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INTEGRAL FORMULAS AND INTEGRAL TESTS FOR SERIES OF POSITIVE MATRICES

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The main results of this paper are integral formulas which generalize that used by Siegel to prove the Minkowski-Hlawka theorem in the geometry of numbers. The main application is the derivation of an integral test for Dirichlet series of several complex variables defined by sums over integer matrices. Such an integral test yields an easy proof of the convergence of Eisenstein series, whose analytic continuations are important in harmonic analysis on Minkowski's fundamental domain for the positive $n \times n$ real matrices modulo $n \times n$ integer matrices of determinant ± 1 (i.e., $O(n)\backslash GL(n, \boldsymbol{R})/GL(n, \boldsymbol{Z})$). These integral tests can also be used to analyze the analytic continuation of Eisenstein series as sums of higher dimensional incomplete gamma functions.

For example, the easiest case of the integral formulas (that due to Siegel) implies Theorem 1, which says that for every s in the interval $(0, n/2)$ there is a positive $n \times n$ matrix Y such that the Epstein zeta function of Y and s takes on any sign. Epstein's zeta function is the simplest of the Eisenstein series for $GL(n)$. Theorem 8 gives modified incomplete gamma expansions of Eisenstein series using a method of Selberg involving invariant differential operators on the space of $n \times n$ positive matrices. Our formulas (given in Theorems 4 and 5) can be applied to show that the expansion in Theorem 8 only provides an analytic continuation of the Eisenstein series in the last complex variable. The integral formulas given here can also be used to prove Theorems 3 and 6, which generalize the Minkowski-Hlawka result on the size of the minima of quadratic forms over the integer lattice.

The *general linear group* $GL(n, D)$ for a domain D consist of all invertible $n \times n$ matrices A such that A and A^{-1} have entries in D. The symmetric space for $GL(n, \boldsymbol{R})$ is the homogeneous space $O(n)\backslash GL(n, \boldsymbol{R})$, where $O(n)$ is the orthogonal group. This homogeneous space can be identified with $\mathscr{P}_n$, *the space of all positive definite symmetric $n \times n$ real matrices*, via the map sending the coset $O(n)g$ to the matrix $Y = {}^t gg$ for g in $GL(n, \boldsymbol{R})$. Here ${}^t g$ denotes the transpose of g. References for the general theory of such symmetric spaces are [16] and [44]. Note that A in $GL(n, \boldsymbol{R})$ acts on Y in $\mathscr{P}_n$ via $Y[A] = {}^t AYA$. There have been many applications of analysis on $\mathscr{P}_n$ in physics (cf. [6]), statistics (cf. [10] and [18]), and number theory (cf. [8], [14], [26], [33], [35], [44]).

Hecke's theory of the correspondence between modular forms and Dirichlet series with functional equations (cf. [14] and [29]) is central to much of modern number theory. The basic mechanism here is the Mellin transform and its inversion. For example, if $\theta(z) = \sum_{n \in Z} \exp(i\pi n^2 z)$, $\operatorname{Im} z > 0$, then the Mellin transform is:

$$2\pi^{-s}\Gamma(s)\zeta(2s) = \int_0^\infty y^{s-1}(\theta(iy) - 1)\, dy\ .$$

Many problems in number theory (e.g., the study of L-functions for algebraic number fields) lead to automorphic forms like theta on higher rank symmetric spaces. One might expect that the generalization of Hecke's results (e.g., to Siegel modular forms for $Sp(n, \boldsymbol{Z})$) would require a Mellin inversion formula for the symmetric space $\mathscr{P}_n$. Such an inversion formula was obtained by Harish-Chandra and Helgason in the mid-60's for any symmetric space (cf. [16, p. 60]). However Siegel modular forms f yield functions $f(iY)$ such that $f(iY) = f(iY[A])$ for all Y in $\mathscr{P}_n$ and A in $GL(n, \boldsymbol{Z})$. Thus the generalization of Hecke's results requires Mellin inversion for the fundamental domain $\mathscr{P}_n/GL(n, \boldsymbol{Z})$. Kaori Imai (cf. [17]) has shown how to generalize Hecke's results to Siegel modular forms using such Mellin inversion formulas. When the genus is 2 (i.e., the group is $Sp(2, \boldsymbol{Z})$), the Mellin inversion formula needed is the Roelcke-Selberg spectral resolution of the Laplacian on the upper half plane modulo $SL(2, \boldsymbol{Z})$ (cf. [3], [12], [20, p. 62], [22], [34], [44]).

