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# THE STABILITY OF THE AXIALLY SYMMETRIC PENDENT DROP

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**This paper analyzes the stability of the axially symmetric pendent drop for three different physical arrangements; problem A, constant pressure, fixed circular opening; Problem B, constant volume, prescribed angle of contact with a horizontal plate; Problem C, constant volume and fixed circular opening. As examples, the following results are established. Relative to Problem B we prove that for any angle of contact, $\alpha$ $(0<\alpha<\pi)$, as the volume increases an inflection point will appear on the profile curve before instability occurs. For Problem C we show that if the opening is narrow enough to support a stable drop with a bulge then as the volume is increased drops with both a neck and a bulge will appear before instability occurs.**

## 1. Introduction.

This paper is a study of the axially symmetric pendent drop as it may be realized in a variety of physical situations. We shall be interested in those drops which, besides being in a state of equilibrium, are also stable relative to small perturbations. In particular we shall examine how the stability criterion affects drop formation for the following three physical settings.

PROBLEM A. Constant Pressure, Fixed Circular Opening.

Here the drop is to protrude downward from a fixed horizontal circular orifice of given radius (e.g., the end of a pipette held vertically) with a pressure prescribed at the opening. This configuration is most easily realized by taking a flexible circular tube, filling it with the fluid and immersing one end of it in a resevoir to form a siphon, and bending the tube so that the exposed end opens downward. The potential energy of such a configuration is given by

$$E(\Lambda) = \sigma A(\Lambda) + \rho g \iiint z dv \ . \tag{1.1}$$

Here $\Lambda$ is the liquid-air interface, $\sigma$ is the surface tension of this interface, $\rho$ is the density of the fluid, and $g$ is the gravitational constant. $A(\Lambda)$ denotes the area of the interface and the integral is over the exposed volume of the fluid. If the exposed end is set at the level of the fluid in the resevoir we have the zero pressure situation and if the tube is narrow enough a stable equilibrium is attained with the liquid-air interface, $\Lambda$, perfectly flat. By lowering the tube, the pressure at the opening is increased linearly and a family of stable pendent drops is formed. At some point the limit of stability will be reached, and any further increase in pressure will cause the fluid to flow out.

PROBLEM B. Constant Volume, Prescribed Angle of Contact with a Horizontal Plate.

Here the angle of contact, $\alpha$, of the liquid-air interface with the horizontal plate, measured interior to the fluid, is determined by the energy functional which is

$$E(\Lambda) = \sigma A(\Lambda) + \rho g \iiint z dv - \sigma \lambda A(\Sigma) . \tag{1.2}$$

Here $\Sigma$ is the region of contact of the liquid drop with the horizotal plate, $A(\Sigma)$ is the area of $\Sigma$, and $\lambda = \cos \alpha$.

For small volumes the surface tension predominates over the gravitational potential and the stable pendent drops resemble spherical caps. Allow the volume to increase, by inserting fluid through a small hole in the plate for instance, and a family of stable drops is generated until a maximum stable configuration is attained.

PROBLEM C. Constant Volume, Fixed Circular Opening.

Here we have the "medicine dropper" filled with a fluid and possessing a circular orifice. As in Problem A, if the diameter of the tube is sufficiently small, the horizontal slice, $u \equiv 0$, will yield a stable configuration. The potential energy for this problem is

$$E(\Lambda) = \sigma A(\Lambda) + \rho g \iiint z dv . \tag{1.3}$$

We can now increase the exposed volume by slowly squeezing the dropper, thus forming a family of stable pendent drops until the limit of stability is reached at some maximum volume.

For each of these problems the Euler equation for the drop interface, $\Lambda$, requires that the mean curvature be a linear function of height. If we also assume axial symmetry then the Euler equation becomes an ordinary differential equation for the profile curve. In § 2 we discuss in some detail the solutions of this O. D. E., stating needed previous results, and adding some new remarks.

In § 3 we analyze each of the three problems. The procedure is similar in each case. We show the existence of a smooth one-parameter family of symmetric, equilibrium solutions which contains a stable configuraton. We run through the family until a maximum value of the constraint is reached. This determines the limit of symmetric stability, and except for one extra condition that needs to be satisfied in Problem C, the limit of stability is general. The validity of this procedure is discussed in § 4.

The condition for stability used in this paper and discussed in § 4 is determined by the second variation. For example, with Problem B an equilibrium configuration, $\Lambda$, is determined by the requirement that the first variation of the energy functional, $\partial E_\Lambda(N)$, vanish for every smooth normal perturbation, $N$, for which the first variation of volume, $\partial V_\Lambda(N)$, also vanishes. (See Formulas 4.3, 4.4.) By the method of Lagrange multipliers this means that $\partial(E - cV)_\Lambda(N) = 0$ for some constant, $c$, and all smooth normal perturbations, $N$. An equilibrium configuration, $\Lambda$, is said to be stable if the second variation, $\partial^2(E - cV)_\Lambda(N, N)$, is positive for all nontrivial normal perturbations satisfying $\partial V_\Lambda(N) = 0$. (See Formula 4.7.) We do not restrict ourselves to symmetric perturbations. Also, we do not discuss the question of whether or not our condition for stability implies that any such stable drop in equilibrium is actually a local minimum of the energy functional compared to other drops of the same volume. This last question could bear further investigation.

The results (qualitative in nature) may be summarized as follows.

PROBLEM A. There is a value, $r_0$, such that for tube radius greater than $r_0$, no stable configuration is possible. If the tube radius, $a$, is less than $r_0$, then the horizontal slice for $\Lambda: u \equiv 0$ is stable at zero pressure. For each $a < r_0$, there is a maximum pressure, obtained by lowering the tube, beyond which the drops are unstable. This maximum pressure increases monotonically as $a \to 0$. For values of $a$ near $r_0$, the profile curves for stable drops will develop an inflection point before instability is reached. If $a$ is very small, then all stable drops will have purely convex profile curves. Furthermore, all stable configurations may be expressed in nonparametric form, $u = u(x, y)$, (Theorem 3.1 and Figures 1, 2).

PROBLEM B. For any value of the angle of contact, $0 < \alpha < \pi$, drops of very small volume are stable. As the volume is increased and before the point of instability is reached, the profile curves will develop an inflection point, (Theorem 3.2). For $\alpha = \pi/2$, a consequence of this theorem is that before instability is reached the drops will develop a neck where it contacts the plate. It will also follow that if $\alpha$ is slightly less than $\pi/2$ the drop will develop a neck away from the plate before the point of instablity is reached, (Theorem 3.2).

For $0 < \alpha < \pi$, as the volume increases from zero the area of contact with the horizontal plate will initially increase, but before maximum volume is attained the area will decrease. Drop height will increase monotonically with volume as long as the configurations remain stable. Finally, for any stable configuration, the profile curve will contain at most one inflection point, (Theorem 3.6).

For $\alpha = 0$ it is clear that the profile curve of any drop must contain an inflection point. We show here that the profile curve for any stable drop can be expressed nonparametrically, (i.e., the profile curve will contain no vertical tangent), (Theorems 3.3, 3.8), (see Figures 1, 3, 6, 7 at the end of the paper).

PROBLEM C. As in Problem A, there is a maximum radius, $r_1$, beyond which no stable configuration can exist. If the radius, $a$, is less than $r_1$, then $\Lambda$: $u = 0$ is a stable configuration. If we increase the volume, then before the point of instability is reached the profile curve will develop an inflection point, (Theorem 3.4). Furthermore, if the radius, $a$, is small enough stable drop configurations will arise whose profile curve will possess a vertical tangent. These drops will then develop a bulge. If this occurs, then before the limit of stability is reached the drop will develop a neck, (a second vertical tangent on the profile curve). At some point *after* the formation of the neck instability will occur, (Theorem 3.5), (see Figure 3).

As in Problem B drop height will increase monotonically with volume until the limit of stability is reached. Also, any drop whose profile curve contains a second bulge must be unstable, (Theorem 3.7).

For a drop which is symmetrically stable to be stable as well, the following condition must be satisfied. The profile curve cannot contain a horizontal tangent away from the drop tip, (Figures 1, 4, 5, 8).

There is an extensive literature relating to the axially symmetric pendent drop. An early treatise (1883) was that of F. Bashforth and J.C. Adams [1] who developed numerical methods for solving the O.D.E.'s determining the pendent and sessile drop. Further discussion may be found in the book of W. Thomson [15], and recently P. Concus and R. Finn have produced a series of papers to which we shall refer in § 2.

There also has been considerable work on the question of stability. E. Pitts [14] studied all three problems. For example he established that for Problem C, the maximum volume criterion determines the limits of symmetric stability. His proof assumes that drop height can be used as the parameter for the family of pendent drops; an assumption supported by his numerical calculations and which we verify here. E. A. Boucher and M. J. B. Evans in [5], and Boucher, Evans and H. J. Kent in [6] present the implications derived from careful numerical calculations as applied to Problems B and C. They use the maximum volume principle for finding the limit of stability. Their conclusions support the theorems of this paper. A. K. Chesters [7] studies Problem C for a very narrow tube. In particular, he proves that for sufficiently narrow tube openings (in which case the bulbous drop will be nearly spherical), that the point of instability occurs after formation of the neck. (This is our Theorem 3, 5 in limiting form.) The special condition for asymmetric instability in Problem C mentioned above and discussed in § 4 seems to have been first analyzed by D. H. Michael and P. G. Williams in [13].

I would like to express my thanks to Frederic Brulois for his many helpful suggestions.

**2. The profile curves of the pendent drop.** The Euler equations for any of the systems described in the introduction give the condition

$$2H = -ku + c \tag{2.1}$$

where $k = \rho g/\sigma$, $H$ is the mean curvature with normal pointing into the drop, and $c$ is a constant which arises as a Lagrange multiplier. By a normalization we may allow $k = 1$ and by a vertical translation of coordinates we can make $c = 0$, reducing (2.1) to the condition $2H = -u$. If the surface is axially symmetric about the vertical $u$-axis the differential equation for the profile curve becomes

$$(2.2)\qquad \begin{aligned} &\text{(a)}\quad r'(s) = \cos\psi && r(0) = 0 \\ &\text{(b)}\quad u'(s) = \sin\psi && u(0) = u_0 = -2\kappa \\ &\text{(c)}\quad \psi'(s) = -u - (\sin\psi/r) && \psi(0) = 0\ , \end{aligned}$$

Here $\psi$ is the inclination angle, $s$ is arc length measured from the drop tip, and $u_0$ is the height at the tip.

When $r'(s) = \cos\psi \neq 0$ the solution to (2.2) may be expressed nonparametricly with $r$ as the independent variable and satisfying the differential equation

$$(r\sin\psi)_r + ru = 0,\ u(0) = u_0\ . \tag{2.3}$$

It will be convenient to work with the function $v(r) = \sin\psi(r)$ allowing us to rewrite (2.3),

$$(rv)_r + ru = 0,\ u(0) = u_0,\ v(0) = 0\ . \tag{2.4}$$

This equation may be written in the form

$$v' + (v/r) + u = 0 \tag{2.5}$$

where $v'(r)$ represents the meridonal curvature and $v/r$ the longitudinal curvature. It follows (as Concus and Finn have shown) that $v'(0) = -u_0/2 = \kappa$. For this reason we choose $\kappa$, which is the curvature at the drop tip, as a parameter in (2.2b).

According to P. Concus and R. Finn [9] the basic existence and uniqueness theorem for the system (2.2) or (2.3) was proved by Lohnstein [12] in 1891 using the method of majorants. In a more recent paper [11] W. E. Johnson and L. M. Perko proved an existence and uniqueness theorem for the sessile drop, $(r \sin \psi)_r - ru = 0$.

We shall need the fact that the solutions to (2.2) and (2.3) are analytic and that the solutions depend analytically on the initial conditions. Presumably the method of Johnson and Perko could be used to establish the result, however it seems easier to use the method of majorants. Since Lohnstein's thesis is not readily available we shall present a short proof here.

If we differentiate (2.4) with respect to $r$, $u(r)$ can be eliminated from the equation giving

$$\begin{aligned} &\text{(a)} \quad (rv')' + \left(\frac{r}{W} - \frac{1}{r}\right)v = 0 \ \text{ if } |\psi| < \pi/2 \\ &\text{(b)} \quad (rv')' - \left(\frac{r}{W} + \frac{1}{r}\right)v = 0 \ \text{ if } \pi/2 < |\psi| < \pi , \\ &\qquad W = (1 - v^2)^{1/2} . \end{aligned} \tag{2.6}$$

LEMMA 2.1. *The pair* $u(r)$, $v(r)$ *is a solution to* (2.4) *with* $u(0) = -2\kappa$, $v(0) = 0$ *if and only if* $v(r)$ *is a solution to* (2.6a) *with* $v(0) = 0$, $v'(0) = \kappa$.

*Proof.* The proof is straightforward. If $v(r)$ solves (2.6a), one then obtains $u(r)$ using (2.5).

THEOREM 2.1. *Given* $K > 0$ *there exists an* $R > 0$ *and a function* $v(r, \kappa)$ *analytic in* $r$ *and* $\kappa$ *in the region* $|\kappa| < K$, $|r| < R$ *satisfying the differential equation* (2.6a) *and the initial conditions* $v(0, \kappa) = 0$, $v_r(0, \kappa) = \kappa$.

*Proof.* (2.6a) may be rewritten in the form $L(v) = M(v)$ where if $v(r) = \sum_{n=1}^{\infty} a_n r^n$, then $L(v) = (rv')' - v/r = 3a_2 r + 8a_3 r^2 + \cdots + (n^2 - 1)a_n r^{n-1} + \cdots$ and

$$\begin{aligned} M(v) &= -rv/(1-v^2)^{-1/2} = -r\left[v + \sum_{n=2}^{\infty} c_n v^n\right] \\ &= -[a_1 r^2 + (a_2 + a_1^2 c_2) r^3 + \cdots + p_n(a_1, \cdots, a_{n-1}, \\ &\qquad c_1, \cdots, c_{n-1}) r^n + \cdots] \end{aligned}$$

where $p_n(a_1, \cdots, a_{n-1}, c_1, \cdots, c_{n-1})$ is a polynomial of degree $(n-1)$ in $a_1, \cdots, a_{n-1}$ and is affine in $c_1, \cdots, c_{n-1}$ with positive coefficients.

The series for $M(v)$ has radius of convergence 1. Thus for any positive $\rho$ less than one, there is a constant $C$, with $|c_n| < C/\rho^n$ and $c_1 = 1$.

Now set $L_1(v) = v'(r) - v'(0)$. If $v(r)$ is expanded as above, then formally we have $L_1(v) = \sum_{n=2}^{\infty} n a_n r^{n-1}$. We now define $M_1(v) = \mathrm{Cr}\,[(v/\rho)/(1-(v/\rho))]$. The formal solution to $L_1(v) = M_1(v)$ with $v(0) = 0$, $v'(0) = b_1 = |a_1|$ majorizes the corresponding solution to $L(v) = M(v)$. That is, if $w(r, b_1) = b_1 r + \sum_{n=2}^{\infty} b_n r^n$ is a formal solution to $L_1(w) = M_1(w)$, $w'(0) = b_1$ and if $v(r)$ is a formal solution to $L(v) = M(v)$ with $v'(0) = a_1$ then if $b_1 \geqq |a_1|$ we will have $b_n \geqq |a_n|$ for all $n$.

However, if we differentiate the system $L_1(w) = M_1(w)$ we are led to the second order differential equation

$$w'' = C\frac{d}{dr}[r(w/\rho)/(1-(w/\rho))]$$

with $w(0) = 0$, $w'(0) = b_1$. This is a differential equation without singular point at $r = 0$. Therefore, given $B_1 > 0$, there is an analytic solution with $w(0) = 0$, $w'(0) = B_1$ in some interval $|r| \leqq r_1$. Our formal series expression must coincide with this solution. Choose $\bar{r}$ with $0 < \bar{r} < r_1$. The series $w(\bar{r}, B_1)$ must converge absolutely. Therefore the series for $w(r, \kappa)$ and $v(r, \kappa)$ represents analytic functions in the domain $|r| < \bar{r}$, $|\kappa| < B_1$.

*Consequence.* As shown by Concus and Finn [9] any solution to (2.2) exists for all $s \geqq 0$. For small $s$, the solutions may be expressed nonparametricly. We may apply the above theorem to conclude that the solutions to (2.2) $\langle r(s, \kappa), u(s, \kappa), \psi(s, \kappa)\rangle$ are analytic in $s$, and in the initial parameter $\kappa = -u_0/2$.

We shall refer to $\langle r(s, \kappa), u(s, \kappa), \psi(s, \kappa)\rangle$ as the parametric form for the equation of the profile curves. If no confusion should result, we shall often write $r(s)$ for $r(s, \kappa)$ if $\kappa$ is a fixed parameter. In this case we would write $r'(s)$ for $r_s(s, \kappa)$. On the other hand, differentiation with respect to $\kappa$ will always be explicitely indicated.

We shall have frequent need to consider the derivatives of solutions to (2.2) with respect to $\kappa$. We set $\langle\rho(s,\kappa), \nu(s,\kappa), \omega(s,\kappa)\rangle = \langle r_\kappa(s,\kappa), u_\kappa(s,\kappa), \psi_\kappa(s,\kappa)\rangle$. In particular we find

$$(2.7)\quad \begin{aligned} &\text{(a)}\quad \rho_s = (-\sin\psi)\omega && \rho(0,\kappa) = 0\\ &\text{(b)}\quad \nu_s = (\cos\psi)\omega && \nu(0,\kappa) = -2\\ &\text{(c)}\quad \omega_s = (-\cos\psi/r)\omega + (\sin\psi/r^2)\rho - \nu && \omega(0,\kappa) = 0\,. \end{aligned}$$

If we express the surface nonparametricly, $u = u(r,\kappa)$, with $v(r,\kappa) = \sin\psi$, then we set

$$(2.8)\qquad \varphi(r,\kappa) \equiv u_\kappa(r,\kappa) \quad\text{and}\quad h(r,\kappa) \equiv v_\kappa(r,\kappa)\,.$$

In particular, we obtain from (2.5) and (2.6)

$$(2.9)\quad \begin{aligned} &\text{(a)}\quad h_r + (h/r) + \varphi = 0\\ &\text{(b)}\quad (rh_r)_r + (r/W^3 - 1/r)h = 0,\ h(0,\kappa) = 0,\ h_r(0,\kappa) = 1 \end{aligned}$$

if $|\psi| < \pi/2$, while for $|\psi| > \pi/2$ we obtain

$$\text{(c)}\qquad (rh_r)_r = (r/W^3 + 1/r)h\,.$$

The following theorem states several key results of P. Concus and R. Finn [8, 9] pertaining to solutions of (2.2) and (2.3).

THEOREM 2.2. (Concus and Finn, [8] *and* [9]).

(a) *There is a largest value* $\bar{u}_0 < -2$ (*smallest* $\bar{\kappa} > 1$) *so that the corresponding profile curve,* $u(r,\bar{\kappa})$, *attains a vertical tangent at a point* $(\bar{r}_1, \bar{u}_1)$. *For this curve* $\bar{r}_1\bar{u}_1 = -1$ *and is an inflection point for the profile curve if we continue it past* $(\bar{r}_1, \bar{u}_1)$ *using* (2.2).

$$u_0 \cong -2.5678,\ \bar{r}_1 \cong .\,9176, \quad \textit{and} \quad \bar{u}_1 \cong -1.0894\,.$$

(b) *If* $\bar{u}_0 \leqq u_0 \leqq 0$, *then the corresponding profile curve,* $u(r,\kappa)$ *is analytic for all* $r$. *As* $r$ *increases the curve is oscillatory in behavior and is asymptotic to* $u_0J_0(r)$ *as* $u_0$ *approaches* 0.

(c) *If* $u_0 < -2\sqrt{2}$ *then there is a point of first vertical tangency* $(r_1, u_1)$ *with*

$$u_1 < u_0/2[1 + (1 - 8/u_0^2)^{1/2}] < 0 \quad \textit{and} \quad r_1 < R$$

*where* $1/R = -u_0/4[1 + (1 - 8/u_0^2)^{1/2}]$.

