sanin.

THEOREM 2.2. *The space $S = \omega(X, \mathscr{D})$ is uniquely determined (in the usual sense) among T_1-compactifications of X by its properties*

(a) $\{\mathrm{cl}_S D : D \in \mathscr{D}\}$ *is a base for the closed sets of* $\omega(X, \mathscr{D})$.

(b) *For F_1, F_2 in $\mathscr{D}$,* $\mathrm{cl}_S F_1 \cap \mathrm{cl}_S F_2 = \mathrm{cl}_S(F_1 \cap F_2)$.

According to the Proposition 2.1 if any Hausdorff compactification of X satisfies (a) and (b), then $\mathscr{D}$ is a normal base on X.

III. Disconnectedness of $\omega(X, Z(\mathscr{A}))$. The next result is a necessary and sufficient condition for the disconnectedness of $\omega(X, Z(\mathscr{A}))$ being $\mathscr{A}$ a Wallman ring on X.

THEOREM 3.1. *Let $\mathscr{A}$ be a Wallman ring on a space X. Then $\omega(X, Z(\mathscr{A})$ is disconnected if and only if there is a Wallman ring $\mathscr{A}^\circ$, equivalent to $\mathscr{A}$* (i.e., $\omega(X, Z(\mathscr{A})) = \omega(X, Z(\mathscr{A}^\circ))$), *which is the direct sum of any two proper ideals.*

Proof. The sufficiency has been proved in ([1], Theorem 3.14) with $\mathscr{A} = \mathscr{A}^\circ$. Necessity. Suppose that $S = \omega(X, Z(\mathscr{A}))$ is disconnected. Then there exist nonempty disjoint closed subsets A and B of S whose union is S. Since A is a closed set of S,

$$A = \bigcap \{\mathrm{cl}_S Z : A \subset \mathrm{cl}_S Z,\ Z \in Z(\mathscr{A})\} .$$

It follows from $A \cap B = \varnothing$ that $\{B, \mathrm{cl}_S Z : A \subset \mathrm{cl}_S Z,\ Z \in Z(\mathscr{A})\}$ does not have the finite intersection property. Therefore $B \cap \mathrm{cl}_S Z_1 \cap \cdots \cap \mathrm{cl}_S Z_n = \varnothing$, for some $Z_i \in Z(\mathscr{A})$, $A \subset \mathrm{cl}_S Z_i$, $1 \leqq i \leqq n$. This implies $A = \bigcap \{\mathrm{cl}_S Z_i : 1 \leqq i \leqq n\} = \mathrm{cl}_S \cap \{Z_i : 1 \leqq i \leqq n\}$. So $A = \mathrm{cl}_S Z(f)$ where

$f \in \mathscr{A}$. In the same way we find that $B = \mathrm{cl}_S Z(g)$, $g \in \mathscr{A}$.

The set $\mathscr{A}^\circ = \{h/s: h, s \in \mathscr{A}, Z(s) = \emptyset\}$ is a subring of $C(X)$ such that $Z(\mathscr{A}) = Z(\mathscr{A}^\circ)$. So $\mathscr{A}^\circ$ is a Wallman ring on X equivalent to $\mathscr{A}$. The functions $h_1 = f^2/(f^2 + g^2)$, $h_2 = g^2/(f^2 + g^2)$ belong to $\mathscr{A}^\circ$ and they are the characteristic functions of the zero-sets $Z(g)$ and $Z(f)$, respectively. Since $Z(f) \cap Z(g) = \emptyset$, the ideal (h_i) of $\mathscr{A}^\circ$ generated by h_i is proper, $1 \leqq i \leqq 2$. On the other hand, $1 = h_1 + h_2$ implies that $\mathscr{A}^\circ = (h_1) \oplus (h_2)$.

The following is an example of Wallman ring which cannot be expressed as the direct sum of nontrivial ideals.

EXAMPLE 3.2. Let $X = [0, 1) \cup [2, 3)$, $\mathscr{B} = \{f \in C(X)$: for some compact set $K \subset X$, f is an integer constant on $X \sim K\}$.

Since X is locally compact, $Z(\mathscr{B})$ is a disjunctive base for the closed sets of X.