Mellin inversion on $\mathscr{P}_n$ involves the expansion of an arbitrary function on $\mathscr{P}_n$ in Fourier integrals of eigenfunctions of the Laplacians on $\mathscr{P}_n$. Such eigenfunctions of the invariant differential operators on the symmetric space are known as spherical functions (which are generalizations of the conical functions $P_{-1/2+it}(x)$). Mellin inversion on $\mathscr{P}_n/GL(n, \boldsymbol{Z})$ involves analogous functions to the spherical functions called Eisenstein series (as well as the more mysterious cusp forms). References for the spherical functions and the Eisenstein series are [3], [11], [12], [16], [20], [22], [24], [34], [44].

The Eisenstein series are functions of two variables—one variable a vector s in $\boldsymbol{C}^q$ and the other an element Y of $\mathscr{P}_n$. As functions of Y they are automorphic forms for $GL(n)$ (cf. [5, pp. 199–210]), meaning that they are eigen for all the $GL(n, \boldsymbol{R})$-invariant differential operators on $\mathscr{P}_n$ and are invariant under $GL(n, \boldsymbol{Z})$.

These Eisenstein series can themselves be viewed as higher dimensional Mellin transforms of Siegel modular forms. Thus they provide important examples for the understanding of the problems of analytically continuing higher dimensional Mellin transfyrms of Siegel modular forms (which are not cusp forms). These problems were considered by Koecher in [19]. See also the review by Maass in [25]. The integral formulas to be proved in § 2 help to analyze what is going wrong here. Recently Arakawa has obtained a solution to the problem of analytic continuation in a fairly general case (cf. [2]). There has been much work on generalizing Hecke's correspondence to Siegel modular forms, though none but Imai [17] make use of higher dimensional Mellin inversion formulas. Koecher did not obtain an inverse correspondence in [19] since he was only considering Mellin transforms of one complex variable. Work on related matters can be found in [1], [24], [28], [31], [32], [44].

Now define the Eisenstein series for $GL(n)$. Suppose that n is partitioned as $n = n_1 + n_2 + \cdots + n_q$ with n_j in $\mathbf{Z}^+$. Then the *parabolic subgroup* $P(n_1, n_2, \cdots, n_q)$ of $GL(n)$ is defined to be the group of matrices with block form

$$\begin{pmatrix} U_1 & & * \\ & \ddots & \\ 0 & & U_q \end{pmatrix}, \qquad U_j \in GL(n_j) .$$

The *power function* or (left) spherical function $p_s(Y) = p_s^P(Y)$ is defined for s in $\mathbf{C}^q$ and Y in $\mathscr{P}_n$ by

$$(1.1) \qquad p_s(Y) = \prod_{j=1}^{q} |Y_j|^{s_j} \qquad \text{where} \quad Y = \begin{pmatrix} Y_j & * \\ * & * \end{pmatrix}$$
$$\text{with} \quad Y_j \in \mathscr{P}_{N_j} , \quad N_j = n_1 + n_2 + \cdots + n_j .$$

Here $|Y|$ denotes the *determinant* of Y. The general theory of such power functions is in [16] and [44]. The most important property of p_s is that it is eigen for the $GL(n, \mathbf{R})$-invariant differential operators on $\mathscr{P}_n$ (cf. [24, p. 69] or [44]).

The *Eisenstein series* $E_P(Y, s)$ for Y in $\mathscr{P}_n$ and s in $\mathbf{C}^q$ is defined by:

$$(1.2) \qquad E_P(Y, s) = \sum_{A \in GL(n, \mathbf{Z})/P} p_{-s}(Y[A]) ,$$
$$\text{for} \quad \operatorname{Re} s_j > (n_{j+1} + n_j)/2 , \quad j = 1, \cdots, q-1 .$$

The integral formulas of Theorems 4 and 5 will imply an easy integral test for the convergence of these series, providing a simplification of the classical argument (cf. [5, p. 207], [12], [20], and [24, Ch. 10]).