(d) *Any solution to* (2.2) *with drop tip at* $u = u_0 < 0$ *will increase with* $u$ *until the* $r$-*axis is crossed. If* $|u_0|$ *is large the profile curve will oscillate in* $r$ *describing roughly a series of bubbles whose mean curvature is approximately* $-u/2$. *Once the axis is crossed, the solution may once again be expressed nonparametrically* $u = u(r)$ *with behavior as in* (b). *The bulges of the profile curve always occur between the curves* $ru = -1$ *and* $ru = -2$, *while* *the necks always occur above* $ru = -1$. *The curve remains above the curve* $ru = -2$ *until it crosses the u-axis.*

For our purpose it will frequently be convenient to work with the slope function $v(r) = \sin\psi$ and satisfying (2.6a) and $v(0) = 0$, $v'(0) = \kappa \geqq 0$. Theorem 2.2 gives the following information regarding the function $v(r)$.

COROLLARY TO THEOREM 2.2.

(a) *If* $0 \leqq v'(0) < 1$, *then* $v(r)$ *exists as an analytic solution to* (2.6a) *for all* $r > 0$.

(b) *If* $v'(0) = \kappa > \sqrt{2}$ *then* $v(r)$ *is defined over an interval* $[0, r_1]$ *with* $v(r_1) = 1$.

(c) *By continuity, there is a first solution to* (2.6a) *which attains the value* 1 *at the finite* $\bar{r}_1$. *Let* $\bar{\kappa} = v'(0) = \bar{u}_0/2 \cong 1.2839$ *be the corresponding initial derivative. If* $0 < \kappa < \bar{\kappa}$ *and* $v'(0) = \kappa$ *then condition* (a) *applies.*

We shall show shortly that if $\kappa > \bar{\kappa}$ and $v'(0) = \kappa$, then condition (b) applies.

At this point we should note that $v'(r) = \psi'(s)$ is the curvature of the profile curve. An inflection point along the profile curve occurs where $v'(r) = 0$. We will also need the following result of Concus and Finn [9, p. 314].

LEMMA 2.2. *Let* $v(r)$ *be a solution to* (2.6a) *with* $v(0) = 0$ *and* $v'(0) = \kappa > 0$. *Suppose* $v'(r) \geqq 0$ *for* $0 \leqq r \leqq \bar{r}$. *Then* $v''(r) < 0$ *for* $0 < r \leqq \bar{r}$.

We have the following useful identities.

LEMMA 2.3. *Let* $v(r)$ *be a solution to* (2.6a) *on an interval* $[0, r_1]$ *with* $v(0) = 0$. *We have*

$$\text{(2.10a)} \qquad \begin{gathered} \frac{1}{2}[r_1 v'(r_1)]^2 - r_1^2 W(r_1) + 2\int_0^{r_1} r\,W(r)dr = v^2(r_1)/2\,, \\ W = (1 - v^2)^{1/2}\,. \end{gathered}$$

*Suppose further that* $v'(r) > 0$ *on the interval* $[0, r_1]$. *Then the identity may be rewritten*

$$\text{(2.10b)} \qquad \frac{1}{2}[r_1 v'(r_1)]^2 + \int_0^{v_1} r^2(v)[v/W]dv = v^2(r_1)/2$$

*where* $v_1 = v(r_1)$.

*Proof.* Identity (2.10a) is obtained by multiplying (2.6a) by $rv'$, integrating the result from 0 to $r_1$, and performing an integration by parts. (2.10b) then follows from (2.10a) by a change in the variable of integration.

We can use these identities to obtain refinements of some of the results of Concus and Finn. For example, consider a profile curve where $u_0 > -2$ so that the curve may be expressed in the nonparametric form, $u = u(r)$ for $r \geqq 0$. In [9] it is shown that the successive maxima and minima for $u(r)$ decrease in absolute value. In particular, if $(r_1, u_1)$ is the location of the first maximum, then $0 < u_1 < |u_0|$. We now show

THEOREM 2.3. *If $-2 < u_0 < 0$ then the first maximum, $u_1$, of the profile curve $u(r)$ satisfies $u_1 < \sqrt{2}$.*

*Proof.* Let the first maximum occur at $r = r_1$. This means that $v(r_1) = \sin \psi(r_1) = 0$ and $v(r) > 0$ for $0 < r < r_1$. We know, (2.4), that $(rv)' + ru = 0$. Thus if $v(r_1) = 0$ we will have $v'(r_1) = -u(r_1)$. Now apply (2.10a) and we derive the identity,

$$u^2(r_1) = 2 - (4/r_1^2)\int_0^{r_1} r W(r) dr < 2 .$$

In fact, the conclusion of Theorem 2.5 is true for any initial value $u_0$. The proof, which we shall omit, follows directly from the following identity, similar to (2.10a).

LEMMA 2.4. *Let $\langle r(s), u(s), \psi(s)\rangle$ be a solutiou to the system (2.2) with $u(0) = u_0$. Let $\langle r_1, u_1, \psi_1\rangle$ be the values of $\langle r(s), u(s), \psi(s)\rangle$ at $s = s_1$.*

$$(2.11) \qquad \frac{1}{2}[r_1\psi'(s_1)]^2 = \frac{(\sin^2 \psi_1)}{2} + r_1^2 \cos \psi_1 - \int_0^{s_1} 2r \cos^2 \psi ds .$$

*Proof.* Take the D. E. (2.2c) multiply it by $r(s)$ and differentiate. This gives

$$(r\psi')' = -r \sin \psi + (\sin \psi \cos \psi)/r .$$

Now multiply through by $r\psi'$, integrate from 0 to $s_1$, and follow this by an integration by parts.

Let a point $(\bar{r}, \bar{u})$ lie on a profile curve. The volume of the generated drop which lies below the plane $u = \bar{u}$ is given by

$$V = 2\pi\int_0^{\bar{s}}(\bar{u} - u)rr_s ds = \pi r^2\bar{u} - 2\pi\int_0^{\bar{s}} urr_s ds .$$

However, by (2.2) we find that $(r \sin \psi)_s = -ru \cos \psi = -urr_s$. This gives us

$$V = \pi\bar{r}[\bar{r}\bar{u} + 2 \sin \bar{\psi}] = \pi\bar{r}[\bar{r}\bar{u} + 2\bar{v}] . \tag{2.12}$$

Furthermore, if the profile curve does not have a vertical tangent at $(\bar{r}, \bar{u})$ then using (2.5),

$$V = \pi\bar{r}[\bar{v} - \bar{r}v'(\bar{r})] . \tag{2.13}$$

Suppose $v(r)$ is a solution to (2.6a) with $v(0) = 0$, $v'(0) = \kappa > 0$. By Lemma 2.2 we know that $v(r)$ is convex up until a maximum is attained. The following lemma bears on its behavior after the maximum.

LEMMA 2.5. *Suppose $v(r)$ attains a maximum at $r_1$ with $v(r_1) = v_M < 1$. At the next critical point $r_2 > r_1$ for $v(r)$ we will have $v'(r_2) = 0$ and $v(r_2) < 0$.*

*Proof.* By Lemma 2.2 $v''(r_1) < 0$ so $v(r)$ is initially decreasing after $r_1$. Since $v(r_1) < 1$ it follows that $v(r)$ is a solution to (2.6a) for all $r \geqq 0$. By Theorem 2.2b, $u(r)$ has an infinite number of maxima and minima implying that $v(r)$ has an infinite number of zeros on $(0, +\infty)$. Let $r_3 > r_1$ be the first positive zero of $v(r)$. We need to show that $v'(r) < 0$ for $r_1 < r \leqq r_3$. If not there will be a $b$, $r_1 < b \leqq r_3$ with $v(b) = \bar{v} > 0$, $v'(b) = 0$ and $v'(r) < 0$ for $r_1 < r < b$. There is also a unique $a$, $0 \leqq a < r_1$ with $v(a) = \bar{v}$, $v'(a) > 0$, and $v(r) > \bar{v}$ for $a < r < b$.

Now apply the integral identity (2.10a) on the interval $(a, b)$. We obtain

$$[av'(a)]^2 = 4\int_a^b rW(r)dr - 2(b^2 - a^2)\bar{W}$$

where $W(r) = (1 - v^2(r))^{1/2}$ and $\bar{W} = W(a) = W(b)$. But $W(r) < \bar{W}$ for $a < r < b$ from which it follows that $[av'(a)]^2 < 0$, a contradiction.

THEOREM 2.4. *Let $u(r)$ with $u(0) = u_0$ be a profile curve which does not possess a vertical tangent. Let the first positive maximum for $u(r)$ occur at $r = r_1$. On the interval $(0, r_1)$, $u(r)$ posseses exactly one inflection point.*

*Proof.* This is a corollary to Lemma 2.5. $r_1$ is the first positive zero of the slope function $v(r)$, and so $v'(r)$ vanishes exactly once on $(0, r_1)$ which is the sole inflection point for $u(r)$.

The rest of this section is devoted to a comparison of different profile curves of the family (2.2). We shall be especially interested in the envelope of this family (Theorem 2.6). As is well known, this amounts to finding the conjugate points of the family relative to the drop tip, which in turn determines the limits of stability for the configurations of Problem A.

LEMMA 2.6. *Let $v(r)$, $w(r)$ be solutions to* (2.6a) *with $v(0) = w(0) = 0$ and $0 < v'(0) < w'(0)$. Suppose that for $0 \leqq r < a$, $v'(r) > 0$ and set $v(a) = v_M \leqq 1$. (Note: If $v(a) < 1$ then $v'(a) \geqq 0$ while if $v(a) = 1$ then $v'(a) = \lim v'(r)$ is the curvature of the profile curve at the vertical tangent.) There is a $b \in (0, a)$ so that $w'(r) > 0$ for $0 \leqq r < b$ and $w(b) = v(a) = v_M$. The graph of $w(r)$, $0 < r \leqq b$ lies to the left of $v(r)$, $0 < r \leqq a$ and given $\bar{v}$, $0 < \bar{v} \leqq v_M$ if we determine $r_1 < r_2$ with $w(r_1) = v(r_2) = \bar{v}$, then $r_1 w'(r_1) > r_2 v'(r_2) > 0$.*

*Proof.* The proof is based on the identity (2.10b). Choose a $c > 0$ so that $w'(r) > 0$ for $0 \leqq r < c$ and $w(c) \leqq v_M$. Initially this curve lies to the left of $v(r)$, $0 < r \leqq a$. We claim that the entire curve segment does. If not then there is a $\sigma$, $0 < \sigma \leqq c$ such that the curves intersect for the first time at $\sigma$, $w(\sigma) = v(\sigma)$ and $0 \leqq w'(\sigma) \leqq v'(\sigma)$. Both $v(r)$ and $w(r)$ have inverse functions $r(v)$ and $\tilde{r}(v)$ respectively such that $0 < \tilde{r}(v) < r(v)$ if $0 < v < v(\sigma)$. Let $0 < \bar{v} \leqq v(\sigma)$, determine $r_1$, $r_2$ so that $w(r_1) = v(r_2) = \bar{v}$ and apply (2.10b).

$$[r_1 w'(r_1)]^2 = \bar{v}^2 - \int_0^{\bar{v}} \tilde{r}^2(v)(v/W)dv$$

and

$$[r_2 v'(r_2)]^2 = \bar{v}^2 - \int_0^{\bar{v}} r^2(v)(v/W)dv, \quad W = (1 - v^2)^{1/2} .$$

We conclude that $r_1 w'(r_1) > r_2 v'(r_2) \geqq 0$. If we choose $\bar{v} = v(\sigma) = w(\sigma)$, then $r_1 = r_2 = \sigma$ and we conclude $w'(\sigma) > v'(\sigma)$, a contradiction. Thus $w(r)$ stays to the left of $v(r)$ and if $w(r_1) = v(r_2)$ then $r_1 w'(r_1) > r_2 v'(r_2)$.

The proof is completed by choosing the largest interval $[0, b]$ for which $w'(r) > 0$, $0 < r < b$ and for which $w(b) \leqq v_M$. If follows that $w(b) = v(a) = v_M$.

THEOREM 2.5. *Let $\bar{u}_0 < 0$ be that largest value for $u_0$ such that the corresponding profile curve with initial value $\bar{u}_0$ possesses* *a vertical tangent (say at* $(\bar{r}_1, \bar{u}_1)$*), and also an inflection point there. Any profile curve with initial value* $u_0 < \bar{u}_0$ *will have a point of vertical tangency at a point* $(r_1, u_1)$ *which is a bulge in the drop profile. As* $u_0$ *decreases to* $-\infty$, $r_1$ *will decrease to* 0, *and* $u_1$ *will decrease to* $-\infty$.

*Proof.* This theorem is a corollary of the previous lemma. If $v(r)$ is the slope function for the profile curve with initial value $\bar{u}_0$, then $v(\bar{r}_1) = 1$ and $v'(\bar{r}_1) = 0$. Thus if $w(r)$ is the slope function for the profile curve whose initial value $u_0 < \bar{u}_0$, then $w(r)$ lies to the left of $v(r)$ and $w(r_1) = 1$, $r_1 w'(r_1) > 0$ implying that the profile has positive curvature at the point of vertical tangency.

Also by Lemma 2.6 it is clear that $r_1$ decreases as $u_0$ decreases, and $r_1 v'(r_1)$ increases. But by (2.4) $r_1 v'(r_1) + 1 + r_1 u_1 = 0$ showing that $u_1$ must decrease. That $u_0 \to -\infty$ and $r_1 \to 0$ as $u_0 \to -\infty$ follows from Theorem 2.2c.

LEMMA 2.7. *Suppose* $u(r)$, $\tilde{u}(r)$ *are two profile curves with initial values* $u_0$, $\tilde{u}_0$ *which attain a vertical tangent at* $(r_1, u_1)$ *and* $(\tilde{r}_1, \tilde{u}_1)$ *respectively. The two curve segments* $u(r)$, $0 \leqq r \leqq r_1$ *and* $\tilde{u}(r)$, $0 \leqq r \leqq r_1$ *intersect at most once. If* $\tilde{u}_0$ *is sufficiently close to* $u_0$ *then the two curves will intersect once transversally.*

*Proof.* Suppose $\tilde{u}_0 < u_0$. By Theorem 2.5 we know that $\tilde{r}_1 < r_1$, $\tilde{u}_1 < u_1$, and that the slope function $\tilde{v}(r)$, $0 \leqq r \leqq \tilde{r}_1$ lies to the left of $v(r)$, $0 \leqq r \leqq r_1$. Therefore if $0 \leqq r \leqq \tilde{r}_1$ then $\tilde{v}(r) > v(r)$ implying that $\tilde{u}'(r) > u'(r)$. It follows that the two curves intersect at most once.

To complete the proof we fix $u_0 = -2\kappa$ and examine the differential equation satisfied by the accessary function to $u(r) \equiv u(r, \kappa)$, namely $\phi(r) \equiv \phi(r, \kappa) = u_\kappa(r, \kappa)$. Since $u(0, \kappa) = -2\kappa$, $u_r(0, \kappa) = 0$ we know that $\phi(0) = -2$, $\phi'(0) = 0$ and satisfies the D. E.

$$(r\phi'/J^3)' + r\phi = 0,\ J = (1 + u'^2)^{1/2} . \tag{2.14}$$

This is seen be differentiating (2.3) with respect to $\kappa$. Since $J > 0$, (2.14) is nonsingular for $r > 0$. Thus whenever $\phi(r) = 0$ we must have $\phi'(r) \neq 0$. Therefore, to prove the lemma it suffices to show that $\phi(r)$ vanishes at least once.

Suppose this were not so. Then $\phi(r)$ is negative for $0 \leqq r < r_1$ with $\phi'(0) \equiv 0$, $\phi(0) = -2$. By integrating (2.14) we conclude that $\phi'(r) > 0$ for $0 < r < r_1$. It follows that $-2 \leqq \phi(r) < 0$ for $0 \leqq r < r_1$ and thus is bounded.

We now compute $\phi(r)$ in terms of the triple of functions $\langle r(s, \kappa),$

$u(s, \kappa),\ \psi(s, \kappa)\rangle$ satisfying (2.2) using the chain rule.

$$\phi = (D_1u)s'(\kappa) + D_2u,\ 0 = (D_1r)s'(\kappa) + D_2r\ .$$

Here $D_1$, $D_2$ denote differentiation with respect to the two variables, $s$ and $\kappa$. Recalling that $D_1u = \sin\psi$ and $D_1r = \cos\psi$ we obtain $\phi = (D_2u) - (\tan\psi)D_2r$. As we pass through the point of vertical tangency both $D_2u$ and $D_2r$ remain analytic. Therefore, $\phi$ remains bounded as $r \to r_1$ only if $D_2r = 0$ when $\psi = \pi/2$.

If we set $h(r) = v_\kappa(r, \kappa)$, then by differentiating (2.4) with respect to $\kappa$ we obtain $(rh)' + r\phi = 0$. Integrate this expression from 0 to $\bar{r}$. We get

$$\bar{r}h(\bar{r}) = \int_0^{\bar{r}} - r\phi(r)dr$$

from which we conclude that as $\bar{r} \to r_1$ limit $h(\bar{r}) = h(r_1) > 0$. However, if we compute $h(r)$ implicitely as we did $\phi(r)$ we get (recalling that $v = \sin\psi$)

$$h(r) = -(D_2r)(D_1\psi) + (\cos\psi)D_2\psi\ .$$

But if $D_2r = 0$ when $\psi = \pi/2$ then $h(r_1) = 0$. Therefore $\phi(r)$ must vanish at least once on the interval $(0, r_1)$.

We are now ready to prove the desired theorem on the envelope of the family $u = u(r, \kappa)$.

DEFINITION 2.1. The envelope of the family of profile curves, $u(r, \kappa)$, is the set of points $(\bar{r}, \bar{u})$ satisfying $\bar{u} = u(\bar{r}, \bar{\kappa})$ and $u_\kappa(\bar{r}, \bar{\kappa}) = 0$ for some $\bar{\kappa}$.

THEOREM 2.6. *The first envelope, $\Gamma$, of the family of profile curves $u(r, \kappa)$ is given by an increasing function $e(r)$, $0 < r \leqq \alpha_0$ where $\alpha_0$ is the first zero of the Bessel function, $J_0(r)$. $e(r)$ is continuous and differentiable with* $\text{limit}_{r\to 0+} e(r) = -\infty$, $e(\alpha_0) = 0$, $\text{limit}_{r\to 0} e'(r) = +\infty$, *and* $e'(\alpha_0) = 0$. *Furthermore, the following are true.* (Figure 2.)

(i) *If a profile curve attains a vertical tangent at a point $(r_1, u_1)$, then $\Gamma$ will touch the profile curve at a point to the left and below this point.*

(ii) *If the profile curve does not have a vertical tangent, then $\Gamma$ will touch the profile curve to the left of the first zero of $u(r)$.*

*Proof.* Assertion (i) is contained in the proof of Lemma 2.7, so let $u = u(r, \kappa)$ be a profile curve which does not possess a vertical tangent. Denote by $b$ the first positive zero for $u(r, \kappa)$, and as before $\phi(r) = u_\kappa(r, \kappa)$. We are to show that the first zero of $\phi(r)$ occurs to the left of $b$. We know that $\phi(0) = -2$, $\phi'(0) = 0$ and $\phi(r)$ satisfies (2.14).

Suppose (ii) is not true. Then for some $\bar{\kappa} > 0$ we would have $u(b) = 0$, $u(r) < 0$ for $0 \leqq r < b$, $u'(r) > 0$ for $0 < r \leqq b$, while $\phi(r) < 0$ for $0 \leqq r < b$, $\phi(b) \leqq 0$ and $\phi'(r) > 0$ for $0 < r \leqq b$. Now equation (2.3) may be rewritten in the form

$$(2.15) \qquad (ru'/J)' + ru = 0,\ J = (1 + u'^2)^{1/2} .$$

Equations (2.14) and (2.15) allow us to evaluate the integral of $ru\phi$ two ways. Carry out the integrations, equate the result, perform an integration by parts, and note that $(1/J) - (1/J^3) = (u')^2/J^3$, $u(b) = 0$. We find

$$\frac{bu'(b)\phi(b)}{J(b)} = \int_0^b \frac{ru'^3\phi'}{J^3} dr .$$

The left hand side nonpositive while the right hand side is positive, a contradiction, thus proving (ii).