Consider the following functions in $C(X)$

$$\begin{aligned} \varphi_1(x) &= e\,, \quad x \in [0, 1) \qquad \varphi_1(x) = 0\,, \quad x \in [2, 3) \\ \varphi_2(x) &= 0\,, \quad x \in [0, 1) \qquad \varphi_2(x) = e\,, \quad x \in [2, 3)\,. \end{aligned}$$

Let $\mathscr{A}$ be the subring of $C(X)$ generated by $\mathscr{B} \cup \{\varphi_1, \varphi_2\}$. Since $\varphi_1\varphi_2 = 0$, a function of $\mathscr{A}$ will be of the form

$$f = g_{00} + g_{10}\varphi_1 + g_{20}\varphi_1^2 + \cdots + g_{m0}\varphi_1^m + g_{01}\varphi_2 + \cdots + g_{0j}\varphi_2^j$$

where g_{ik} belong to $\mathscr{B}$ and m, j are nonnegative integers.

From the definition of $\mathscr{B}$, there exist compact sets $K_1 \subset [0, 1)$ and $K_2 \subset [2, 3)$ such that if $x \in X \sim (K_1 \cup K_2)$ then $g_{ik}(x) = \alpha_{ik} \in Z$ (the set of integer numbers). Therefore

$$(*) \qquad \begin{aligned} f(x) &= \alpha_{00} + \alpha_{10}e + \cdots + \alpha_{m0}e^m\,, \qquad x \in [0, 1) \sim K_1 \\ f(x) &= \alpha_{00} + \alpha_{01}e + \cdots + \alpha_{0j}e^j\,, \qquad x \in [2, 3) \sim K_2\,. \end{aligned}$$

Since $Z(\mathscr{B}) \subset Z(\mathscr{A})$ it follows that $Z(\mathscr{A})$ is a disjunctive base for the closed sets of X and a ring of sets.

Now, we will show that $K = [0, 1] \cup [2, 3]$ is a compactification of X equivalent to $\omega(X, Z(\mathscr{A}))$. According to Theorem 2.2 it suffices to show that: (a) The family $\{\mathrm{cl}_K Z: Z \in Z(\mathscr{A})\}$ is a base for the closed sets of K (b) For Z_1, Z_2 in $Z(\mathscr{A})$, $\mathrm{cl}_K(Z_1 \cap Z_2) = \mathrm{cl}_K Z_1 \cap \mathrm{cl}_K Z_2$.

(a) If C is a closed set in K and $1 \notin C$, then the set $C \cap [0, 1]$ is compact and $1 \notin C \cap [0, 1]$. Let β be a point in $[0, 1)$ such that $C \cap [\beta, 1] = \emptyset$. Then, there exists a function $f \in C(K)$ such that $f([\beta, 1] \cup [2, 3]) = \{1\}$ and $f(C \cap [0, 1]) = \{0\}$. If g is the restriction of f to X, then $g \in \mathscr{B}$, $h = \varphi_1 g \in \mathscr{A}$, $C \subset \mathrm{cl}_K Z(h)$ and $1 \notin \mathrm{cl}_K Z(h)$. With the point 3 a similar argument can be used (also in (b)).

(b) Let $f, g \in \mathscr{A}$ and suppose that $1 \in \mathrm{cl}_K Z(f) \cap \mathrm{cl}_K Z(g)$. From (*) there exists $\beta \in [0, 1)$ such that $f(x) = m_1$ and $g(x) = m_2$ for every $x \in [\beta, 1)$. By our assumption $m_1 = m_2 = 0$, therefore $1 \in \mathrm{cl}_K(Z(f) \cap Z(g))$.

Then $K = \omega(X, Z(\mathscr{A}))$, hence $Z(\mathscr{A})$ is a normal base on X and $\mathscr{A}$ is a Wallman ring.

Now, we will show that the characteristic function of the interval $[0, 1)$ is not in $\mathscr{A}$. Let $h \in \mathscr{A}$. From (*), there exist $\beta \in [0, 1)$, $\gamma \in [2, 3)$ and $\alpha_{ik} \in Z$, $0 \leqq i \leqq m$, $0 \leqq k \leqq j$ such that

$$h(x) = \alpha_{00} + \alpha_{01}e + \cdots + \alpha_{0j}e^j , \quad x \in [\gamma, 3)$$
$$h(x) = \alpha_{00} + \alpha_{10}e + \cdots + \alpha_{m0}e^m , \quad x \in [\beta, 1) .$$

If $[2, 3) \subset Z(h)$, then $\alpha_{00} = \alpha_{01} = \cdots = \alpha_{0j} = 0$ because e is a transcendental number. Therefore $h(x) = \alpha_{10}e + \cdots + \alpha_{m0}e^m \neq 1$ if $x \in [\beta, 1)$.