It is clear that $E_P(Y, s)$ is an automorphic form for $GL(n)$, since $E_P(Y[A], s) = E_P(Y, s)$, for all A in $GL(n, \mathbf{Z})$. Moreover, $E_P(Y, s)$ is an eigenfunction for all the $GL(n, \mathbf{R})$-invariant operators on $\mathscr{P}_n$, since p_s is.

The analytic continuation of $E_P(Y, s)$ to all s in $\mathbf{C}^q$ is important for harmonic analysis on $\mathscr{P}_n/GL(n, \mathbf{Z})$. Many authors have obtained the analytic continuation (cf. [3], [22], [24], [39], [40]), but it does not appear that anyone has obtained an explicit formula in the spirit of [23], except in the special case $q = 2$ in [2]. Instead of one functional equation, there are many. For example, if $q = n$, there are $n!$ functional equations corresponding to the permutations of the right variables (the Weyl group of the symmetric space).

There is also a sense (cf. [39]) in which the case $q = n$ of (1.2) contains all the others as specializations upon setting the appropriate variables equal to zero. Harmonic analysis on $\mathscr{P}_n/GL(n, \boldsymbol{Z})$ makes use of the Fourier expansions of Eisenstein series. Often only the constant term is required (cf. [5, pp. 237–238]). The complete expansion for the case $q = 2$ is obtained in [41].

There are many open questions. One would like to know the dimensions of spaces of automorphic forms for $GL(n, \boldsymbol{Z})$ with given eigenvalues for the $GL(n, \boldsymbol{R})$-invariant differential operators. For some systems of eigenvalues, one would expect only Eisenstein series. The only known results are for $SL(2, \boldsymbol{Z})$. References are [42], [44], [20]. Of course the discrete spectrum of the invariant differential operators is not even well understood for $SL(2, \boldsymbol{Z})$.

The special case $q = 2$ and $n_1 = 1$ of (1.2) is essentially *Epstein's zeta function* defined by

$$Z_1(Y, s) = \frac{1}{2} \sum_{a \in \boldsymbol{Z}^n - 0} Y[a]^{-s}, \qquad \text{for} \quad \operatorname{Re} s > n/2 . \tag{1.3}$$

For it is easy to pull out the g.c.d. of the components of a to obtain:

$$Z_1(Y, s) = \zeta(2s) E_{P(1, n-1)}(Y, (s, 0)) . \tag{1.4}$$

The Epstein zeta function for $n = 3$ gives the potential of a crystal (cf. [6]). Applications to number theory derive from Hecke's integral formula writing the Dedekind zeta function of a number field as a finite sum of integrals of Epstein zeta functions (cf. [42, formula (21)] and [14, p. 198ff.]). It follows (cf. [21, pp. 260–262]) that an easy proof of the Brauer-Siegel theorem on the growth of certain invariants of algebraic number fields would exist if one knew that $Z_1(Y, s) \leqq 0$ for all s in $(0, n/2)$ and certain Y in $\mathscr{P}_n$. Unfortunately life is not so simple, since one can prove:

THEOREM 1. *Given s in $(0, n/2)$, there exist quadratic forms Y in $\mathscr{P}_n$ such that $Z_1(Y, s) > 0$. There are also Y in $\mathscr{P}_n$ such that $Z_1(Y, s) < 0$ or $= 0$.*

This result is old for $n = 2$ (cf. [4] and [7]), though it is rather surprising that the Epstein zeta functions do not behave the way one conjectures Dedekind zeta functions behave. Of course, one does not know how to prove that Dedekind zeta functions are negative in (0, 1), even for quadratic fields (cf. [37] and [43]).

The case $q = 2$ of (1.2) was studied by Koecher in [19] (cf. [25]) in an attempt to generalize Hecke theory to $Sp(n, \boldsymbol{Z})$. Thus it is natural to call the case $q = 2$ of (1.2) the *Koecher zeta function*. Siegel used the analytic continuation of this function in [35, Vol. 1, pp. 459–468; Vol. 3, pp. 328–333] to prove Minkowski's formula for the volume of the fundamental domain of $\mathscr{P}_n/GL(n, \boldsymbol{Z})$.