As $\kappa \to 0$, $u(r, \kappa)$ converges uniformly to $u \equiv 0$ in $r$, and $\phi(r, \kappa)$ converges uniformly to $\phi_0(r)$ where $\phi_0(r)$ is a solution to the limit of (2.14), $(r\phi')' + r\phi = 0$. This means that $\phi_0(r) = -2J_0(r)$ showing that $\Gamma$ passes through the point $(\alpha_0, 0)$ where $\alpha_0$ is the first positive zero of $J_0(r)$.

We next show that $\Gamma$ is a regular curve. Let $(\bar{r}, \bar{u}) \in \Gamma$ where $\bar{u} = u(\bar{r}, \bar{\kappa})$ and $u_\kappa(\bar{r}, \bar{\kappa}) = 0$ for some $\bar{\kappa} > 0$. $\Gamma$ will be regular if $u_{\kappa\kappa}(\bar{r}, \bar{\kappa}) \neq 0$. As above $\phi(r) = u_\kappa(r, \bar{\kappa})$ and also set $\beta(r) = u_{\kappa\kappa}(r, \bar{\kappa})$. By differentiating (2.14) with repsect to $\kappa$, we find that

$$(2.16) \qquad (r\beta'/J^3)' + r\beta = (3ru'\phi'^2/J^5)' .$$

$\beta(0) = 0$ and since $\beta(r)$ is analytic in $r$, there is an $\varepsilon > 0$ so that either $\beta$ and $\beta'$ are both positive or both negative on $(0, \varepsilon)$. An integration of (2.16) tells us that $\beta$ and $\beta'$ are initially positive.

Let $\bar{r}$ be the first zero of $\phi(r)$ and let $a$ be the first positive zero of $\beta(r)$ if such exists. If $a$ does not exist there is nothing to prove. Otherwise we shall show that $\bar{r} < a$.

Suppose $a$ exists. We may suppose that $u'(r) > 0$ for $0 < r \leqq a$, for otherwise $u(r)$ would have a zero to the left of $a$ and then so would $\phi(r)$. Thus we assume that $\bar{r} \geqq a$ and $u'(r) > 0$, $0 < r \leqq a$. As before we then have $\phi(0) = -2$, $\phi(r) < 0$ for $0 < r < a$, and $\phi'(r) > 0$ for $0 < r \leqq a$. We now use (2.14) and (2.16) to evaluate the integral of $r\phi(r)\beta(r)$ two ways. After some manipulations we arrive at

$$\frac{a\phi(a)\beta'(a)}{J^3(a)} = \frac{3au'(a)\phi(a)(\phi'(a))^2}{J^5(a)} - \int_0^a \frac{3ru'(r)(\phi'(r))^3}{J^5(r)} dr .$$

But $\phi(a) \leqq 0$ and $\beta'(a) \leqq 0$ so the left hand side is nonnegative. However, our assumptions also imply that the right hand side is negative, a contradiction. Therefore $\Gamma$ is a regular curve, defined by a function $e(r)$, $0 < r \leqq \alpha_0$.

As is well known, $e(r)$, is tangent to each profile curve at contact. It follows that $e'(r) > 0$, $0 < r < \alpha_0$ and that $e'(\alpha_0) = 0$.

Finally we wish to show that limit $e'(r) = +\infty$ as $r \to 0$. For large $\kappa$, let $u(r, \kappa)$ be a profile curve with a vertical tangent at $(r_1, u_1)$. The associated slope function $v(r, \kappa)$ is convex up on $[0, r_1]$ with $v_r(0, \kappa) = \kappa$. Therefore $v(r, \kappa) < \kappa r$. This result, along with the estimate of Theorem 2.2c imply that $1/\kappa < r_1 < 1/\kappa + 0(1/\kappa^3)$. Let $r(\kappa)$ be the first zero of $\phi(r, \kappa)$, $0 < r(\kappa) < r_1$. We shall show that $v[r(\kappa), \kappa] \to 1$ as $\kappa \to +\infty$.

To do this we normalize each curve by setting $U = \kappa u$ and $R = \kappa r$, thus obtaining $U(R, \kappa) = \kappa u(R/\kappa, \kappa)$ and $V(R, \kappa) = v(R/\kappa, \kappa)$. If follows that $V(R, \kappa)$ is increasing convex up, on the interval $[0, R_1]$ with $V(0, \kappa) = 0$, $V_R(0, \kappa) = 1$, $V(R_1, \kappa) = 1$ where $1 < R_1(\kappa) < 1 + 0(1/\kappa^2)$. A direct calculation reveals that

$$(RV_R)_R + (R/\kappa^2 W - 1/R)V = 0, \ W = (1 - V^2)^{1/2} .$$

This equation yields an integral identity of the form (2.10).

$$\frac{1}{2}[R_1 V'(R_1)]^2 + \frac{1}{\kappa^2}\int_0^{V_1} R^2(V)(V/W)dV = V^2(R_1)/2 . \tag{2.17}$$

Using the style of proof in Lemma 2.6 we may use (2.17) to show that if $\kappa_1 > \kappa_2$, then $V(R, \kappa_1)$ lies to the left of $V(R, \kappa_2)$. From this it follows that $V(R, \kappa)$ is a monotonically increasing sequence of functions which converges uniformly to $V_\infty(R) \equiv R$ on $[0, 1]$.

We now normalize the accessary function $\phi(r, \kappa) = u_\kappa(r, \kappa)$ by setting $\Phi(R, \kappa) = \phi(R/\kappa, \kappa)$. It satisfies the differential equation

$$\begin{aligned} &[R\Phi_R/J^3]_R + (R/\kappa^2)\Phi = 0, \ J^2 = 1 + U_R^2, \ \Phi(0) = -2 , \\ &\Phi'(0) = 0 . \end{aligned} \tag{2.18}$$

Let $R(\kappa)$ be the first zero $\Phi(R, \kappa)$. We claim that $R(\kappa) \to 1$ as $\kappa \to +\infty$. First, since $R(\kappa) < R_1(\kappa)$ where $V[R_1(\kappa), \kappa] = 1$ we conclude that $\limsup R(\kappa) \leqq 1$. Now $V(R, \kappa) \to R$ uniformly on any interval $[0, c]$ where $0 < c < 1$, and since $U_R = V/(1 - V^2)^{1/2}$ it follows that $J^2(R, \kappa) \to 1/(1 - R^2)$ uniformly on $[0, c]$.

Since $\Phi(0, \kappa) = -2$ and $\Phi_R(0, \kappa) = 0$ it follows from (2.18) that $\Phi(R, \kappa) \to -2$ uniformly on $[0, c]$, $0 < c < 1$. Thus $\liminf R(\kappa) \geqq 1$.

It now follows easily that

$$\text{limit}\, v[r(\kappa), \kappa] = \text{limit}\, V[R(\kappa), \kappa] = 1 .$$

**3. The limits of stability.** We now analyze the stability of drop formation for each of the three problems discussed in the introduction. Our method is similar in each case. We start with a known stable configuration. We imbed this stable drop in a smooth one parameter of equilibrium configurations. As long as the constraint parameter is increasing the drop will remain symmetricly stable. This principle will be discussed further in § 4.

PROBLEM A. Let $r = a$ be the radius of the circular opening for the siphon. If $u(r, \kappa)$ is the profile curve for a drop in equilibrium, then the pressure at the mouth of the pipette is $p(\kappa) = -u(a, \kappa)$. The considerations of stability are answered by referring to the first envelope of the family $u(r, \kappa)$, as described in Theorem 2.6.

THEOREM 3.1. (i) *For pressure equal to zero at the mouth of the siphon, the solution $u(r) = 0$ generates a stable configuration if the radius, a, is less than $\alpha_0$ where $\alpha_0$ is the first zero of the Bessel function, $J_0(r)$. If $a > \alpha_0$ then the flat interface is unstable*

(ii) *If $a < \alpha_0$, then the condition $p = -u(a, \kappa)$ determines a one-parameter family of stable configurations, with pressure increasing with $\kappa$, until the first envelope, $e(r)$, of the family is met. At this point $p'(\kappa) = -u_\kappa(a, \kappa) = 0$ and any further increase in $\kappa$ will decrease the pressure, creating instability.* (Figure 2.)

*Proof.* Part (i) follows from the discussion in § 4. Part (ii) follows from Theorem 2.6. The only thing to check is that $p'(\kappa)$ changes sign as our one-parameter family passes through the envelope, $e(r)$. But $p''(\kappa) = -u_{\kappa\kappa}(a, \kappa)$ is not equal to zero when $p'(\kappa) = \phi(a, \kappa) = 0$ since this is the condition that $e(r)$ be differentiable. Thus when the envelope is crossed, the drop of maximum pressure for a given radius, $a$, is produced.

Other immediate consequences of Theorems 2.6 and 3.1 are the following.

(1) The profile curve for any stable configuration can always be given nonparametrically, $u = u(r)$.

(2) As the radius, $a$, of the opening decreases to zero, the shape of the "largest" stable supported drop appears as a hemisphere of radius, $a$.

Let $(\bar{r}_1, \bar{u}_1)$ be the point of vertical tangency for that unique profile curve $u(r, \kappa)$ which has a point of inflection there.

(3) If $a < \bar{r}_1$, then all stable drops are convex downward.

(4) The first inflection point, $\beta_0$, of $J_0(r)$ lies to the left of $\alpha_0$, the first zero of $J_0(r)$. If $\beta_0 < a < \alpha_0$, then every profile curve corresponding to a stable drop contains an inflection point.

PROBLEM B. (Stability until an inflection point appears.) Our main result here is the following theorem. (Figures 3 and 6.)

THEOREM 3.2. *Suppose that $0 < \alpha < \pi$ where $\alpha$ is the prescribed angle of contact of the pendent drop with a horizontal plate, measured interior to the liquid. For small volumes stable configurations are obtained by choosing profile curves for which the drop tip $u_0$ is large and negative, and slicing it at the lowest point where the angle of inclination is $\alpha$. As $u_0$ increases from $-\infty$ there is generated a smooth one-parameter family of stable drops with increasing volume until a value $u_0^*$ is reached, for which the corresponding profile curve has an inflection point at the point of contact with the horizontal plate. This configuration is stable. Further stable configurations of greater volume are generated by decreasing $u_0$ and tracing the corresponding profile curve beyond its first inflection point to where the angle of inclination is again $\alpha$.*

*Proof.* We split the proof into three cases; $0 < \alpha < \pi/2$, $\alpha = \pi/2$, $\pi/2 < \alpha < \pi$.

*Case* 1. $0 < \alpha < \pi/2$. Consider the equation $v(r, \kappa) = v^* \equiv \sin \alpha$ where $0 < v^* < 1$. By continuity, there is a unique positive value $\kappa^*$ such that the slope function $v(r, \kappa^*)$ has $v^*$ as its first positive maximum, say $v(r^*, \kappa^*) = v^*$ and $v_r(r^*, \kappa^*) = 0$. By Lemma 2.6, if $\kappa > \kappa^*$, then $v(r, \kappa)$ increases to a point above $v^*$ while if $\kappa < \kappa^*$, $v(r, \kappa^*) < v^*$.

Our one-parameter family, determined by the equation $v(r, \kappa) = v^*$, will be smooth if grad $v = (v_r, v_\kappa) \neq (0, 0)$ along the solution set. The following lemma will show that when an inflection point first appears, with $\kappa = \kappa^*$, $r = r^*$, and $v_r(r^*, \kappa^*) = 0$, then $v_\kappa(r^*, \kappa^*) \neq 0$.

LEMMA 3.1. *Let $v(r) \equiv v(r, \kappa^*)$ be a solution to (2.6a) with $v(0) = 0$, $v'(0) = \kappa$ and set $h(r) = v_\kappa(r, \kappa^*)$. Suppose that $v'(r) > 0$ for $0 < r < r^*$ and that either $v(r^*) = 1$ or $v(r^*) < 1$ and $v'(r^*) = 0$. Then $h(r) > 0$ for $0 < r < r^*$ and if $v(r^*) < 1$ then $h(r^*) > 0$, $h'(r^*) < 0$. Finally, the first positive zero of $h(r)$ will occur to the left of the first positive zero of $v(r)$.*

*Proof.* We have that $h(r) \equiv h(r, \kappa^*)$ is a solution to (2.9b) satisfying $h(0) = 0$, $h'(0) = 1$. It follows from Lemma 2.6 that $h(r) \geqq 0$ for $0 < r < r^*$. Since $h(r)$ is a nontrivial solution to the linear differential equation, (2.9b), it follows that if $h(r) = 0$ then $h'(r) \neq 0$. We can conclude that $h(r) > 0$ for $0 < r < r^*$. Now we show that if $v(r^*) < 1$, $v'(r^*) = 0$, then $h(r^*) > 0$. Differentiate the integral identity (2.10a) with respect to $\kappa$, remembering that $v'(r^*) = 0$, to obtain

$$v(r^*)h(r^*)[1 - (r^*)^2/W] + \int_0^{r^*}(2rvh/W)dr = 0,\ W = (1 - v^2)^{1/2} .$$

Since $v(r)$ and $h(r)$ are positive on $(0, r^*)$ the integral is positive. Therefore $h(r^*) \neq 0$ and so must be positive.

To see that $h'(r)^* < 0$, we multiply (2.6a) by $h(r)$, multiply (2.9b) by $v(r)$, integrate and arrive at the identity

$$r[v'h - vh']_{r=0}^{r=r^*} = \int_0^{r^*}(rhv^3/W^3)dr . \tag{3.1}$$

The integral is positive, $v'(r^*) = 0$, $v(r^*) > 0$, implying that $h'(r^*) < 0$.

Finally, let $b$ be the first positive zero of $v(r)$. Suppose $h(r) > 0$ for $0 < r < b$ with $h(b) \geqq 0$, $v(b) = 0$, $v'(b) < 0$ and $v(r) > 0$ for $0 < r < b$. The integral in (3.1) would then be positive while the left hand side is not, proving the lemma.

We continue with the proof of Theorem 3.2. For $\kappa_1 > \kappa^*$ our one-parameter family consists of those curve segments $v(r, \kappa_1)$, $0 \leqq r \leqq r_1$ which are increasing on $[0, r_1]$ with $v(r_1, \kappa_1) = v^*$ and $v_r(r_1, \kappa_1) > 0$. The volume of the drop generated by this segment is given by (2.13), $T = V/\pi = r(v - rv')$. For $\kappa > \kappa^*$, the equation $v(r, \kappa) = v^*$ determines $r$ as a function of $\kappa$ with $dr/d\kappa = -v_\kappa/v_r = -h/v'$

$$dT/d\kappa = (dr/d\kappa)(v - rv') - rd(rv')/d\kappa .$$

Now $dr/d\kappa = -h/v' < 0$ by Lemma 3.1 and $d(rv')/d\kappa \geqq 0$ by Lemma 2.6. Therefore $dT/d\kappa < 0$, for $\kappa > \kappa^*$.

At the point of inflection, $\kappa$ cannot be used as a parameter since $v_r = 0$, but $h = v_\kappa \neq 0$ implying that $r$ may be used as a parameter with $\kappa'(r) = -v'/h$. At the point of inflection $\kappa = \kappa^*$, $r = r^*$, $v_r(r^*, \kappa^*) = 0$ and so $\kappa'(r^*) = 0$. It follows, using (2.6a) that $T'(r) = r^2v/W$ at $r = r^*$. Therefore $T'(r^*) > 0$. By continuity $T'(r)$ will be positive in some interval about $r = r^*$, implying stability beyond the appearance of an inflection point in the profile curve.

The stability of small drops will be dealt with in § IV.

*Case* 2. $\alpha = \pi/2$. Since the differential equation for $v(r, \kappa)$ breaks down when $v = 1$, we shall work with the parametric representation of the profile curves (2.2). We need the following lemmas.

LEMMA 3.2. *There is a* $\kappa^* > 0$ *such that the corresponding profile curve has its first vertical tangent at an inflection point. Thus there is an* $s^* > 0$ *so that* $\psi_s(s, \kappa^*) > 0$ *for* $0 \leqq s < s^*$, $\psi(s^*, \kappa^*) = \pi/2$ *and* $\psi_s(s^*, \kappa^*) = 0$.

*If* $\kappa > \kappa^*$ *then* $\psi(s, \kappa)$ *increases to a value greater than* $\pi/2$.

*If* $\kappa_1 > \kappa_2 \geqq \kappa^*$, *then* $\psi(s, \kappa_1)$ *lies to the left of* $\psi(s, \kappa_2)$ *for* $0 < \psi \leqq \pi/2$, *and if* $\psi(s_1, \kappa_1) = \psi(s_2, \kappa_2) = \bar{\psi}$ *where* $0 < \bar{\psi} \leqq \pi/2$, *then* $r(s_1, \kappa_1) < r(s_2, \kappa_2)$ *and* $(r\psi_s)(s_1, \kappa_1) > (r\psi_s)(s_2, \kappa_2)$.

*Proof.* The existence of $\kappa^*$ and the corresponding profile curve follows from Theorem 2.5. The rest of the lemma is a restatement of Lemma 2.6 for if $\kappa_1 > \kappa_2$ and $\psi(s_1, \kappa_1) = \psi(s_2, \kappa_2)$ then $r(s_1, \kappa_1) < r(s_2, \kappa_2)$, $(r\psi_s)(s_1, \kappa_1) > (r\psi_s)(s_2, \kappa_2)$ by Lemma 2.6. This means that $ds_1/d\psi < ds_2/d\psi$ implying that $\psi(s, \kappa_1)$ lies to the left of $\psi(s, \kappa_2)$.

LEMMA 3.3. *Let* $\kappa_1 \geqq \kappa^*$ *be chosen so that* $\psi(s, \kappa_1)$ *is increasing for* $0 \leqq s \leqq s_1$ *where* $\psi(s_1, \kappa_1) = \pi/2$ *and* $\psi_s(s_1, \kappa_1) \geqq 0$. *Let* $\omega(s) = \psi_\kappa(s, \kappa_1)$ *and* $\rho(s) = r_\kappa(s, \kappa_1)$. *Then* $\rho(s) < 0$ *for* $0 < s \leqq s_1$ *and* $\omega(s) > 0$ *for* $0 < s < s_1$. *Furthermore if* $\kappa_1 = \kappa^*$, *and* $s_1 = s^*$ *then* $\omega(s^*) > 0$ *as well.*

*Proof.* By Lemma 3.2, $\psi(s, \kappa)$ lies to the left of $\psi(s, \kappa_1)$ if $\kappa > \kappa_1$, which shows that $\omega(s) \geqq 0$ for $0 \leqq s \leqq s_1$. Now $\omega(0) = 0$, $\omega'(0) = 1$, so $\omega(s)$ is initially positive. We integrate (2.7a) to obtain

$$\rho(s, \kappa_1) = -\int_0^s \omega(\sigma, \kappa_1) \sin \psi(\sigma, \kappa_1) d\sigma$$

from which it follows that $\rho(s, \kappa_1) < 0$ for $0 < s \leqq s_1$.

We next show that $\omega(s) > 0$ for $0 < s < s_1$. If not, there is a smallest positive value, $\bar{s}$, $0 < \bar{s} < s_1$ with $\omega(\bar{s}) = \omega'(\bar{s}) = 0$. The equation $\psi(s, \kappa) = \psi(\bar{s}, \kappa_1) = \bar{\psi}$ determines $s$ as a function of $\kappa$, with $s'(\kappa) = -\omega/\psi_s$.

By Lemma 3.2 we conclude that $d(r\psi_s)/d\kappa \geqq 0$, ($\psi = \bar{\psi}$, fixed). But recall that $dr/d\kappa$ ($\psi$ fixed) is negative, for if $\psi$ is fixed then so is $\bar{v} = \sin \bar{\psi}$ and then $dr/d\kappa = -v_\kappa/v_r = -h/v' < 0$ by Lemma 3.1. If follows that $d(\psi_s)/d\kappa$ is positive. However, a direct calculation yields

$$d\psi_s/d\kappa = \psi_{ss}(-\omega/\psi_s) + \omega_s \ .$$

If $\omega(\bar{s}) = \omega'(\bar{s}) = 0$ then $d\psi_s/d\kappa = 0$, a contradiction. Finally, we wish to show that if $\kappa_1 = \kappa^*$, then $\omega(s^*) \equiv \omega(s^*, \kappa^*) > 0$. Take the integral identity (2.11) and differentiate with respect to $\kappa$. Recalling that $\psi_s(s^*, \kappa^*) = 0$, $\cos\psi(s^*, \kappa^*) = 0$ we find that

$$r^2(s^*, \kappa^*)\omega(s^*) = \int_0^{s^*}[4r\omega \sin\psi \cos\psi - 2\rho \cos^2\psi]ds \,.$$

The integral is positive, hence $\omega(s^*) > 0$.