Finally, we will show that $\mathscr{A}$ cannot be expressed as the direct sum of nontrivial ideals. Suppose that $\mathscr{A} = \mathscr{C} \oplus \mathscr{H}$ where $\mathscr{C}$ and $\mathscr{H}$ are proper ideals of $\mathscr{A}$. Then $1 \in \mathscr{A}$ implies that there exist $f \in \mathscr{C}$ and $g \in \mathscr{H}$ such that $1 = f + g$ and $fg = 0$. Hence $\{Z(g), Z(f)\}$ is a partition on X. On the other hand, since $\mathscr{C}$ and $\mathscr{H}$ are proper ideals, the zero-sets $Z(f)$ and $Z(g)$ are nonempty, so $[0, 1) = Z(f)$ and $[2, 3) = Z(g)$. Therefore $g \in \mathscr{A}$ is the characteristic function of the interval $[0, 1)$, which is a contradiction.

IV. An example of a closed Wallman ring which is not a lattice. Let N denote the set of natural numbers. By a sublattice of $C(X)$ we mean a subset of $C(X)$ which contains the supremum and infimum of each pair of its elements. By a closed subring of $C(X)$ we mean a subring of $C(X)$ which is closed in the uniform topology on $C(X)$.

EXAMPLE 4.1. Let $\mathscr{B}$ be the set $\{f \in C(N)$: for some finite subset $M \subset N$, f is an integer constant on $N \sim M\}$. Then $\mathscr{B}$ is a subring of $C(N)$ and $Z(\mathscr{B}) = \{B \subset N: B$ or $N \sim B$ is finite$\}$. It is well-known that $\mathscr{B}$ is a Wallman ring on N such that $\omega(N, Z(\mathscr{B}))$ is the one-point compactification of N.

Let φ be the function defined $\varphi(2n) = n$, $\varphi(2n-1) = -n$, $n = 1, 2, \cdots$. Let $\mathscr{A}$ be the subring of $C(N)$ generated by $\mathscr{B} \cup \{\varphi\}$. Obviously $Z(\mathscr{B}) \subset Z(\mathscr{A})$. To show that $Z(\mathscr{A}) \subset Z(\mathscr{B})$, let $f \in \mathscr{A}$. Then $f = g_0 + g_1\varphi + \cdots + g_m\varphi^m$, where $g_i \in \mathscr{B}$, $0 \leqq i \leqq m$. From the definition of $\mathscr{B}$, there exist $n_0 \in N$, $\alpha_i \in Z$, $0 \leqq i \leqq m$ such that $g_i(2n-1) = g_i(2n) = \alpha_i$, $0 \leqq i \leqq m$ for every $n \geqq n_0$. If $\alpha_1 = \cdots = \alpha_m = 0$, then $f(2n-1) = f(2n) = \alpha_0$ for every $n \geqq n_0$ and therefore

$f \in \mathscr{B}$. Suppose $\alpha_{i_0} \neq 0$ for some $i_0 \geqq 1$. Then, if $n \geqq n_0$, $f(2n) = \alpha_0 + n\alpha_1 + \cdots + n^m\alpha_m$ and $f(2n-1) = \alpha_0 - n\alpha_1 + \cdots + (-1)^m n^m \alpha_m$. So $Z(f)$ is finite and $Z(f) \in Z(\mathscr{B})$. Hence $\mathscr{A}$ is a Wallman ring on X.

If $\varphi^+ = \varphi \vee 0$, then $Z(\varphi^+) = \{1, 3, 5, \cdots\} \notin Z(\mathscr{A})$. Therefore $\varphi^+ \notin \mathscr{A}$ and $\mathscr{A}$ is not a lattice. Finally, since the functions of $\mathscr{A}$ are integer-valued, it follows that $\mathscr{A}$ is uniformly closed in $C(N)$.

References

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