An outline of the paper follows. Section 2 concerns integral formulas analogous to that used by Siegel in [35, Vol. 3, pp. 39–46] to prove the Minkowski-Hlawka theorem. First the special case $q = 2$ is considered. The integral formulas for general parabolic subgroups of $GL(n)$ are given in Theorems 4 and 5. The amazing thing is that Theorem 4, which is proved by a rather complicated induction, yields an easy proof of the convergence of Eisenstein series, while Theorem 5, which is proved easily using ideas of Weil [49] and the computation of the Jacobian of the partial Iwasawa decomposition (cf. (2.15)), forces a much more complicated proof of the convergence of the Eisenstein series. Aside from these convergence arguments, two kinds of applications of these integral formulas are given. The first is to the size of the minima m_Y^k defined by (2.11) for Y in $\mathscr{P}_n$. Such results appear in Theorems 3 and 6. The second type of application is to the study of properties of the analytic continuation of the Eisenstein series. For such applications one requires incomplete gamma expansions of the Eisenstein series generalizing (2.5) for the special case of Epstein's zeta function. The simplest special case of the integral formulas of Theorems 4 and 5 (Siegel's case) can be combined with formula (2.5) to prove Theorem 2. The latter concerns the vanishing of the integral of Epstein's zeta function over the determinant one surface of Minkowski's fundamental domain for $\mathscr{P}_n/GL(n, Z)$, assuming that the complex variable s lies in the critical strip. Theorem 1 is an easy corollary of Theorem 2. The analytic continuation of the more complicated Eisenstein series requires a study of higher dimensional incomplete gamma functions undertaken at the beginning of § 3. These higher dimensional incomplete gamma functions can be computed rather easily using methods devised by Riho Terras in [45] and [46]. Theorem 7 shows that these functions can, surprisingly perhaps, be realized not only as multi-dimensional integrals but also as one dimensional integrals. However the recursive methods of R. Terras seem to give speedier algorithms for computing such functions than the formula of Theorem 7. An expression for the general Eisenstein series as a modified sum of incomplete gamma functions is obtained in Theorem 8 using methods due to Selberg involving invariant differential operators on $\mathscr{P}_n$ to annihilate the singular parts of theta functions. Theorem 4 (or 5) shows that the formula of Theorem 8 does not converge for all values of all the complex variables.

The higher dimensional incomplete gamma functions of one complex variable have been shown by Lavrik [23] to arise whenever one has a Dirichlet series with functional equations involving multiple gamma factors. Such Dirichlet series include Artin L-functions of number fields. Thus knowledge of the behavior of these higher dimensional incomplete gamma functions is useful in algebraic number theory. There are applications to the Brauer-Siegel theorem on the growth of the product of the class number and the regulator with the discriminant (cf. [42. p. 9]), to the computation of class fields (cf. [38]), and there should be applications to the calculation of Dedekind zeta functions needed to answer the questions raised in [30]. The Eisenstein series require not only incomplete gamma functions of one complex variable but also those involving several complex variables, however.

2. **Integral formulas.** Minkowski found a *fundamental domain* $\mathscr{F}_n$ for $\mathscr{P}_n/GL(n, \boldsymbol{Z})$ (cf. [8], [24, pp. 123–139], [36, Ch. 2]). Define $\mathscr{S}\mathscr{P}_n = \{W \in \mathscr{P}_n \mid |W| = 1\}$ and $\mathscr{S}\mathscr{F}_n = \mathscr{F}_n \cap \mathscr{S}\mathscr{P}_n$. The integral formulas to be derived involve integrals over $\mathscr{F}_n$ and $\mathscr{S}\mathscr{F}_n$. The $GL(n, \boldsymbol{R})$-*invariant volume* element on $\mathscr{P}_n$ is

$$dv_n = |Y|^{-(n+1)/2} dY\,, \qquad \text{where}\quad dY = \text{Lebesgue measure on}\quad \mathscr{P}_n \subset \boldsymbol{R}^{n(n+1)/2}\,.$$