*Proof of Theorem* 3.2, *Case* 2. The proof proceeds as in Case 1. If $\kappa > \kappa^*$ then the curve $\psi(s, \kappa)$ intersects $\psi = \pi/2$ with $\psi_s$ positive. Thus the equation $\psi(s, \kappa) = \pi/2$ allows us to use $\kappa$ as parameter. We write

$$T = (\text{Volume})/\pi = r(\sin\psi - r\psi')$$

and we are to show $dT/d\kappa < 0$. We find $dT/d\kappa = (dr/d\kappa)[\sin\psi - r\psi'] - rd(r\psi')/d\kappa$, where we differentiate with respect to $\kappa$ holding $\psi$ fixed. By Lemma 3.2 we have $d(r\psi_s)/d\kappa \geqq 0$. Furthermore $dr/d\kappa = (r_s)s'(\kappa) + r_\kappa = (\cos\psi)s'(\kappa) + \rho = \rho$ when $\psi = \pi/2$. Therefore, by Lemma 3.3 $dr/d\kappa < 0$ showing that $dT/d\kappa < 0$ for $\kappa > \kappa^*$.

When the point of inflection appears we have $\psi(s^*, \kappa^*) = \pi/2$, $\psi_s(s^*, \kappa^*) = 0$, and $\omega(s^*, \kappa^*) = \psi_\kappa(s^*, \kappa^*) > 0$. As in Case 1 this means that $\kappa$ is not a differentiable parameter, but $s$ is. We easily compute that $\kappa'(s) = -\psi_s/\omega$ so that $\kappa'(s^*) = 0$. We set $T = V/\pi$ and use (2.12) to find

$$dT/ds = 2(dr/ds)(ru + \sin\psi) + r^2(du/ds) \,.$$

But $dr/ds = (\cos\psi) + \rho s'(\kappa)$ and $du/ds = \sin\psi + \nu s'(\kappa)$; so at $s = s^*$ $\psi = \pi/2$ we see that $dr/ds = 0$ and $du/ds = 1$. This means that $dT/ds = r^2 > 0$ implying that the drop is stable when the point of inflection appears.

*Case* 3. $\pi/2 < \alpha < \pi$. For $\kappa$ large a profile curve will have a bulge ($\psi = \pi/2$) at a point $(r_1, u_1)$. The radius $r$ will then decrease to a value $r_2 < r_1$ where a neck forms. Between $r_2$ and $r_1$, $\psi > \pi/2$. From the tip of the profile to $(r_1, u_1)$ the slope function $v(r)$ is a solution to (2.6a) with $v(0) = 0$, $v'(0) = \kappa$ and satisfying $v'(r) > 0$ for $0 < r < r_1$ with $v(r_1) = 1$. For a bulge to exist at $(r_1, u_1)$ it is also necessary that $v'(r_1) > 0$.

Between the bulge and the neck the slope function $w(r)$ is a solution to (2.6b) on the interval $r_2 < r < r_1$ and satisfying $w(r_2) = w(r_1) = 1$, $w'(r_1) = v'(r_1) > 0$.

LEMMA 3.4. *Let $w(r, \kappa)$, $r_2 \leqq r \leqq r_1$ be the slope function for a profile curve between the neck and bulge. Then $w''(r) > 0$ for $r_2 < r < r_1$, $w(r, \kappa)$ has a positive minimum $w_m(\kappa)$ which approaches zero as $\kappa \to +\infty$.*

*Proof.* If we differentiate (2.4), $(rw)' + ru = 0$, and solve for $w''(r)$, we obtain $w'' = -u' + [(w - rw')/r^2]$. But $u' < 0$ as $\psi > \pi/2$ and $w - rw' > 0$ by the volume formula (2.13), showing that $w''(r) > 0$.

At the unique minimum for $w(r)$ at $\bar{r}$, $r_2 < \bar{r} < r_1$ we have $w'(\bar{r}) = 0$, $w''(\bar{r}) > 0$. By substitution into (2.6b) we find that $w(\bar{r}) > 0$.

By applying the integral identity (2.10b) for the slope function $v(r, \kappa)$ between 0 and $r_1$, we find

$$\frac{1}{2}[r_1 v'(r_1)]^2 + \int_0^1 r^2(v)(v/W)dv = 1/2 .$$

As $\kappa \to +\infty$, $r(v) \Rightarrow 0$ showing that $r_1 v'(r_1) \to 1$ as $\kappa \to +\infty$.

Now take the integral formula which corresponds to (2.10a) for $w(r)$ and integrate between $\bar{r}$ and $r_1$. So $w'(\bar{r}) = 0$, $w(r_1) = 1$, $W(r_1) = (1 - w^2(r_1))^{1/2} = 0$. We find

$$\frac{1}{2}[r_1 w'(r_1)]^2 = \bar{r}^2 W(\bar{r}) + \int_{\bar{r}}^{r_1} 2r W(r)dr + (1 - w^2(\bar{r}))/2 . \tag{3.2}$$

But as $\kappa \to +\infty$, $r_1 w'(r_1) \to 1$, $r_1 \to 0$, and $0 \leqq W(r) \leqq 1$. It follows that $w(\bar{r}) \to 0$.

As in Case 1 we shall set $h(r, \kappa) = w_\kappa(r, \kappa)$.

LEMMA 3.5. *Let $w(r, \kappa_1)$, $r_2 \leqq r \leqq r_1$ be the slope function of a profile curve between the neck and bulge, with its positive minimum at $\bar{r}$. Set $w(r) \equiv w(r, \kappa_1)$ and $h(r) \equiv w_\kappa(r, \kappa_1)$. Then $h'(r) > 0$ for $r_2 < r < r_1$ and $h(r)$ vanishes at a point $c$ where $\bar{r} < c$.*

*Proof.* Take the integral identity (3.2) with lower limit $\bar{r}(\kappa_1)$ and upper limit $r_1(\kappa_1)$, differentiate with respect to $\kappa$ to obtain

$$\begin{aligned} &[r_1 w'(r_1)]d(r_1 w'(r_1))/d\kappa + (\bar{r}^2 w(\bar{r})h(\bar{r})/W(\bar{r})) \\ &\qquad + 2\int_{\bar{r}}^{r_1} (rwh/W)dr = -w(\bar{r})h(\bar{r}) . \end{aligned} \tag{3.3}$$

We first note that $h(r_1) > 0$. To see this we compute $h(r_1, \kappa_1)$ in parametric coordinates. Let the corresponding $s$ coordinate be $s_1$, so that $r_1 = r(s_1, \kappa_1)$. A direct computation yields $h(r_1, \kappa_1) = -\rho\psi_s + \omega \cos\psi$ where the right hand side is evaluated at $(s_1, \kappa_1)$ and where $\rho = r_\kappa$, $\omega = \psi_\kappa$ as defined in (2.7). At a bulge $\psi_s > 0$ and $\psi = \pi/2$.

But by Lemma 3.3 $\rho < 0$ and so $h(r_1, \kappa_1) = h(r_1) > 0$.

Suppose that $h(r) > 0$ for $\bar{r} < r < r_1$. By Lemma 2.6 $d(r_1 w'(r_1))/d\kappa \geqq 0$ and $r_1 w'(r_1) > 0$. It follows that the left hand side of (3.3) is positive, hence $h(\bar{r})$ would be negative. Therefore $h(r)$ vanishes at a point $c$, $\bar{r} < c < r_1$, with $h(r)$ positive for $r > c$.

Since $h(r)$ is a solution to the linear differential equation (2.9c) with $h(c) = 0$ and $h(r)$ positive for $r > c$, it follows that $h'(c) > 0$. Now it follows from (2.9c) that $(rh')'$ is positive for $r > c$ and so $rh'$ is increasing. Thus $h'(r) > 0$ for $r \geqq c$.

Finally, we claim that $h'(r) > 0$ for $r < c$ as well. If not then there would exist an $r_3 < r$ with $h(r_3) < 0$, $h'(r_3) = 0$, $h''(r_3) \geqq 0$; an impossibility by (2.9c).

We are now in a position to prove the following extension of Lemma 2.6.

LEMMA 3.6. *Let* $w(r) \equiv w(r, \kappa_1)$ *be a solution to* (2.6b) *on the interval* $r_2 < r < r_1$ *where* $w(r_2) = w(r_1) = 1$. *As in Lemma* 3.5 *suppose* $w'(\bar{r}) = 0$ *and let* $h(c) = 0$ *where* $\bar{r} < c < r_1$. *Let* $\tilde{w}$ *be chosen so that* $w(c) \leqq \tilde{w} < 1$. *Then* $d(rw')/d\kappa$ *with* $w(r, \kappa) = \tilde{w}$ *is positive at* $\kappa = \kappa_1$.

*Proof.* Between $\bar{r}$ and $r_1$, $w'(r) \equiv w_r(r, \kappa_1) > 0$ and the equation $w(r, \kappa) = \tilde{w}$ determines $r$ in terms of $\kappa$ with $\tilde{r} = r(\kappa_1)$. Take the integral identity for solutions to (2.6b) analoguous to (2.10b) integrated between $\tilde{w}$ and 1, and differentiate with respect to $\kappa$ obtaining

$$\tilde{r}w'(\tilde{r})d[\tilde{r}w'(\tilde{r})]/d\kappa = r_1 w'(r_1)d[r_1 w'(r_1)]/d\kappa$$
$$-\int_{\tilde{w}}^{1} 2r(dr/d\kappa)(v/W)dv \ .$$

But $\tilde{w} < w < 1$ means that $c < r < r_1$ and so $dr/d\kappa = -h/w_r < 0$. But by Lemma 2.6 we have $d[r_1 w'(r_1)]/d\kappa \geqq 0$ and so $d[\tilde{r}w'(\tilde{r})]/d\kappa > 0$.

LEMMA 3.7. *Let* $w(r, \kappa)$ *be the solution to* (2.6b) *on* $r_2(\kappa) < r < r_1(\kappa)$ *with* $w(r_2) = w(r_1) = 1$, *and which attains its positive minimum at* $\bar{r}(\kappa)$. *Then* $\bar{r}'(\kappa) < 0$ *and* $d[w(\bar{r}(\kappa), \kappa)]/d\kappa < 0$.

*Proof.* $\bar{r}(\kappa)$ is determined by the equation $w_r(r, \kappa) = 0$. Since $w_{rr}(\bar{r}, \kappa) > 0$ it follows that $\bar{r}(\kappa)$ is differentiable with $\bar{r}'(\kappa) = -h'(\bar{r})/w''(\bar{r}) < 0$, as $h'(\bar{r}) > 0$ by Lemma 3.5. But then $w'(\kappa) = w_r(\bar{r}, \kappa)\bar{r}'(\kappa) + w_\kappa = 0 + h(\bar{r}) < 0$.

*Proof of Theorem* 3.2, *Case* 3. For $\pi/2 < \alpha < \pi$ we have $0 <$

$\bar{w} = \sin \alpha < 1$. The slope function $w(r, \kappa)$, $r_1(\kappa) < r < r_2(\kappa)$ has a minimum $w_m(\kappa)$ which decreases to 0 as $\kappa \to +\infty$. Thus there is a unique $\kappa^*$ such that $w(r, \kappa^*)$ touches the line $w = \bar{w}$ at a point $r^*$ with $w_r(r^*, \kappa^*) = 0$.

If $\kappa > \kappa^*$ then $w(r, \kappa^*)$ will cut the curve $w = \bar{w}$ twice at values $r_2(\kappa) < a(\kappa) < b(\kappa) < r_1(\kappa)$. Initially, the small drops will contain no inflection points which corresponds to the profile curve generated by $w(r, \kappa)$ from $b(\kappa)$ to $r_1(\kappa)$ where the bulge is, and then continuing to the drop tip. As before we set $T = (\text{Volume})/\pi = r(w - rw') > 0$.

As long as $w'(r) \neq 0$ we may use $\kappa$ as the parameter. We now show that $dT/d\kappa < 0$ for $\kappa > \kappa^*$. For large $\kappa$ we have $w_\kappa(b(\kappa), \kappa) = h(b) > 0$. In this case we have $db/d\kappa = -h(b)/w'(b) < 0$. Now $dT/d\kappa = (db/d\kappa)(w - rw') - b(\kappa)d(rw')/d\kappa$. By Lemma 3.6, $d(rw')/d\kappa$ is positive when $h$ is positive. Thus in this case $dT/d\kappa < 0$.

Now suppose that $w_r[b(\kappa), \kappa] = w'(b) > 0$ but that $h(b) \leqq 0$, as will be the case for smaller $\kappa > \kappa^*$. A direct calculation, making use of (2.6b) yields

$$dT/d\kappa = (-r^2 w/W)(-h/w_r) + r(h - rh'), \; W = (1 - w^2)^{1/2} .$$

Under present conditions, we see that $dT/d\kappa < 0$.

At $\kappa = \kappa^*$, $w_r(r^*, \kappa^*) = 0$ so that $\kappa$ is no longer an admissible parameter for the family. However, $w_\kappa(r^*, \kappa^*) = h(r^*) < 0$ implying that we may solve for $\kappa = \kappa(r)$ in a neighborhood of $r = r^*$. In this case we obtain $\kappa'(r) = -w'(r)/h(r)$ and in particular $\kappa'(r^*) = 0$. Now compute $T'(r)$ when $r = r^*$. With the aid of (2.6b) and the fact that $\kappa'(r^*) = 0$ we obtain $T'(r) = -r^2 w(r)/W(r)$ at $r = r^*$. Hence $T'(r^*) < 0$ and the configurations are stable in some interval about $r = r^*$.

For the case when the angle of contact $\alpha = 0$, the process of drop formation is altered. The following theorem covers this case.

THEOREM 3.3. *If $\alpha = 0$, drop formation proceeds by considering the set of profile curves $u(r, \kappa)$, $\kappa > 0$, and for $0 < r < r(\kappa)$ where $u_r[r(\kappa), \kappa] = 0$. As $\kappa$ increases from 0, the volume will also increase until a maximum volume is attained. This point will be reached before the profile curves develop a vertical tangent, and $dr/d\kappa$ is negative.*

*Proof.* For small $\kappa$, $u(r, \kappa) \cong -2\kappa J_0(r)$ so $r(\kappa) \to \beta_0$ the first positive zero of $J_1(r) = J_0'(r)$ as $\kappa \to 0$. In § IV we will show that these configurations are stable.

If $u(r, \kappa)$ does not attain a vertical tangent, then by Lemma 2.1, the second critical point for $v(r, \kappa)$ occurs when $v$ is negative. Therefore, if $v(r_1, \kappa_1) = 0$ where $r_1$ is the first positive zero of $v(r, \kappa)$ we will have $v_r(r, \kappa) < 0$. This implies that $v(r, \kappa) = 0$ may be solved differentiably for $r(\kappa)$ with $dr/d\kappa = -h(r)/v'(r)$.

If $T = (\text{Volume})/\pi$, then a direct calculation yields that $dT/d\kappa = r(h - rh')$ where $v = 0$ is fixed. By Lemma 3.1, the first positive zero for $h(r)$ occurs to the left of the first positive zero of $v(r)$. Also, as $\kappa \to 0$, $v(r)$ and $h(r)$ coalesce. It follows that for small $\kappa$, $dT/d\kappa$ is positive and $dr/d\kappa$ is negative.

Furthermore, for small $\kappa$, $h(r)$ and $h'(r)$ are both negative at $r(\kappa)$. If for some $\kappa_1 > 0$ we have $dr/d\kappa = 0$ it would mean that $h[r(\kappa_1)] = 0$. This would mean that $r(\kappa_1)$ is the second positive zero for $h(r, \kappa_1)$. Since $h(r)$ satisfies the linear D. E. (2.9b) we must have $h'[r(\kappa_1)] > 0$. This means that $dT/d\kappa < 0$. Therefore, as long as $dT/d\kappa > 0$ we must have $dr/d\kappa < 0$.

Finally, it will follow from Theorem 3.8 that by the time a profile curve with a vertical tangent is reached, a maximum volume will have been attained.

PROBLEM C. (Stability through bulge and neck.) Consider a tube with a circular opening of radius, $r_0$, starting with the solution $u \equiv 0$ and exposed volume zero. If the family of profile curves is written in the parametric form (2.2) then the equation $r(s, \kappa) = r_0$ determines the family of profile curves. From this it follows that $\kappa$ may be used as the parameter unless $r_s = \cos \psi = 0$, (i.e., the profile curve has a vertical tangent).

Recall that there is a unique $\bar{u}_0 = -2\bar{\kappa}_0$ such that the corresponding profile curve has a vertical tangent at a point $(\bar{r}_1, \bar{u}_1)$ where $\bar{r}_1\bar{u}_1 = -1$, $\bar{r}_1 \cong .9176$, which is also an inflection point on the curve.

As we shall see, if $r_0 < \bar{r}_1$ then stable configurations with bulges will always occur, while if $r_0$ is substantially larger than $\bar{r}_1$ then all stable configurations may be expressed nonparametricly, $u = u(r, \kappa)$.

THEOREM 3.4. *Consider the one-parameter family $u(r, \kappa)$ of profile curves satisfying* (2.3). *Let $V(r_0, \kappa)$ denote the exposed volume determined by the profile curve $u(r, \kappa)$, $0 \leqq r \leqq r_0$.*

(a) *$V_\kappa(r_0, 0)$ is positive for $0 < r_0 < \gamma$ where $\gamma$ is the smallest positive solution to $rJ_0(r) + 2J_0'(r) = 0$.*

(b) *If $r_0 < \gamma$, then $V_\kappa(r_0, \kappa)$, $\kappa \geqq 0$ will remain positive until after an inflection point appears on the profile curves.* (Figure 4.)

*Proof.* By (2.13) $T = (\text{Volume})/\pi = r(v - rv')$. Since $r = r_0$ we find that $dT/d\kappa = r(h - rh')$ where $h(r)$ is a solution to (2.9b). For a fixed $\kappa$, $dT/d\kappa$ is a function of $r$. A direct calculation using (2.9b) yields $(dT/d\kappa)_r = r^2h/W^3$ where $W = (1 - v^2)^{1/2}$. Since $h(0)=0$, $h'(0) = 1$ we see that $dT/d\kappa$ is positive at least as long as $h(r)$ is. But by Lemma 3.1, the first positive zero of $h(r)$ occurs after the first positive zero of $v'(r)$, proving (b).

If $\kappa = 0$, then $W \equiv 1$, $h(r) = J_1(r)$ and the condition $dT/d\kappa = 0$ reduces to $J_1(r) - rJ_1'(r) = 0$. This is equivalent to the stated condition in (a).

REMARKS. It will be shown in § 4 that due to nonsymmetric perturbations an axially symmetric pendent drop (other than $u \equiv 0$) whose profile curve contains a horizontal tangent away from the axis cannot be stable for Problem C. This has the following implications. (See also [13].)

(1) If $\kappa$ is small, then $u(r, \kappa) \cong -2\kappa J_0(r)$. By Theorem 3.4, this will determine a drop stable with respect to symmetric perturbations if $r_0 < r(\kappa)$ where $r(\kappa)$ is the first positive root of $h-rh'$, with limit $r(\kappa) = \gamma$ as $\kappa \to 0$. For nonsymmetric perturbations, the limit of stability is determined by the condition $u_r(r, \kappa) = 0$ with solution $\tilde{r}(\kappa)$. Limit $\tilde{r}(\kappa)$ as $\kappa \to 0$ is $\beta_1$ where $J_1(\beta_1) = J_0'(\beta_1) = 0$. It is easy to check that $\beta_1 < \gamma$ which shows that for $r_0 > \beta_1$, the nonsymmetric perturbations govern stability.

(2) Suppose we choose $r_0$ slightly smaller than $\beta_1$. Then $u \equiv 0$, $0 \leqq r \leqq r_0$ is initially stable and as we increase $\kappa$ the volume will increase (indicating symmetric stability). In this case, the drop configuration will become unstable, not when the volume ceases to increase, but rather when $u_r(r_0, \kappa) = 0$ for some positive $\kappa$.

THEOREM 3.5. *Suppose $r_0 < \bar{r}_1$ where $(\bar{r}_1, \bar{u}_1)$ are the coordinates of the simultaneous inflection point and vertical tangent for the profile curve $u(r, \bar{\kappa}_0)$.*

*As $\kappa$ increases from 0, the profile curves determine a one-parameter family of stable drops until a values $\kappa_1$ is reached, at which point the profile curve develops a vertical tangent at $r = r_0$. This configuration is stable.*

*A further one-parameter family of stable drops with increasing volume is generated by decreasing $\kappa$, with the drop profile having a bulge whose radius exceeds $r_0$.*

*At a point $\kappa_2$, $0 < \kappa_2 < \kappa_1$, the bulging drop profile will again develop a vertiɔal tangent at the opening, forming a neck. This configuration remains stable. Further stable drops are now formed by once again increasing $\kappa$. The resulting drops possess both a* *neck and a bulge.*

*Finally if* $r_0 = \bar{r}_1$ *the drops* $u(r, \kappa)$, $0 \leqq r \leqq r_0$, $0 \leqq \kappa \leqq \bar{\kappa}_1$ *are stable. In this case, further stable drops of greater volume are generated by increasing* $\kappa$ *beyond* $\bar{\kappa}_1$. *The resulting drop will possess both a bulge and a neck, although the radius of the bulge will be small than* $r_0$. (Figures 4 and 5.)

*Proof.* Let $r_0 \leqq \bar{r}_1$. By Theorem 2.5, there will be a smallest positive value $\kappa_1$ such that the profile curve $u(r, \kappa_1)$ will attain a vertical tangent when $r = r_0$. For $0 < \kappa < \kappa_1$ the slope functions $v(r, \kappa)$ satisfying (2.6a) will be increasing on the interval $0 \leqq r \leqq r_0$, (for any $\kappa$, the first root of $v_r(r, \kappa) = 0$ will be greater than $\bar{r}_1$). It follows that none of the curves $u(r, \kappa)$, $0 \leqq r \leqq r_0$ where $\kappa < \kappa_1$ has an inflection point on the interval. Thus by Theorem 3.4, the volume increases with $\kappa$ and all profile curves define stable drops until $\kappa = \kappa_1$.

At $\kappa = \kappa_1$, we express the profile curve parametricly using (2.2), with $\psi(s_1, \kappa_1) = \pi/2$. The constraint is $r(s, \kappa) = r_0$. Since $r_s(s_1, \kappa_1) = \cos\psi_1 = 0$, $\kappa$ is not an admissible parameter, but by Lemma 3.3 $\rho(s_1) = r_\kappa(s_1, \kappa_1) < 0$ showing that $s$ is an admissible parameter for the family at the point of vertical tangency.

We have $\kappa'(s) = -r_s/r_\kappa = -\cos\psi/\rho(s)$. Thus when $s = s_1$, $\kappa = \kappa_1$ and $\kappa'(s_1) = 0$, we may write $T(s) = (\text{Volume})/\pi = r_0(\sin\psi - r_0\psi'(s))$, and find that $T'(s_1) = -\psi''(s_1)r_0^2 = r_0^2 > 0$, as $\psi''(s) = -1$ whenever $\psi = \pi/2$.

If $r_0 < \bar{r}_1$ then $\psi_s(s_1) > 0$ where $\psi(s_1, \kappa_1) = \pi/2$. It follows that $\kappa'(s)$ will change sign as $s$ increases through $s_1$. Thus stable drops of larger volume are generated by decreasing $\kappa$. On the other hand if $r_0 = \bar{r}_1$ then $\psi_s(\bar{s}_1) = 0$, $\psi(\bar{s}_1, \bar{\kappa}_1) = \pi/2$. In this case a direct calculation shows that $\kappa''(\bar{s}_1) = 0$, $\kappa''(\bar{s}_1) = -1/\rho(\bar{s}_1) > 0$. Thus as $s$ increases through $\bar{s}_1$, $\kappa$ will continue to increase as stable drops of larger volume are formed.

For $r_0 < \bar{r}_1$, once past the vertical tangent our stable drops will have a bulge. Between the neck and the bulge $\pi/2 < \psi < \pi$ so once again we may use $\kappa$ as parameter. The slope function $w(r, \kappa) = \sin\psi$ satisfies (2.6b) for $r_2 < r < r_1$ where $w(r_2) = w(r_1) = 1$. Its properties were discussed in Lemmas 3.4–3.7. If $r_0 \in (r_2, r_1)$ then $T(\kappa) = r_0[w - r_0 w']$, which implies that $dT/d\kappa = r_0[h - r_0 h']$. We claim that $dT/d\kappa < 0$ if $r_2 < r_0 < r_1$.

A direct calculation, using (3.5) shows $(dT/d\kappa)_r = -r^2 h/W^3$ where $W = (1 - w^2)^{1/2}$. Since $h'(r) > 0$ for all $r$ by Lemma 3.5, it follows that $dT/d\kappa$ has a maximum at $r = c$ where $h(c) = 0$. But for $r = c$, $dT/d\kappa = -c^2 h'(c) < 0$. Therefore, $dT/d\kappa < 0$ between the neck and the bulge implying that the drops remain stable.

Finally we show that as the neck forms, the drop still remains stable. Once again $\psi = \pi/2$ so $\kappa$ cannot be used as parameter. We wish to use $s$ as parameter. Let $\psi(s_2, \kappa_2) = \pi/2$, $r(s_2, \kappa_2) = r_2$ so we need to solve $r(s, \kappa) = r_2$. This determines $\kappa = \kappa(s)$ at $s = s_2$ if $r_\kappa(s_2, \kappa_2) = \rho(s_2) \neq 0$, in which case $\kappa'(s) = -\cos\psi/\rho(s)$.

To show that $\rho(s_2) \neq 0$ we compute $h(r, \kappa_2) = w_\kappa(r, \kappa_2)$ in terms of $s$ and $\kappa_2$. Since $w = \sin\psi$ it follows that $h(r, \kappa) = (\cos\psi)(\psi_s s'(\kappa) + \psi_\kappa)$ where $r(s, \kappa) =$ constant implies that $s'(\kappa) = -\rho/\cos\psi$. We find that

$$h(r, \kappa_2) = -\rho(s, \kappa_2)\psi_s(s, \kappa_2) + (\cos\psi)\omega(s, \kappa_2)$$

where $r = r(s, \kappa_2)$, $\rho(s, \kappa) = r_\kappa(s, \kappa)$, and $\omega(s, \kappa) = \psi_\kappa(s, \kappa)$. As $s \to s_2$, $\psi \to \pi/2$ and we get $h(r_2, \kappa_2) = -\rho(s_2, \kappa_2)\psi_s(s_2, \kappa_2)$. But as $r \to r_2$ from the right, $h(r, \kappa_2)$ is negative and decreasing. Hence $h(r_2, \kappa_2) < 0$. It follows that $\psi_s(s_2, \kappa_2) > 0$ (we're at the neck) and $\rho(s_2, \kappa_2) > 0$.

We may use $s$ as a parameter when $s = s_2$, $\psi = \pi/2$. Repeating our calculations at the bulge we get $T'(s_2) = r_2^2 > 0$ and the drop remains stable.

PROBLEMS B AND C. (Monotonicity of Drop Height.) We now prove the remaining assertions concerning drop formation as stated in the introduction. In particular, we show that drop height increases monotonically with volume as long as the drops are stable. We first prove the following lemma.

LEMMA 3.8. *For $\bar{\kappa} > 0$ let $\langle r(s, \bar{\kappa}), u(s, \bar{\kappa}), \psi(s, \bar{\kappa})\rangle$ be a solution to* (2.2) *with its derivative with respect to $\kappa$, $\langle \rho(s, \bar{\kappa}), \nu(s, \bar{\kappa}), \omega(s, \bar{\kappa})\rangle$ a solution to* (2.7), *and let $s_m > 0$ be the first positive zero of $\psi_s(s, \bar{\kappa})$ with $\psi_m = \psi(s_m, \bar{\kappa})$, $0 < \psi_m < \pi$. The following are true.*

(a) *$\psi_{ss} < 0$ if either $0 < s \leqq s_m$ or $\psi(s, \bar{\kappa}) \geqq \pi/2$.*

(b) *If $0 < \bar{\psi} < \psi_m$, then $d(\psi_s)/d\kappa > 0$, ($\psi$ fixed).*

(c) *$\omega(s, \bar{\kappa}) > 0$ for $0 < s \leqq s_m$.*

*Proof.* By differentiating (2.2c) one obtains $\psi_{ss} = [\cos\psi(ru + 2\sin\psi)/r^2] - \sin\psi < 0$ when $\psi \geqq \pi/2$. For $0 < \psi < \pi/2$, $0 < s < s_m$, $\psi_s(s, \bar{\kappa}) > 0$, we recall that $\psi_s(s, \bar{\kappa}) = v_r(r, \bar{\kappa})$ where $r = r(s, \bar{\kappa})$, which implies that $\psi_{ss} = (\cos\psi)v_{rr}$. However, by Lemma 2.2 $v_{rr} < 0$ and (a) follows.

From Lemmas 3.2 and 3.3 we can conclude that if $0 < \bar{\psi} \leqq \pi/2$ and $\bar{\psi} < \psi_m$ then $d(r\psi_s)/d\kappa \geqq 0$, $dr/d\kappa < 0$ where $\psi = \bar{\psi}$ is fixed. We conclude (b) if $0 < \bar{\psi} \leqq \pi/2$.

If $\bar{\psi}$ (and hence $\psi_m$) $> \pi/2$ we appeal to Lemmas 3.4-3.7. We have $w(r, \bar{\kappa}) = \sin\psi(s, \bar{\kappa})$, $h(r, \bar{\kappa}) = w_\kappa(r, \bar{\kappa})$, and there are values

$\bar{r} < c < r_1$ where $w_r(\bar{r}, \bar{\kappa}) = 0$, $w(r, \bar{\kappa}) = \sin\psi_m$, $h(c, \bar{\kappa}) = 0$, and $h_r(r, \bar{\kappa}) > 0$ on the interval. Let $\sin\psi_c = w(c, \bar{\kappa})$. It follows from Lemma 3.6 that if $\bar{\psi} < \psi_c$ then $d(r\psi_s)/d\kappa > 0$ and $dr/d\kappa \leqq 0$ proving (b) in this case. On the other hand if $\bar{\psi} > \psi_c$ we compute directly

$$d\psi_s/d\kappa = d(w_r)/d\kappa = w_{rr}(-h/w_r) + h_r$$

which is positive, proving (b).

The situation is now similar to that of Lemmas 2.6 and 3.2. In particular, (b) allows us to conclude that for $\kappa > \bar{\kappa}$, the graphs of the functions $\psi(s, \kappa)$ lie to the left of $\psi(s, \bar{\kappa})$ and at a given level, $\psi = \bar{\psi}$, possess a greater slope. It follows that $\omega(s, \bar{\kappa}) = \psi_\kappa(s, \bar{\kappa}) \geqq 0$ for $0 \leqq s \leqq s_m$. By repeating the argument in Lemma 3.3 we have that $\omega(s, \bar{\kappa}) > 0$ for $0 < s < s_m$.

Finally, if $\psi_m = \pi/2$ then $\omega(s_m, \bar{\kappa}) > 0$ by Lemma 3.3. Otherwise we use the identity

$$h(r, \bar{\kappa}) = -\rho\psi_s + \omega\cos\psi .$$

If $\psi_s = 0$, $\psi_m < \pi/2$ then $h > 0$ by Lemma 3.1 while if $\psi_s = 0$, $\psi_m > \pi/2$ then $h < 0$ by Lemma 3.5. In either case we conclude that $\omega(s_m, \bar{\kappa}) > 0$.

THEOREM 3.6. *Let $0 < \alpha < \pi$ and consider the one-parameter family of pendent drops as described in Theorem 3.2. The following are true.*

(a) *Drop height increases monotonically with volume throughout the range of stability. (In fact, drop height is a differentiable parameter for the entire family of stable drops.)*

(b) *The area of contact of the drop with the horizontal plate initially increases with volume, but will start to decrease before the maximum volume configuration is reached.*

(c) *The profile curve of a stable pendent drop never contains more than one inflection point.*

*Proof. Step* 1. Monotonicity of drop height, $H$, through the appearance of an inflection point on the profile curve.

Let $\psi(s, \bar{\kappa})$, $\bar{\kappa} > 0$ satisfy $\psi_s > 0$ for $0 \leqq s < s_m$ and $\psi_s(s_m, \bar{\kappa}) = 0$, $\psi(s_m, \bar{\kappa}) = \psi_m$. Let $0 < \alpha < \psi_m$ and suppose $\alpha = \psi(\bar{s}, \bar{\kappa})$ where $0 < \bar{s} < s_m$. We have, using (2.2b),

$$H(\alpha, \bar{\kappa}) = \int_0^\alpha (\sin\psi/\psi_s)d\psi . \tag{3.4}$$

Since $\psi_s > 0$, $s$ is regarded as a function of $(\psi, \kappa)$ with $\psi_s(\psi, \kappa) \equiv$

$\psi_s[s(\psi, \kappa), \kappa]$. This leads to

$$dH/d\kappa(\psi = \alpha \text{ fixed}) = \int_0^\alpha (-\sin\psi/\psi_s^2)(d\psi_s/d\kappa)d\psi$$

which, by Lemma 3.8, is negative.

At the point of inflection $\psi_s(s_m, \bar{\kappa}) = 0$, $\psi(s_m, \bar{\kappa}) = \psi_m = \alpha$. By Lemma 3.8 $\omega(s_m, \bar{\kappa}) = \psi_\kappa(s_m, \bar{\kappa}) > 0$ so that $s$ is a differetiable parameter for the family. Starting from $H = u - u_0 = u + 2\kappa$, one finds that $dH/ds(\psi = \alpha \text{ fixed}) = (\sin\alpha) + (u_\kappa + 2)\kappa'(s)$. But at $s = s_m$, $\kappa'(s_m) = -\psi_s/\omega = 0$ and so $dH/ds = \sin\alpha > 0$, proving Step 1.

*Step* 2. Beyond the first inflection point, $\alpha \geqq \pi/2$.

Let $(\bar{s}, \bar{\kappa})$ with $\psi(\bar{s}, \bar{\kappa}) = \alpha$ define a member of our one-parameter family after the first inflection point has appeared. Thus $\psi(s, \bar{\kappa})$, $0 \leqq s \leqq \bar{s}$, $\bar{\kappa} > 0$ increases to a maximum greater than $\alpha$ and, by Lemma 3.8, $\psi_{ss} < 0$ for $0 < s \leqq \bar{s}$. It follows that no profile curve of our one-parameter family can contain a second inflection point, so (c) is automatically true in this case.

Since $\psi_s(\bar{s}, \bar{\kappa}) < 0$ we use $\kappa$ as parameter. For $\alpha > \pi/2$ we compute $dr/d\kappa$ using the nonparametric equations. We find $dr/d\kappa$ $(v = \sin\alpha \text{ fixed}) = -h/v_r$. By Lemma 3.5 we find that $h < 0$, $v_r < 0$ and so $dr/d\kappa < 0$. If $\alpha = \pi/2$ we compute $dr/d\kappa$ using the parametric form

$$dr/d\kappa(\psi = \pi/2) = -(\omega\cos\psi + \rho\psi_s)/\psi_s\ .$$

Now the expression in parentheses is just $h(r, \kappa)$ written parametrically, and by Lemma 3.5 the limiting value of $h(r, \kappa)$ as $\alpha \to \pi/2$ is negative (were at the neck). Thus $dr/d\kappa < 0$ in this case also. This proves (b) for $\alpha > \pi/2$.

Moreover, if we set $T = (\text{Volume})/\pi$ we find from (2.12) that

$$dT/d\kappa(\psi = \alpha) = 2(dr/d\kappa)(ru + \sin\alpha) + r^2[(dH/d\kappa) - 2] \tag{3.5}$$

observing that since $H = u - u_0 = u + 2\kappa$ we have $dH/d\kappa = (du/d\kappa) + 2$. But $ru + \sin\alpha = -r\psi_s > 0$ from (2.2c). Since $dr/d\kappa < 0$, it follows that $dH/d\kappa \geqq 2$ if $dT/d\kappa \geqq 0$. This proves (a) for $\alpha \geqq \pi/2$.

*Step* 3. Beyond the first inflection point, $0 < \alpha < \pi/2$.

Let the equation $\psi(\bar{s}, \bar{\kappa}) = \alpha$ determine a profile curve with just one inflection point. Thus $\psi(s, \bar{\kappa})$, $0 \leqq s \leqq \bar{s}$ has a single critical point at $s = s_1$ with $\psi_s(s_1, \bar{\kappa}) = 0$, $\psi(s_1, \bar{\kappa}) = \psi_m > \alpha$, and $\psi_s(s, \bar{\kappa}) < 0$ for $s_1 < s \leqq \bar{s}$.

For $\alpha < \psi_m < \pi/2$ the profile curve can be expressed nonparametrically $u = u(r, \bar{\kappa})$, $0 \leqq r \leqq \bar{r}$ with the critical point at $r = r_1$. We have $h(r_1, \bar{\kappa}) > 0$ by Lemma 3.1. We show first that if $h(r, \bar{\kappa}) \geqq 0$ for $r_1 \leqq r \leqq \bar{r}$ then $dH/d\kappa(v \text{ fixed})$ and $dV/d\kappa(v \text{ fixed})$ are both positive. Note that $dr/d\kappa = -h/v_r \geqq 0$. Since $H = u + 2\kappa$ and $T = V/\pi = r(v - rv')$ we find that

$$dT/d\kappa(v \text{ fixed}) = (r^2v/W)(-h/v') + r(h - rh')$$
$$dH/d\kappa(v \text{ fixed}) = (v/W)(-h/v') + (\Phi + 2)\ .$$

From (3.1) we see that if $v' < 0$, $h \geqq 0$, and $v > 0$, then $h' < 0$, showing that $dT/d\kappa$ is positive. One also finds using (2.9a) and (2.9b) that $\phi_r(r, \bar{\kappa}) = h/W^3$ which is nonnegative. Therefore $\phi(r, \bar{\kappa}) + 2 > \phi(0, \bar{\kappa}) + 2 = 0$ showing that $dH/d\kappa$ is also positive. On the other hand, if $h < 0$ then $dr/d\kappa = -h/v' < 0$ and from (3.5) we conclude that if $dT/d\kappa \geqq 0$ then $dH/d\kappa \geqq 2$.

By Lemma 2.5 if $\psi_m < \pi/2$, the second critical point of $v(r, \bar{\kappa})$ occurs when $v$ is negative so that as long as $\psi_m < \pi/2$ a second inflection point cannot develop on the profile curve.

Now suppose that $\psi_m \geqq \pi/2$. Since $\alpha < \pi/2$ and our profile curve is assumed to possess a single inflection point, the profile curve must contain both a neck and a bulge (or a similtaneous vertical tangent and inflection point if $\psi_m = \pi/2$). Let the neck occur at $r = r_1$ and let $v(r, \bar{\kappa})$, $r \geqq r_1$ be the slope function for the profile curve beyond the neck. We assume that $v_r < 0$ for $r_1 < r \leqq \bar{r}$ where $v(\bar{r}, \bar{\kappa}) = \sin\alpha$ so that $\kappa$ is an admissible parameter. At the neck $h(r_1, \bar{\kappa}) \leqq 0$ and $h(r, \bar{\kappa}) < 0$ initially for $r > r_1$. We claim that if the profile curve determines a stable drop out to $(\bar{r}, \bar{u})$ then $h$ and $h'$ must be negative for $r_1 < r \leqq \bar{r}$. To see this we observe that for any $r \in (r_1, \bar{r}]$ the corresponding profile curve must be stable for Problem C. This means that $dT/d\kappa(r \text{ fixed}) = r(h - rh') > 0$ for $r_1 < r \leqq \bar{r}$. Near $r_1$, this is true by Theorem 3.5, so that $h'$ is initially negative. But should $h'$ vanish for some $r \in (r_1, \bar{r}]$ we would have $h - rh' < 0$ implying instability of our pendent drop for Problem C and hence also for Problem B. We may now repeat the previous argument using (3.5). We have $dr/d\kappa = -h/v' < 0$ and so $dT/d\kappa(v \text{ fixed}) \geqq 0$ implies that $dH/d\kappa \geqq 2$.