An $SL(n, \boldsymbol{R})$-invariant volume element dW on $\mathscr{S}\mathscr{P}_n$ can be defined by setting $t = |Y|$ and $Y = t^{1/n} W$ with $W \in \mathscr{S}\mathscr{P}_n$ to obtain:

$$dv_n = |Y|^{-(n+1)/2} dY = t^{-1} dt dW\,.$$

Note that, setting $G = SL(n, \boldsymbol{R})$ and $\Gamma = SL(n, \boldsymbol{Z})$, the quotient G/Γ has a G-invariant volume element $d\bar{g}$ which is unique up to a constant multiple (cf. [15], [35, Vol. 3, pp. 39–46], [48]). It follows that there is a constant c_n such that:

$$\int_{\mathscr{S}\mathscr{F}_n} f(W) dW = c_n \int_{G/\Gamma} f({}^t g g) d\bar{g}\,,$$

for Γ-invariant functions f on $\mathscr{S}\mathscr{P}_n$. The constant c_n will cancel out of the integral formulas.

The simplest integral formula to be discussed here is that which Siegel employed to prove the Minkowski-Hlawka theorem (cf. [8], [35, Vol. 3, pp. 39–46], and [49]). The formula says that if h is sufficiently nice with $h: \boldsymbol{R}^n \to \boldsymbol{C}$, then

$$\text{(2.1)} \qquad \operatorname{vol}(G/\Gamma)^{-1} \int_{G/\Gamma} \sum_{a \in \boldsymbol{Z}^n - 0} h(ga) d\bar{g} = \int_{\boldsymbol{R}^n} h(x) dx .$$

If h is radial; i.e., if $h(x) = f({}^t xx)$ then Siegel's integral formula becomes:

$$\text{(2.2)} \qquad \operatorname{vol}(\mathscr{S}\mathscr{F}_n)^{-1} \int_{\mathscr{S}\mathscr{F}_n} \sum_{a \in \boldsymbol{Z}^n - 0} f(W[a]) dW = \int_{\boldsymbol{R}^n} f({}^t xx) dx .$$

Three applications of (2.2) come to mind. The first is that made by Siegel. Set

$$\text{(2.3)} \qquad m_Y = \min \{ Y[a] \mid a \in \boldsymbol{Z}^n - 0 \text{ for } Y \in \mathscr{P}_n \} .$$

Then if $k_n = n/2\pi e$, one sees that there exist Y in $\mathscr{P}_n$ such that

$$\text{(2.4)} \qquad m_Y > k_n |Y|^{1/n} , \quad \text{for } n \text{ sufficiently large.}$$

Hilbert's problem 18 includes that of finding W in $\mathscr{P}_n$ such that m_W is maximal for fixed n. It is only solved for $n = 1, 2, \cdots, 8$ (cf. [9], [26], [27]).

The second application of (2.2) is to find an integral test for the convergence of Epstein's zeta function. Formula (2.2) implies that

$$\operatorname{vol}(\mathscr{S}\mathscr{F}_n)^{-1} \int_{W \in \mathscr{S}\mathscr{F}_n} \sum_{\substack{a \in \boldsymbol{Z}^n - 0 \\ W[a] \geqq 1}} W[a]^{-s} dW = \int_{\substack{x \in \boldsymbol{R}^n \\ {}^t xx \geqq 1}} ({}^t xx)^{-s} dx .$$

The integral on the right is easily evaluated as $2\pi^{n/2}\Gamma(n/2)(s - n/2)^{-1}$, provided that $\operatorname{Re} s > n/2$. Then Fubini's theorem says that the series being integrated on the left must converge for almost all W in $\mathscr{S}\mathscr{F}_n$. This series differs from $Z_1(W, s)$ by at most a finite number of terms. Thus $Z_1(W, s)$ converges for $\operatorname{Re} s > n/2$ and almost all W in $\mathscr{S}\mathscr{F}_n$. In order to deduce the convergence for all Y in $\mathscr{P}_n$ note that there is a positive constant c such that: $cI[a] \leqq Y[a] \leqq c^{-1}I[a]$ for all a in $\boldsymbol{Z}^n$, where I is the identity matrix.