Finally we must show that when a profile curve develops a second inflection point, the drop of maximum volume will already have been formed. Since $\psi_{ss} < 0$ for $\psi \geqq \pi/2$ a second inflection point can only develop at a point beyond the neck with $\psi_0 < \pi/2$. This means that the slope function $v(r, \bar{\kappa})$, $r \geqq r_1$ beyond the neck will satisfy $v_r(r, \bar{\kappa}) \leqq 0$ for $r_1 < r \leqq \bar{r}$ where $v(\bar{r} \cdot \bar{\kappa}) = \sin\alpha$. Again we may assume that $h(r, \bar{\kappa})$ and $h_r(r, \bar{\kappa})$ are negative for $r_1 < r \leqq \bar{r}$ for otherwise the drop is unstable by the previous argument.

At $r = \bar{r}$, we have $v_r(\bar{r}, \bar{\kappa}) \leqq 0$ but $h(\bar{r}, \bar{\kappa}) < 0$ which means that we can use $r$ as parameter of the family. Now $\kappa'(r) = -v_r/h \leqq 0$. Since $\kappa$ was an increasing parameter for $\kappa < \bar{\kappa}$ we see that $r$ is a decreasing parameter at $r = \bar{r}$. Therefore, in order to prove instability, we need to show that $dT/dr(v = \sin\alpha) > 0$ at $r = \bar{r}$.

A direct computation gives

$$dT/dr(v \text{ fixed}) = (r^2 v/W) + r(h - rh')(-v'/h) \equiv G(r, \bar{\kappa}) .$$

There is an initial value $r_2$, $r_1 < r_2 \leqq \bar{r}$ with $v_r(r_2, \bar{\kappa}) = 0$. At this point $G(r_2, \bar{\kappa}) > 0$. Now write $F(r, \bar{\kappa}) = h(r, \bar{\kappa})G(r, \bar{\kappa})$ so that $F(r_2, \bar{\kappa}) < 0$. We must show that $F(\bar{r}, \bar{\kappa}) < 0$ as well. A direct computation using (2.6) and (2.9) yields

$$F_r(r, \bar{\kappa}) = (3rvh/W) + [v' - (v/r)](h - rh')$$

which is negative for $r_1 \leqq r \leqq \bar{r}$. It follows that $F(\bar{r}, \bar{\kappa}) < 0$ and the drop is unstable.

THEOREM 3.7. *For Problem C consider the one-parameter family of pendent drops as described in Theorems* 3.4 *and* 3.5. *The following are true.*

(a) *Drop height increases with volume throughout the range of stability, and may be used as a differentiable parameter for the entire family of stable drops.*

(b) *The profile curve of a stable pendent drop contains at most two points where the tangent line is vertical,* (i.e., *a stable pendent drop cannot have a second bulge*).

*Proof. Case* 1. The profile curve can be expressed nonparametrically $u = u(r, \bar{\kappa})$.

Here, $\kappa$ is an increasing parameter. Let $\phi(r, \bar{\kappa}) \equiv u_\kappa(r, \bar{\kappa})$ and let $0 < r_1 < r_2$ be the first positive zeros of $\phi(r, \bar{\kappa})$. First we show that $dH/d\kappa(r \text{ fixed})$ is positive for $0 < r \leqq r_2$. As before $H = u - u_0 = u + 2\kappa$ implying that $dH/d\kappa = \phi + 2$. But $\phi(0) = -2$ and $\phi_r(r, \bar{\kappa}) > 0$ for $0 < r \leqq r_1$ by (2.14) showing that $dH/d\kappa > 0$ for $0 < r \leqq r_1$, while for $r_1 < r \leqq r_2$, $\phi(r, \bar{\kappa}) \geqq 0$ showing that $dH/d\kappa \geqq 2$.

On the other hand we show that $dV/d\kappa < 0$ at $r = r_2$. Differentiate the volume formula (2.12) setting $T = V/\pi$. One obtains $dT/d\kappa(r \text{ fixed}) = r[r\phi + 2(\cos^3\psi)\phi_r]$. At $r = r_2$, $\phi = 0$ and $\phi_r < 0$ showing that $dT/d\kappa$ is negative. Thus as long as $dT/d\kappa$ is positive we have $dH/d\kappa$ positive.

*Case* 2. The profile curve out to the first bulge.

In this case we know that $\phi(r, \bar{\kappa})$ vanishes exactly once out to the first vertical tangent, (see proof of Lemma 2.7). The argument is as in Case 1, so that $dT/d\kappa > 0$ and $dH/d\kappa > 0$.

*Case* 3. At the neck or the bulge.

Here we must use the parametric equation with arc length as parameter. By Theorem 3.5 we have that $dT/ds$ ($r$ fixed) is positive at both the neck and bulge. But $dH/ds = u_s + (u_\kappa + 2)\kappa'(s)$, where $u_s = \sin\psi$ and $\kappa'(s) = -(\cos\psi)/\rho$. It follows that when $\psi = \pi/2$, $dH/ds = 1$.

*Case* 4. Between the neck and the bulge.

Here, as in Theorem 3.5, $\kappa$ is a decreasing parameter. The configurations are stable so $dT/d\kappa < 0$. We are to show that $dH/d\kappa$ is negative as well. From Case 1 we have $dH/d\kappa(r \text{ fixed}) = \phi + 2 = 2-(h/r) - h'$ using (2.9a). Suppose we have $r_1 < r < r_2$ where the neck and bulge occur at $r_1$ and $r_2$ respectively. We will show that the maximum value of $dH/d\kappa$ on the interval $(r_1, r_2)$ is negative. By differentiation we obtain, using (2.9c), $(dH/d\kappa)_r = -h/w^3$. By Lemma 3.5 we know that $h'(r)$ is positive on the interval so that $(dH/d\kappa)$ has its maximum at $r = c$, when $h(c, \bar{\kappa}) = 0$.

We compute $dH/d\kappa$ at $r = c$. Again referring to Lemma 3.5 we find that the profile curve for this situation does not contain an inflection point, showing that the parametric slope function $\psi(s, \bar{\kappa})$ satisfies $\psi_s(s, \bar{\kappa}) > 0$ for $0 \leqq s \leqq \bar{s}$ where $r(\bar{s}, \bar{\kappa}) = c$. Since $\psi_s > 0$ we can let $\psi$ be the independent variable, and express the height, $H$, as an integral, (3.4).

$$H(\kappa) = \int_0^{\tilde{\psi}} (\sin\psi/\psi_s)d\psi \ .$$

Here $\psi_s$ is regarded as a function of $\psi$ and $\kappa$, and the condition $c = r(\tilde{s}, \tilde{\kappa})$ determines $\tilde{s} = s(\tilde{\kappa})$ with $s'(\tilde{\kappa}) = -\rho/\cos\tilde{\psi}$. The upper limit of integration is then $\tilde{\psi} = \tilde{\psi}(\tilde{\kappa}) \equiv \psi[s(\tilde{\kappa}), \tilde{\kappa}]$. Now $h(c, \bar{\kappa}) = 0$ and $h(r, \bar{\kappa}) = v_\kappa(r, \bar{\kappa}) = (\cos\psi)(d\tilde{\psi}/d\kappa)$ so that $d\tilde{\psi}/d\kappa(r = c) = 0$ when $\kappa = \bar{\kappa}$. We differentiate (3.4).

$$dH/d\kappa = (\sin\tilde{\psi}/\psi_s)(d\tilde{\psi}/d\kappa) - \int_0^{\tilde{\psi}} (\sin\psi/\psi_s^2)(d\psi_s/d\kappa)d\psi \ .$$

The first term on the right hand side vanishes when $\kappa = \bar{\kappa}$ and by Lemma 3.8 $d\psi_s/d\kappa$ is positive. It follows that $dH/d\kappa$ (is negative) when $h(c, \bar{\kappa}) = 0$.

*Case* 5. Beyond the neck.

Beyond the neck, the profile curve can again be expressed nonparametrically $u = u(r, \bar{\kappa})$, $r > r_1$. It is easy to check that $\phi(r, \bar{\kappa})$ is initially positive when $r > r_1$. This implies that $dH/d\kappa \geqq 2$ until $\phi(r, \bar{\kappa})$ vanishes again. As in Case 1, $dV/d\kappa$ will be negative when this occurs. Thus as long as $dV/d\kappa > 0$ we will have $dH/d\kappa \geqq 2$.

*Case* 6. No second bulge.

We need to show that $\phi(r, \bar{\kappa}) = u_\kappa(r, \bar{\kappa})$ vanishes at least once between the neck and second bulge (if such exists). Let the neck occur at $(r_1, u_1)$ on the profile curve while the second bulge is located at $(r_2, u_2)$. Letting $u$ be the independent variable we have

$$V(u_2) - V(u_1) = \pi\int_{u_1}^{u_2} r^2 du \tag{2.6}$$

where $V(u_i)$ is the volume of the generated drop below the level $u = u_i$. Now differentiate (3.6) with respect to $\kappa$, holding $u_1$ and $u_2$ fixed. We obtain, recalling (2.12)

$$2(dr/d\kappa)(ru + \sin\psi) + 2r\cos\psi(d\psi/d\kappa)\Big|_{u_1}^{u_2} = \int_{u_1}^{u_2} 2r(dr/d\kappa)du \,.$$

For $r$ near $r_1$ we have $du/d\kappa(r \text{ fixed}) > 0$. This implies that $dr/d\kappa$ ($u$ fixed) $< 0$ for $u$ near $u_1$. If $dr/d\kappa$ ($u$ fixed) is negative for $u_1 < u < u_2$, then the right hand side of our equation is negative. However, $\cos\psi = 0$ at $u_1$ and $u_2$ and $ru + \sin\psi = -r\psi_s$. It follows that the left hand side is nonnegative. Therefore, $dr/d\kappa(u \text{ fixed})$ vanishes at least once between the neck and the bulge. The same must be true for $\phi(r, \bar{\kappa}) = u_\kappa(r, \bar{\kappa})$.

PROBLEM B. The Limit of Stability when $\alpha = 0$.

Consider that profile curve with a single vertical tangent at $(r^*, u^*)$ which is simultaneously an inflection point. It follows from Theorems 2.2 and 2.5 that there is exactly one such curve. The nonparametric representation $u = u(r, \kappa^*)$ is continuous for all $r \geqq 0$, differentiable for $r \neq r^*$, and we have $r^* \cong .9176$, $u^* \cong -1.0894$, $r^*u^* = -1$, $u_0^* = -2\kappa^* \cong -2.5678$.

LEMMA 3.9. *Let $v(r, \kappa^*) \equiv v(r)$ be the slope function for the profile curve just described. Let $r_1(v)$, $0 \leqq v \leqq 1$ be the inverse function for $v(r)$, $0 \leqq r \leqq r^*$ and let $r_2(v)$ be the inverse function for $v(r)$, $r^* \leqq r \leqq a$ where $v(a) = 0$. Let $(\bar{r}, \bar{v})$, $0 \leqq \bar{v} < 1$, be a point on the decreasing portion of $v(r)$. We have the following formula*

$$\begin{aligned}[\bar{r}v'(\bar{r})]d[\bar{r}v'(\bar{r})]/d\kappa &= 2\int_{\bar{v}}^{1} r_2(dr_2/d\kappa)(v/W)dv \\ &\quad -2\int_0^1 r_1(dr_1/d\kappa)(v/W)dv\end{aligned} \tag{3.7}$$

*where differentiation is with respect to $\kappa$ holding $v$ fixed.*

*Proof.* For $\kappa$ near $\kappa^*$, $\kappa < \kappa^*$ the equation $v(r, \kappa) = \bar{v}$ determines $r = r(\kappa)$ with $r(\kappa^*) = \bar{r}$. For each $\kappa < \kappa^*$ take the integral identity (2.10a) from 0 to $r(\kappa)$, differentiate it with respect to $\kappa$, let $\kappa \to \kappa^*$ to get

$$[\bar{r}v'(\bar{r})]d[\bar{r}v'(\bar{r})]/d\kappa = 2\int_0^{\bar{r}} rh(v/W)dr .$$

However, by differentiating $v(r, \kappa) = \bar{v}$ we get $dr/d\kappa = -h/v'$ if $v' \neq 0$. Then result follows by changing the variable of integration.

Our main assertions pertaining this profile curve will follow from the following lemma.

LEMMA 3.10. *Let* $v(r) = v(r, \kappa^*)$ *be as in Lemma* 3.8 *with* $v(r^*) = 1$, $v'(r^*) = 0$ *and let* $a > r^*$ *to be the first positive zero for* $v(r)$. *For some* $c$, $r^* < c < a$ *we will have* $h(c) - ch'(c) = 0$.

*Proof.* As in the proof of Theorem 3.2, Case II we recall that

$$dr/d\kappa = -h/v' = -(\cos\psi/\psi_s)\omega(s) + \rho(s) \text{ if } \psi_s \neq 0 \ (v \text{ fixed}) .$$

An application of L'Hospital's rule shows that limit $(\cos\psi/\psi_s) = 0$ as $s \to s^*$. Hence at $s = s^*$, $dr/d\kappa = \rho(s^*) < 0$ by Lemma 3.3. It follows that $h(r)$ is continuous across $r = r^*$ with $h(r^*) = 0$, $h(r) > 0$ for $r < r^*$, and $h(r) < 0$ initially if $r > r^*$.

We know that the profile curve determines a stable drop for Problem C to some point beyond the vertical tangent, (Theorem 3.5). Therefore, initially for $r > r^*$, $dT/d\kappa = r(h - rh')$ is positive, where $T = (\text{Volume})/\pi$.

If $h'(\tilde{r}) = 0$ for some $\tilde{r}$, $r < \tilde{r} \leqq a$ and $h(\tilde{r}) < 0$ as well, then $h - rh' < 0$ at $r = \tilde{r}$ proving the lemma in this case.

Now suppose that $h(r) < 0$ and $h'(r) < 0$ for $r^* < r \leqq a$. We first show that for some $r_1$, $r^* < r_1 < a$, $d(rv')/d\kappa(v \text{ fixed}) = 0$.

By Lemma 2.6 we have $d(rv')/d\kappa \geqq 0$ and $dr/d\kappa < 0$ for $0 < r < r^*$, which implies that $d(v')/d\kappa > 0$ on the same interval. A direct calculation in terms of the parameter $s$ gives

$$\psi_s d(r\psi_s)/d\kappa = r(-\omega\psi_{ss} + \psi_s\omega_s) + \psi_s(-\omega\cos\psi + \rho\psi_s) .$$

The right hand side is positive when $\psi_s = 0$. Since $\psi_s = v'(r)$ it follows that initially for $r > r^*$, $d(rv')/d\kappa < 0$.

Suppose that $d(rv')/d\kappa$ is negative for $r^* < r < a$. Since $d(rv')/d\kappa = r(dv'/d\kappa) + (dr/d\kappa)v' < 0$ and $dr/d\kappa = -h/v' < 0$ it follows that $dv'/d\kappa < 0$ for $r^* < r < a$. However, we also have $v'(dr/d\kappa)_r = -dv'/d\kappa$. But $dv'/d\kappa$ and $v'$ are either both positive or both negative. Thus $(dr/d\kappa)_r < 0$ when $v' \neq 0$. However, at the start of the proof we showed that $dr/d\kappa$ was continuous across $r = r^*$ with $dr/d\kappa = \rho(s^*) < 0$. Thus $dr/d\kappa$ is a negative decreasing function of $r$ for $0 < r \leqq a$.

Since $r_2(v) > r_1(v)$ it follows from (3.7) that $(rv')d(rv')/d\kappa$ is negative when $r = a$, implying that $d(rv')/d\kappa$ is positive there, contradicting our assumption.

Thus for some $r_1$, $r^* < r_1 < a$ we will have $d(rv')/d\kappa = 0$. By (3.7) this means that the difference of the two integrals when $\bar{v} = v(r_1)$ is zero. But for $r_1 < r < a$ we are still assuming that $dr_2/d\kappa = -h/v' < 0$. It follows that if we set $\bar{v} < v(r_1)$ in (3.7) the expression becomes negative.

In particular when $\bar{v} = 0$ and $r = a$ we find that $d(rv')/d\kappa$ is positive. But $d(rv')/d\kappa = -(h - rh') - rhv''/v'$. If $v(a) = 0$, $v'(a) < 0$, then $v''(a) > 0$ by (2.6a). Since we are assuming $h(a) < 0$, it follows that $h(a) - ah'(a)$ is negative.

We are now in a position to prove the following result.

THEOREM 3.8. *Let $\kappa^*$ be the parameter value for which the corresponding profile curve has a vertical tangent and inflection point at $(r^*, u^*)$, as in Lemma* 3.10. *Let $c$, $r^* < c < a$, where $u'(a) = 0$, be the first positive root of $h - rh'$.*

(i) *If $r_0 > c$ then the drop profile $u(r, \kappa^*)$, $0 \leqq r \leqq r_0$ will be beyond the point of instability for Problem* C.

(ii) *For Problem* B *with $\alpha = 0$ instability will occur before the drop generated by the profile curve $u(r, \kappa^*)$ is produced.*

*Proof.* At $r = c$, $h - rh' = 0$ and $h - rh' > 0$ for $r < c$. It is easy to check that $h - rh'$ must change sign at $r = c$. Therefore $dT/d\kappa$ ($r$ fixed) $= r_0(h - r_0h') < 0$ for $r_0 > c$ initially. This means that for $r_0 > c$, $r_0$ near $c$, the configuration is unstable for Problem C. But this implies that for all $r_0 > c$, the corresponding drop is unstable for Problem B as well.

If $\kappa > 0$ is small, then $h - rh'$ will be positive for $0 < r \leqq a(\kappa)$ where $a(\kappa)$ is the first positive root of $v(r, \kappa)$. If $\kappa = \kappa^*$ then we have $r^* < c < a(\kappa^*) = a$ and $h - rh' = 0$ at $c$. Since $h - rh'$ must change sign at a zero we can conclude that there is a value $\kappa_1$, $0 < \kappa_1 < \kappa^*$ such that the first root of $r - rh'$ coincides with the first root of $v(r, \kappa_1)$. Referring back to Theorem 3.3, we see that $u(r, \kappa_1)$ is the limiting drop for Problem B, $\alpha = 0$.

**4. Analysis of the stability criterion.** In this section we give a justification of the procedure used in § 3 for determining stable configurations. We shall focus our discussion on the constant volume, prescribed angle of contact problem.

Denote by $\Sigma$ the rigid surface to which the pendent drop is adhering. In our case $\Sigma$ is the horizontal plane, $z = 0$. Let $\Lambda$

denote the liquid-air interface of the drop. We suppose that $\Sigma$ is an oriented manifold of class $C^2$ imbedded in $R^3$ and that $\Lambda$ is represented by a regular mapping $x\colon \bar{G} \to R^3$ where $G$ is a smooth bounded domain in $R^2$ with $x \in C^2(\bar{G})$ and $\partial\Lambda \subset \Sigma$. Let $\eta$ be the unit normal on $\Sigma$ chosen to point into the liquid and $\xi$ the *outward* unit normal on $\Lambda$.

The potential energy, suitably normalized, of the drop is given by (1.2) which we rewrite

$$E(x) = A(\Lambda) + \iiint_V z dv - \lambda A(\Sigma_\Lambda) \tag{4.1}$$

where $\lambda$ is a constant determined by the properties of the fluid and $A(\Sigma_\Lambda)$ is the area of contact of the drop with $\Sigma$.

Suppose the drop, $\Lambda$, is embedded in a smooth one-parameter family of surfaces, $\Lambda(\varepsilon)$ with $\partial\Lambda(\varepsilon) \subset \Sigma$ described by mappings $x(u, v, \varepsilon)\colon \bar{G} \to R^3$ satisfying

$$x(u, v, \varepsilon) = x(u, v) + \varepsilon[N(u, v)\xi(u, v) + \partial x_T(u, v)] + o(\varepsilon) . \tag{4.2}$$

Here $\xi(u, v)$ is the outward unit normal on $\Lambda = \Lambda(0)$, $N(u, v) = \xi\cdot(\partial x/\partial\varepsilon)$ at $\varepsilon = 0$ is the normal component of the perturbation and $\partial x_T(u, v)$ is the tangential component. We suppose that the variation is smooth so that $N(u, v) \in C^2(\bar{G})$.

*Note.* Since $x\colon \bar{G} \to R^3$ is an embedding of $\bar{G}$ onto $\bar{\Lambda}$, there is a smooth map $\tilde{N}\colon \bar{\Lambda} \to R^3$ with $\tilde{N}\circ x = N$. For convenience we shall identify $\tilde{N}$ and $N$. No confusion should result.

The first variation of the energy is given by

$$\partial E_x(N) = \iint_\Lambda (z - 2H)NdS + \oint(-\lambda \csc\alpha + \cot\alpha)Nd\sigma \tag{4.3}$$

Here $dS$ is the element of area on $\Lambda$, $d\sigma$ is the element of arc length on $\partial\Lambda$, $\alpha$ is the interior angle of contact of $\Lambda$ with $\Sigma$ along $\partial\Lambda$, and $H$ is the mean curvature of $\Lambda$ with respect to the outward normal, $\xi$, on $\Lambda$. The first variation of the volume is

$$\partial V_x(N) \equiv \iint_\Lambda NdS . \tag{4.4}$$

If $\Lambda = \Lambda(0)$ is in equilibrium, then $\partial E_x(N) = 0$ for all $N \in C^2(\bar{\Lambda})$ satisfying $\partial V_x(N) = 0$. This gives the necessary equilibrium conditions

$$\begin{aligned} &\text{(a)}\quad 2H = z - c \text{ for some constant } c \\ &\text{(b)}\quad \lambda = \cos\alpha . \end{aligned} \tag{4.5}$$

For $0 < \alpha < \pi$, we must have $-1 < \lambda < 1$.

In this paper we considered axially symmetric drops whose profile curves satisfied (2.2). It follows that the set of all possible symmetric drops satisfying the Euler equation (4.5a) may be represented parametrically by a mapping $x\colon \bar{B}_1 \to R^3$ where (using polar coordinates $\rho, \theta$) we have

$$x(\rho, \theta, \bar{s}, \bar{\kappa}, \bar{c}) = \langle r(\rho\bar{s}, \bar{\kappa}) \cos\theta,\ r(\rho\bar{s}, \bar{\kappa}) \sin\theta,\ u(\rho\bar{s}, \bar{\kappa}) + \bar{c}\rangle \ . \tag{4.6}$$

The condition that the boundary of this surface lies on $\Sigma$ is just $u(\bar{s}, \bar{\kappa}) + \bar{c} = 0$, while the angle of contact condition (4.5b) is $\psi(\bar{s}, \bar{\kappa}) = \alpha$.

If $A$ is an equilibrium surface satisfying (4.5) then the second variation is given by

$$\begin{aligned} E(N, N) &\equiv \partial^2(E - cV)(N, N) \\ &= \iint_A [\partial(z - 2H)(N)]N dS + \oint - [\partial\alpha(N)]N d\sigma \\ &= \iint_A N(-\Delta N + RN) dS + \oint N(N_1 + pN) d\sigma \ . \end{aligned} \tag{4.7}$$

Here $\Delta$ = Laplace operator on $A$ and

$$R = -2(2H^2 - K) + \xi_3$$

where $H$ = mean curvature, $K$ = Gaussian curvature, $N_1$ = outward mormal directional derivative of $N$ along $\partial A$.

$p = K_A \cot\alpha - K_\Sigma \csc\alpha$, where $K_A$ is the curvature of $A \cap \Pi$ relative to the normal vector, $\xi$, and $\Pi$ is the normal plane to $\partial A$. $K_\Sigma$ is the corresponding curvature of $\Sigma \cap \Pi$ relative to the normal, $\eta$.

$$\xi_3 \text{ is the vertical component of } \xi = (\xi_1, \xi_2, \xi_3) \ .$$

Essentials of this formula may be found in Blaschke [4]. A complete derivation may be found in [16].

DEFINITION 4.1. An equilibrium drop for which $A$ satisfies (4.5) is said to be stable if $E(N, N)$ is positive for all $N \in C^2(\bar{A})$, $N \neq 0$ with $\partial V_x(N) = 0$.

*Note.* In the case that $\Sigma$ is a horizontal plane, a horizontal translation of $A$ leaves both the potential energy and the volume unchanged. This induces a two-dimensional subspace of (nonsymmetric) normal perturbations for which $E(N, N) = 0$ and $\partial V_x(N) = 0$. The definition of stability must be altered in the obvious manner.

The quadratic form, $E(N, N)$, can be written in several different ways. If we set

$$(4.8)\qquad \begin{aligned}&\text{(a)}\quad (F\cdot G)_A = \iint_A FGdS\\ &\text{(b)}\quad (F\cdot G)_{\partial A} = \oint FGd\sigma\end{aligned}$$

then

$$(4.9)\qquad E(F, G) = (F\cdot \mathfrak{L}G)_A + (F\cdot \mathfrak{b}G)_{\partial A}$$

where $\mathfrak{L}G = -\Delta G + RG$ on $A$

$$\mathfrak{b}G = G_1 + pG \text{ on } \partial A\ .$$

With the aid of Green's theorem we obtain

$$(4.10)\qquad E(F, G) = [F, G]_1 + \iint_A FG(R-1)dS + \oint FGpd\sigma$$

where

$$(4.11)\qquad [F, G]_1 = \iint_A (\nabla F\cdot\nabla G)dS + \iint_A FGdS\ .$$

Since $x\colon \bar{B}_1 \to R^3$ is given by a $C^2(\bar{B}_1)$ regular mapping, it follows that $[F, G]_1$ is an inner product on $C^1(B_1)$ which is norm equivalent to the standard inner product

$$(4.12)\qquad (F\cdot G)_1 \equiv \iint_{B_1} (F_uG_u + F_vG_v + FG)dudv\ .$$

$E(F, G)$ is thus a continuous, symmetric bilinear functional on $W_1(B_1)$, the Hilbert space completion of $C^1(\bar{B}_1)$ in the norm (4.12). Therefore, we may apply the developed theory for eigenfunctions and eigenvalues to conclude the following theorem. See [2] for example.

THEOREM 4.1. *There exists a sequence of eigenfunctions* $\phi_n \in C^2(\bar{A})$ *and eigenvalues* $\mu_n$ *such that*

(a) $\mathfrak{L}\phi_n = \mu_n\phi_n$, $\mu_1 < \mu_2 \leqq \mu_3 \leqq \cdots$.

(b) $\mathfrak{b}\phi_n = 0$ *on* $\partial A$.

(c) $\{\phi_n\}$ *is a complete orthonormal sequence on* $L_2(A)$.

In this paper, the surfaces $A$ are axially symmetric in the form (4.6). It follows that all of the eigenfunctions of Theorem 4.1 may be obtained by separation of variables. This reduces Theorem 4.1 to a Sturm-Liouville problem for ordinary differential equations. [See 3, Chap. X], and in fact one could use this approach to prove Theorem 4.1 in the axially symmetric case.

We have $x(\rho, \theta) = [A(\rho)\cos\theta,\ A(\rho)\sin\theta,\ B(\rho)]$ where $A(\rho) =$

$r(\rho\bar{s}, \bar{\kappa})$ and $B(\rho) = u(\rho\bar{s}, \bar{\kappa}) + \bar{c}$, and $A(0) = 0$, $A'(0) = 0$, $A'(\rho)^2 + B'(\rho)^2 = \bar{s}^2$. The coefficients of the first fundamental form $\mathfrak{e}, \mathfrak{f}, \mathfrak{g}$ are given by $\mathfrak{e} = \bar{s}^2$, $\mathfrak{f} = 0$, and $\mathfrak{g} = A^2(\rho)$.

If $N(\rho, \theta) = f(\rho)$ is symmetric, then

$$(4.13)\quad \begin{aligned} &\text{(a)}\quad E(N, N) = 2\pi e_0(f, f) = 2\pi\int_0^1 [af'^2 + RWf^2]d\rho + 2\pi\bar{p}\bar{r}f^2(1) \\ &\qquad \text{where } a(\rho) = (\mathfrak{g}/\mathfrak{e})^{1/2} = A(\rho)/\bar{s}, \quad W = (\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2)^{1/2} = \bar{s}A(\rho), \\ &\qquad \bar{r} = r(\bar{s}, \bar{\kappa}), \text{ and } \bar{p} = K_1 \cot\alpha = \psi_s(\bar{s}, \bar{\kappa}) \cot\alpha. \\ &\text{(b)}\quad \partial V(N) \equiv \partial v(f) = 2\pi\int_0^1 f\,W d\rho. \\ &\text{(c)}\quad (N\cdot N)_A = 2\pi\int_0^1 f^2 W d\rho = 2\pi(f\cdot f)_0. \end{aligned}$$

If $N(\rho, \theta) = f(\rho)\cos\theta(f(\rho)\sin\theta)$, then

$$(4.14)\quad \begin{aligned} &\text{(a)}\quad E(N, N) = \pi e_k(f, f) \\ &\qquad = \pi\int_0^1 \left[ af'^2 + \frac{k^2}{a} f^2 + RWf^2 \right] d\rho + \pi\bar{p}\bar{r}f^2(1). \\ &\text{(b)}\quad \partial V(N) = 0. \\ &\text{(c)}\quad (N\cdot N)_A = \pi\int_0^1 f^2 W d\rho = \pi(f\cdot f)_0. \end{aligned}$$

Now set

$$(4.15)\quad \begin{aligned} &\text{(a)}\quad L(f) = -[(af')'/W] + Rf. \\ &\text{(b)}\quad b(f) = f'(1) + \bar{p}\bar{s}f(1). \end{aligned}$$

We then have the following restatement of Theorem 4.1.

THEOREM 4.2. *If $A$ is axially symmetric in the form* (4.6) *then the complete set of eigenfunctions and eigenvalues are given with separated variables as follows.*

*If $k = 0$ (symmetric case), there is an infinite sequence of eigenfunctions $\{N_{j0}(\rho, \theta) = f_{j0}(\rho)\}$ and eigenvalues $\{\lambda_{j0}\}$ with*

$$(4.16)\quad \begin{aligned} &\text{(a)}\quad L(f_{j0}) = \lambda_{j0} f_{j0},\ b(f_{j0}) = 0. \\ &\text{(b)}\quad \lambda_{00} < \lambda_{10} < \lambda_{20} < \cdots. \end{aligned}$$

*If $k \geqq 1$, there is an infinite sequence of eigenvalues $\{\lambda_{jk}\}$ and a corresponding two-dimensional space of eigenfunctions spanned by $f_{jk}(\rho)\cos k\theta$ and $f_{jk}(\rho)\sin k\theta$ satisfying*

$$(4.17)\quad \begin{aligned} &\text{(a)}\quad L(f_{jk}) + [k^2/aW]f_{jk} = \lambda_{jk} f_{jk},\ b(f_{jk}) = 0. \\ &\text{(b)}\quad \lambda_{0k} < \lambda_{1k} < \lambda_{2k} < \cdots. \end{aligned}$$

*Furthermore $\lambda_{00} < \lambda_{01} < \lambda_{02} < \cdots$ and $\lambda_{01} = 0$.*

*Proof.* A straight forward application of the Sturm-Liouville theory for ordinary differential equations with a regular singular point at $\rho = 0$. It is important to note that the systems (4.16) and (4.17) have a regular singular point at $\rho = 0$ whose indicial equation has roots $k$, $-k$. (*Note*: If $A$ is the surface $u = 0$, the solutions to (4.16), (4.17) are just the Bessel functions, $J_k(\sqrt{\lambda}\rho)$.)

The fact that the listing of eigenvalues in (4.16b) and (4.17b) are all distinct follows from the Sturm-Liouville theory, while the fact that $\lambda_{00} < \lambda_{01} < \lambda_{02} < \cdots$ is a consequence of the formulas (4.13), (4.14) and the characterization of the smallest eigenvalue, $\lambda_{0k}$, by its minimizing property for the Rayleigh quotient, $e_k(f, f)/(f \cdot f)_0$.

Finally, as noted by Concus and Finn, a horizontal translation of $A$ leaves both the energy, and volume unchanged. Therefore, a differentiation yields a normal perturbation satisfying (4.17) with $k = 1$ and $\lambda_{01} = 0$.

Consider the one-parameter family of axially symmetric pendent drops as constructed in § 3 determined by the condition $\psi(\bar{s}, \bar{\kappa}) = \alpha$, $0 < \alpha < \pi$. For small volumes $\kappa$ itself could be chosen as parameter with the volume $\rightarrow 0$ as $\kappa \rightarrow +\infty$. In general if $\varepsilon$ is the parameter the condition $\psi_s s'(\varepsilon) + \psi_\kappa \kappa'(\varepsilon) = 0$ most be satisfied. The construction of a smooth one-parameter family relies on the condition grad $\psi = (\psi_s, \psi_\kappa) \neq (0, 0)$. This was established in § 3. Therefore $(s'(\varepsilon), \kappa'(\varepsilon))$ is unique up to a scale factor. Suppose that we have parametrized our family of pendent drops satisfying $\psi(\bar{s}, \bar{\kappa}) = \alpha$ by $\varepsilon(\varepsilon > 0)$ where

$$\text{(4.17)} \qquad \begin{array}{ll} (1) & \text{limit } V(\varepsilon) = 0 \text{ as } \varepsilon \longrightarrow 0 \\ (2) & V'(\varepsilon) > 0 \text{ for } 0 < \varepsilon < \bar{\varepsilon} \\ (3) & V'(\bar{\varepsilon}) = 0 \end{array}$$

where $V'(\bar{\varepsilon}) = 0$ occurs after the appearance of an inflection point on the profile curves. As noted in the introduction, the following theorem was proven by E. Pitts [14] in the symmetric case.

THEOREM 4.3. *The family of pendent drops, $A(\varepsilon)$, determined by the condition $\psi(\bar{s}, \bar{\kappa}) = \alpha$, $0 < \alpha < \pi$, and for which condition* (4.17) *is satisfied are all stable for $0 < \varepsilon < \bar{\varepsilon}$. Furthermore if $V'(\varepsilon)$ changes sign as $\varepsilon$ increases through $\bar{\varepsilon}$, then $A(\varepsilon)$ is unstable for $\varepsilon > \bar{\varepsilon}$.*

*Proof.* As $\varepsilon \rightarrow 0$, $\kappa \rightarrow +\infty$, and volume $\rightarrow 0$, the drop surface, $A(\varepsilon)$, resembles a spherical cap of radius $1/\kappa$. Let $x(\rho, \theta, \varepsilon)$ be the representation of $A(\varepsilon)$ in the axially symmetric form (4.6). Let

$y(\rho, \theta, \varepsilon) \equiv \kappa(\varepsilon)x(\rho, \theta, \varepsilon)$. By following the argument used in the proof of Theorem 2.6, $y(\rho, \theta, \varepsilon)$ will converge uniformly as $\varepsilon \to 0$ along with its first two derivatives to a parametric representation $y(\rho, \theta, 0) \equiv y_0$ of a spherical cap, $\Lambda_\alpha$, with radius 1 and cutting the plane $z = 0$ at an angle $\alpha$. Such a cap is an equilibrium configuration in a gravity free environment and is well known to be stable. This means that

$$m_\alpha = \text{minimum } E_{y_0}(N, N)/(N \cdot N)_{\Lambda_\alpha} > 0 \tag{4.18}$$

where the minimum is taken over all $N \in C^2(\bar{\Lambda}_\alpha)$ satisfying $\partial V(N) = 0$ and $(N \cdot N_a)_{\Lambda_\alpha} = (N \cdot N_b)_{\Lambda_\alpha} = 0$ where $N_a$, $N_b$ are the normal components of variations arising from horizontal translations in two independent directions.

(*Note*: Because of the invariance of our configuration under horizontal translations, we are forced to alter Definition 4.1 in the above manner.)

From the convergence of $y(\rho, \theta, \varepsilon)$ to $y(\rho, \theta, 0)$ we can infer that $m[y(\varepsilon)]$, the minimum of the corresponding Rayleigh quotient for $y(\rho, \theta, \varepsilon)$, converges to $m_\alpha$ as $\varepsilon \to 0$. But it is also easy to check that $m[x(\varepsilon)] = m[y(\varepsilon)]$. Therefore, for small $\varepsilon$, $m[x(\varepsilon)] > 0$ and the drops determined by $\Lambda(\varepsilon)$ are stable.

As observed in Theorem 4.2 all of the eigenvalues associated with nonsymmetric perturbations are positive and satisfy $\partial V(N) = 0$; except for $\lambda_{01} = 0$, the eigenvalue whose eigenfunctions correspond to horizontal translations. We may therefore conclude that if $\Lambda(\varepsilon)$ is stable with respect to symmetric perturbations, then it is stable.

We have $V'(\varepsilon) > 0$ for $0 < \varepsilon < \bar{\varepsilon}$ and we know that $\Lambda(\varepsilon)$ is stable for small $\varepsilon$. We next show that $\Lambda(\varepsilon)$ is stable for $0 < \varepsilon < \bar{\varepsilon}$. For each $\varepsilon > 0$ let $m(\varepsilon)$ be the minimum of the Rayleigh quotient, (4.18), over all normal perturbations $N \in C^2[\overline{\Lambda(\varepsilon)}]$ for which $\partial V(N) = 0$, $(N \cdot N_a) = (N \cdot N_b) = 0$. It is known that $m(\varepsilon)$ is continuous in $\varepsilon$ and that the minimum will be attained for suitable $N$.

If our claim is false, there is an $\varepsilon_1$, $0 < \varepsilon_1 < \bar{\varepsilon}$ with $m(\varepsilon) > 0$ for $\varepsilon < \varepsilon_1$ and $m(\varepsilon_1) = 0$. There corresponds a symmetric normal perturbation, $N_1$, achieving this minimum. By applying the method of Lagrange multipliers to (4.18) and recalling (4.7), it follows that $N_1(\rho, \theta)$ will satisfy

$$\begin{aligned} &\text{(a)} \quad \mathfrak{L}(N_1) = \gamma(\text{some constant, } \gamma) \text{ on } \Lambda. \\ &\text{(b)} \quad \mathfrak{b}(N_1) = 0 \text{ on } \partial\Lambda. \\ &\text{(c)} \quad \partial V(N_1) = 0. \\ &\text{(d)} \quad (N \cdot N)_{\Lambda(\varepsilon_1)} = 2\pi. \end{aligned} \tag{4.19}$$

Observe that this implies that $E(N_1, N_1) = (N_1 \cdot \mathfrak{L} N_1)_\Lambda + (N_1 \cdot \mathfrak{b} N_1)_{\partial\Lambda} = \gamma(1 \cdot N_1)_\Lambda = \gamma\partial V(N_1) = 0$. Since $N_1$ is symmetric we have $N_1(\rho, \theta) = g(\rho)$ where

$$(4.20) \qquad \begin{aligned} &(a) \quad L(g) = \gamma. \\ &(b) \quad b(g) = 0. \\ &(c) \quad \partial v(g) = 0. \\ &(d) \quad (g \cdot g)_0 = 1. \end{aligned}$$

Now consider the normal perturbation corresponding to $dx/d\varepsilon$, which is $(dx/d\varepsilon)\cdot\xi$. By (4.6), noting that $(\partial x/\partial \bar{s})\cdot\xi = 0$ we obtain

$$(4.21) \qquad \hat{N} \equiv N(\rho, \varepsilon_1) \equiv (dx/d\varepsilon)\cdot\xi = N_\kappa \bar{\kappa}'(\varepsilon_1) + N_c \bar{c}'(\varepsilon_1)\ .$$

Here $N_\kappa = (\partial x/\partial\kappa)\cdot\xi$ and $N_c = (\partial x/\partial c)\cdot\xi$. For example, from (4.6) one has $\partial x/\partial c = (0, 0, 1)$ from which it follows that $N_c(\rho, \theta) = -\cos\psi(\rho\bar{s}, \bar{\kappa})$. The triple $\langle s'(\varepsilon_1), \kappa'(\varepsilon_1), c'(\varepsilon_1)\rangle$ is determined by the conditions $\psi(\bar{s}, \bar{\kappa}) = \alpha$, $u(\bar{s}, \bar{\kappa}) + \bar{c} = 0$ yielding

$$(4.22) \qquad \begin{aligned} &(a) \quad \psi_s(\bar{s}, \bar{\kappa})s'(\varepsilon_1) + \psi_\kappa(\bar{s}, \bar{\kappa})\kappa'(\varepsilon_1) = 0. \\ &(b) \quad u_s(\bar{s}, \bar{\kappa})s'(\varepsilon_1) + u_\kappa(\bar{s}, \bar{\kappa})\kappa'(\varepsilon_1) + c'(\varepsilon_1) = 0. \end{aligned}$$

Since $\mathfrak{L}(N) = \partial(z - 2H)(N)$ it follows that $\mathfrak{L}(N_\kappa) = 0$ and $\mathfrak{L}(N_c) = 1$. We also have $0 = (\partial\alpha)(\hat{N}) = \mathfrak{b}(\hat{N}) = \kappa'(\varepsilon_1)\mathfrak{b}(N_\kappa) + c'(\varepsilon_1)\mathfrak{b}(N_c)$. If we set $N_\kappa(\rho, \theta) \equiv f_\kappa(\rho)$, $N_c(\rho, \theta) \equiv f_c(\rho)$ and $\hat{N}(\rho, \theta) \equiv \hat{f}(\rho)$ we find

$$(4.23) \qquad \begin{aligned} &(a) \quad L(f_\kappa) = 0,\ L(f_c) = 1. \\ &(b) \quad 0 = b(\hat{f}) = \kappa'(\varepsilon_1)b(f_\kappa) + c'(\varepsilon_1)b(f_c). \\ &(c) \quad v'(\varepsilon_1) = \kappa'(\varepsilon_1)\partial v(f_\kappa) + c'(\varepsilon_1)\partial v(f_c). \end{aligned}$$

From (4.20) and (4.23) it follows that $L(g - \gamma f_c) = 0$. But $L$ is a second order differential operator with regular singular point at $\rho = 0$, and $g - \gamma f_c$ is a bounded solution at $\rho = 0$. Therefore $g - \gamma f_c = \tau f_\kappa$ for some $\tau$, or $g = \tau f_\kappa + \gamma f_c$ with $(\tau, \gamma) \neq (0, 0)$.

We compare $\hat{f} = \bar{\kappa}'(\varepsilon_1)f_\kappa + \bar{c}'(\varepsilon_1)f_c$ with $g$, where $L(\hat{f}) = c'(\varepsilon_1)$, $L(g) = \gamma$ and $b(\hat{f}) = 0$, $b(g) = 0$. The triples $\langle\bar{s}'(\varepsilon_1), \bar{\kappa}'(\varepsilon_1), \bar{c}'(\varepsilon_1)\rangle$ and $\langle\sigma, \tau, \gamma\rangle$ are both solutions to (4.22) if we choose $\sigma$ so that (4.22b) is satisfied. But $(\bar{\psi}_s, \bar{\psi}_\kappa) \neq (0, 0)$ implies that the solution space for (4.22) is one-dimensional. From this we can infer that $(\tau, \gamma) = m(\kappa', c')$ for some $m \neq 0$, and thus $g(\rho) = m\hat{f}(\rho)$. However, by (4.20) we have $\partial v(g) = 0$ and by our assumption at $\varepsilon = \varepsilon_1$ we have $\partial v(\hat{f}) = v'(\varepsilon_1) > 0$, which is a contradiction.

Finally we suppose that $v'(\varepsilon) > 0$ for $0 < \varepsilon < \bar{\varepsilon}$, $v'(\bar{\varepsilon}) = 0$, and $v'(\varepsilon) < 0$ for $\varepsilon > \bar{\varepsilon}$. We are to show that $\Lambda(\varepsilon)$ is unstable for $\varepsilon > \bar{\varepsilon}$.

Once again we have $\hat{f}(\varepsilon) = \kappa'(\varepsilon)f_\kappa + c'(\varepsilon)f_c$ with

$$\text{(4.24)}\qquad \begin{aligned} &\text{(a)}\quad L[\hat{f}(\varepsilon)] = \kappa'(\varepsilon)L(f_\kappa) + c'(\varepsilon)L(f_c) = c'(\varepsilon). \\ &\text{(b)}\quad b(\hat{f}) = 0. \\ &\text{(c)}\quad v'(\varepsilon) = \kappa'(\varepsilon)\partial v(f_\kappa) + c'(\varepsilon)\partial v(f_c). \end{aligned}$$

We first show that $c'(\bar{\varepsilon}) \neq 0$. If $c'(\bar{\varepsilon}) = 0$ then we would have $\kappa'(\bar{\varepsilon}) \neq 0$ and $\hat{f}(\bar{\varepsilon}) = \kappa'(\bar{\varepsilon})f_\kappa$. This, along with the assumption $v'(\bar{\varepsilon}) = (\hat{f}(\bar{\varepsilon})\cdot 1)_0 = 0$ implies that $f_\kappa$ satisfies the system $L(f_\kappa) = 0$, $b(f_\kappa) = 0$, and $\partial v(f_\kappa) = (f_\kappa \cdot 1)_0 = 0$. We may now apply the Fredholm alternative for $L$ with boundary conditions $b$, [2, p. 183]. Since $L$ is self-adjoint and the function 1 is orthogonal to the kernel of $L$, there is a function $h(\rho)$ satisfying $L(h) = 1$, $b(h) = 0$. We may conclude that $h = \tau f_\kappa + f_c$ for some constant $\tau$. Now $h$ and $f_\kappa$ are both solutions to $L(y) =$ constant, $b(y) = 0$. But the condition $(\psi_s, \bar{\psi}_\kappa) = \operatorname{grad} \psi \neq 0$ guarantees that the solution space is one-dimensional. But $h = \tau f_\kappa + f_c$ and $f_\kappa$ are clearly linearly independent. Thus $c'(\bar{\varepsilon}) \neq 0$.

For each $\varepsilon$ near $\bar{\varepsilon}$ choose $G_\varepsilon$ in $C^2(\bar{\Lambda})$, symmetric, and with $\partial V(G_\varepsilon) = 1$. Let $Z_\varepsilon = N_\xi - \partial V(N_\xi)G_\varepsilon$, so that $\partial V(Z_\xi) = 0$. If we now compute $E(Z_\varepsilon, Z_\varepsilon)$ we get

$$E(Z_\varepsilon, Z_\varepsilon) = v'(\varepsilon)[-c'(\varepsilon) + v'(\varepsilon)E(G_\varepsilon, G_\varepsilon)]\,. \tag{4.25}$$

Since $c'(\bar{\varepsilon}) \neq 0$ and $v'(\varepsilon)$ changes sign at $\bar{\varepsilon}$ we may conclude that $E(Z_\varepsilon, Z_\varepsilon) < 0$ for $\varepsilon > \bar{\varepsilon}$.

REMARK. As we have just seen, the order of the eigenvalues as listed in Theorem 4.2 implied that for Problem B the limits of stability are determined by the symmetric perturbations.

EXAMPLE. Consider a drop pendent from a horizontal plate, with no volume constraint, but a constant pressure, as might be arranged by connecting a drop hanging from a plate to a water source by a siphon through a small hole in the plate. In this case stability is determined by the smallest eigenvalue in the listing. But $\lambda_{00} < \lambda_{01} = 0$. This configuration is always unstable.

Now consider the pendent drop of Problem C. Here stability is determined by considering normal perturbations, $N \in C^2(\bar{\Lambda})$ for which $\partial V(N) = 0$ and $N = 0$ on $\partial\Lambda$. This means that the boundary term will disappear in the expression (4.7) for the second variation, leading to the following definition.

DEFINITION 4.2. An equilibrium surface, $\Lambda$, is stable for Problem C if

$$E(N, N) = (\mathfrak{L}N, N)_\Lambda = \iint_\Lambda (|\nabla N|^2 + RN^2)dS \tag{4.26}$$

is positive for all $N \in C^2(\bar{A})$, $N = 0$ on $\partial A$, and $\partial V(N) = 0$.

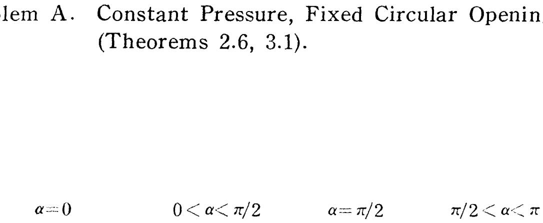
As in Theorem 4.2 there is a listing of eigenfunctions and eigenvalues by separation of variables, $(g_{jk}(\rho), \mu_{jk})$ where

$$\begin{gathered} \mu_{0k} < \mu_{1k} < \mu_{2k} < \cdots \text{ and} \\ \mu_{00} < \mu_{01} < \mu_{02} < \cdots . \end{gathered} \tag{4.27}$$

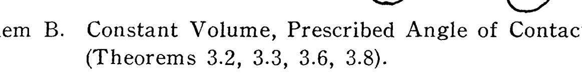
However, it need not be true that $\mu_{01} = 0$. The horizontal translation determines a normal perturbation which vanishes on the boundary only if the profile curve for $A$ has a horizontal tangent on the boundary. Therefore if the profile curve for $A$ does not possess a horizontal tangent then stability is determined by the symmetric perturbations. Once a horizontal tangent appears, the configuration is unstable since then $\mu_{01} \leqq 0$. (See also [13].)

For $u = 0$, the eigenfunctions are the Bessel functions and it is well known that $u = 0$ is stable for Problem C if $r_0 < \beta_1$ where $\beta_1$ is the first positive zero of $J_1(r)$.

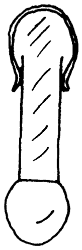
Finally we note that for Problem B with angle of contact $\alpha = 0$, (4.3) breaks down because of the boundary expression. The appropriate set of perturbations is to require $N = 0$ on $\partial A$, yielding the corresponding eigenvalues (4.27). In this case $u'(r) = 0$ on the boundary, hence $\mu_{01} = 0$. Thus, as in Theorem 4.3, stability is determined by the symmetric perturbations.




Problem A. Constant Pressure, Fixed Circular Opening. (Theorems 2.6, 3.1).

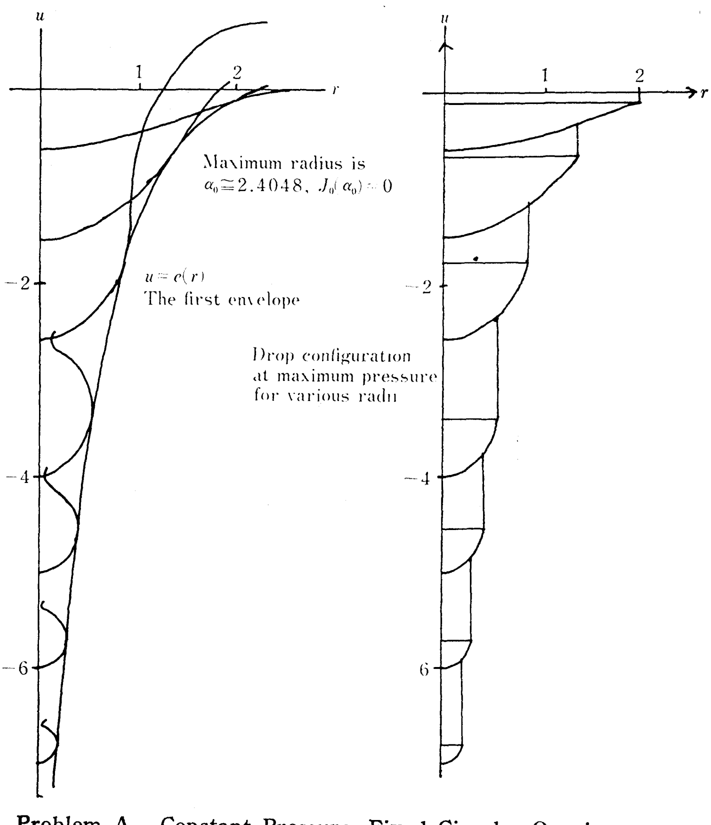



Problem B. Constant Volume, Prescribed Angle of Contact, $\alpha$. (Theorems 3.2, 3.3, 3.6, 3.8).



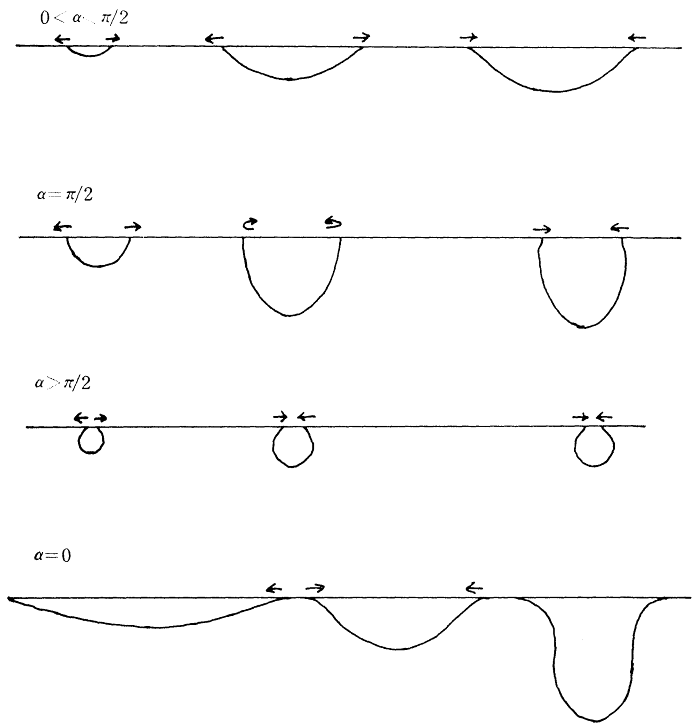
Problem C. Constant Volume, Fixed Circular Opening. (Theorems 3.4, 3.5, 3.7, 3.8).

FIGURE 1



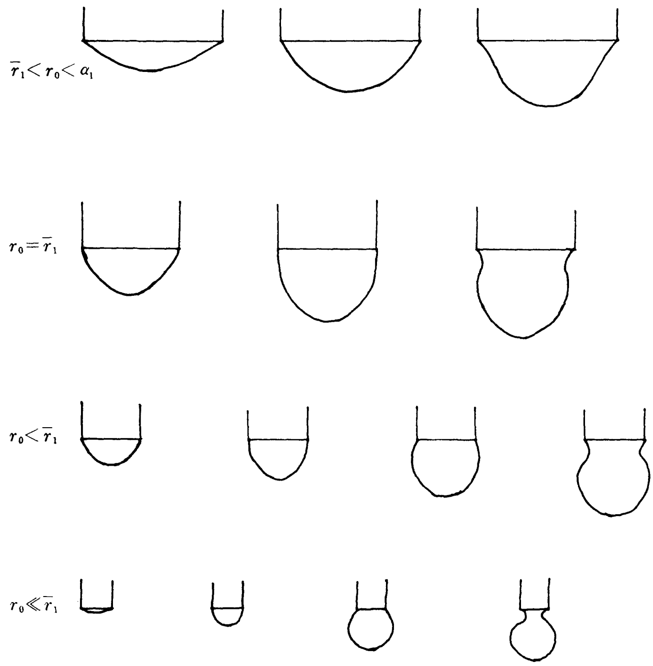

Problem A. Constant Pressure, Fixed Circular Opening. (Theorems 2.6 and 3.1)

FIGURE 2




Problem B. Costant Volume, Prescribed Angle of Contact. For $\alpha > 0$, the sketches represent stable drops of increasing volume, through the appearance of an inflection point on the profile curve, (Theorem 3.2). Also indicated is direction of contact radius with plate, (Theorem 3.6). For $\alpha = 0$, two stable drops are sketched, while the third must be unstable, (Theorem 3.8). Drop height increases monotonically with volume, (Theorem 3.6).

FIGURE 3




Problem C. Constant Volume, Fixed Circular Opening. Below are sketched stable drops configurations for openings of various radii, $r_0$: $r_0 < \bar{r}_1$, $r_0 = \bar{r}_1$, $\bar{r}_1 < r_0 < \alpha_1$. Here $\alpha_1 \cong 3.8317$ where $J_1(\alpha_1) = 0$ represents maximum possible radius. $\bar{r}_1 \cong .9176$ is the largest radial value for which a profile curve can possess a vertical tangent, (Theorems 3.4, 3.5). Drop height increases monotonically with volume, (Theorem 3.7).

FIGURE 4

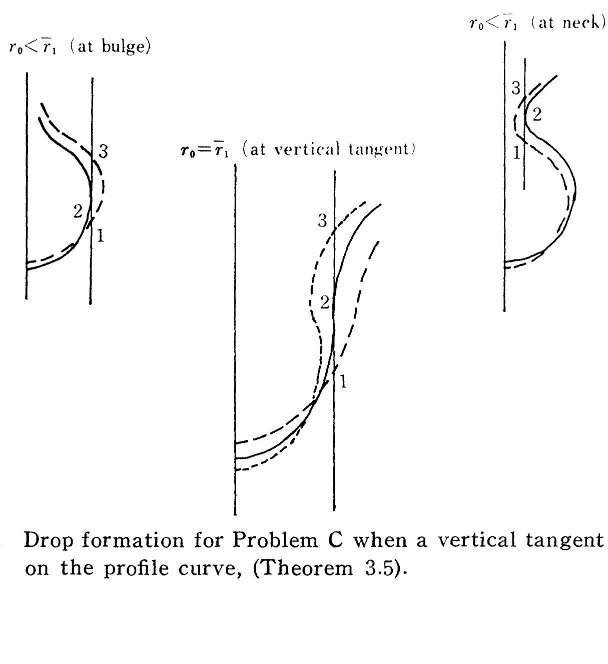



FIGURE 5. Drop formation for Problem C when a vertical tangent appears on the profile curve, (Theorem 3.5).

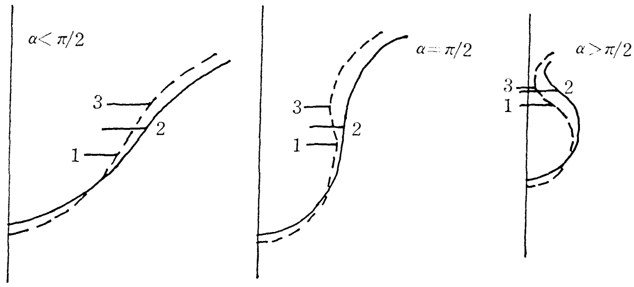



FIGURE 6. Drop formation at inflection points for Problem B, (Theorem 3.2).

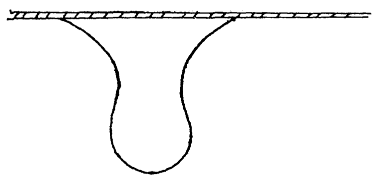


FIGURE 7. The drop is unstable for Problem B since the profile curve contains a second inflection point (Theorem 3.6).

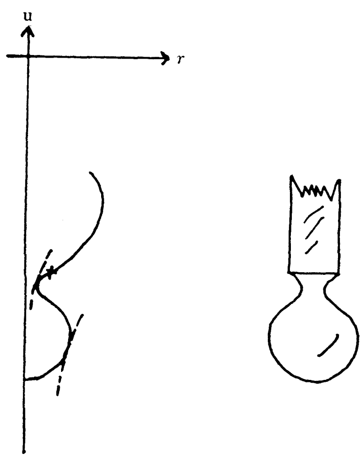



FIGURE 8. The second envelope contacts the profile curve between the neck and second bulge. The profile curve for the drop below contains the contact point with the second envelope. It must be unstable for Problem C. (Theorem 3.7, Case 6).

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