The last application of (2.2) is to prove Theorem 1 which is an easy corollary of

THEOREM 2. *For all s with $0 < \operatorname{Re} s < n/2$,* $\displaystyle\int_{\mathscr{S}\mathscr{F}_n} Z_1(W, s) dW = 0.$

Proof. Since Epstein's zeta function is the Mellin transform of a theta function one has (cf. [42, p. 6])

$$2\pi^{-s}\Gamma(s)Z_1(Y, s) = |Y|^{-1/2}\left(s - \frac{n}{2}\right)^{-1} - \frac{1}{s} + \sum_{a \in Z^n - 0}\left\{G(s, \pi Y[a]) + |Y|^{-1/2}G\left(\frac{n}{2} - s, \pi Y^{-1}[a]\right)\right\}, \tag{2.5}$$

where the incomplete gamma function is defined by:

$$G(s, a) = \int_1^\infty t^{s-1}e^{-at}dt \qquad \text{for} \quad \mathrm{Re}\, a > 0 . \tag{2.6}$$

It is now easy to prove Theorem 2 by combining (2.2) and (2.5).

Next consider the more general integral formula stated by Siegel in [35, Vol. 3, p. 46]. Suppose that $1 \leqq k < n$. Then for a sufficiently nice function h on the space of real $n \times k$ matrices:

$$\mathrm{vol}\,(G/\Gamma)\int_{G/\Gamma} \sum_{\substack{N \in Z^{n\times k} \\ \mathrm{rank}\ N = k}} h(gN)d\bar{g} = \int_{R^{n\times k}} h(X)dX . \tag{2.7}$$

The proof of (2.7) can be carried out very easily using the arguments of [49]. For the applications to integral tests, note that if $h(X) = f({}^tXX)$, then:

$$\mathrm{vol}\,(\mathscr{S}\mathscr{F}_n)^{-1}\int_{\mathscr{S}\mathscr{F}_n} \sum_{\substack{N \in Z^{n\times k} \\ \mathrm{rank}\ N = k}} f(W[N])dW = \int_{R^{n\times k}} f({}^t(XX)dX . \tag{2.8}$$

The integral on the right in (2.8) was evaluated by the statistician Wishart (cf. [10, Ch. 4]) as:

$$\int_{R^{n\times k}} f({}^tXX)dX = c_{n,k}\int_{\mathscr{P}_k} f(Y)|Y|^{n/2}dv_k , \tag{2.9}$$

where

$$c_{n,k} = \prod_{j=n-k+1}^{n} \pi^{j/2}\Gamma(j/2)^{-1} . \tag{2.10}$$

As for (2.2), there are three applications of formula (2.8). The first is a result on minima of $Y \in \mathscr{P}_n$ defined by

$$m_Y^k = \{\min |Y[A]| \mid A \in \boldsymbol{Z}^{n\times k},\ \mathrm{rank}\ A = k\} . \tag{2.11}$$

Then one has:

THEOREM 3. *Suppose that k is fixed with $1 \leqq k < n$ and that r is less than $(n/2\pi e)^k$ with n sufficiently large depending on k. Then there exist quadratic forms $Y \in \mathscr{P}_n$ such that $m_Y^k > r|Y|^{k/n}$.*

Proof. Let $\chi_{[0,r]}(t)$ denote the function which is 1 for t in the interval $[0, r]$ and 0 otherwise. Then (2.8) implies that

$$\operatorname{vol}(\mathscr{S}\mathscr{F}_n)^{-1}\int_{\mathscr{S}\mathscr{F}_n}\sum_{\substack{N\in Z^{n\times k}/GL(k,Z)\\ rk(N)=k}}\chi_{[0,r]}(|W[N]|)dW = c_{n,k}\int_{\substack{Y\in\mathscr{P}_k\\ |Y|\leqq r}}|Y|^{n/2}dv_k$$

$$= \frac{2}{n}c_{n,k}\operatorname{vol}(\mathscr{S}\mathscr{F}_k)r^{n/2}\,.$$

Stirling's formula and the formula for $c_{n,k}$ imply that if r is less than $(n/2\pi e)^k$, then the product on the right-hand side of the above equality approaches zero as n goes to infinity (holding k fixed). Thus, for sufficiently large n, the average value of the function $\sum\chi_{[0,r]}(|W[N]|)$, summed over rank k matrices N in $Z^{n\times k}$ modulo $GL(k, Z)$, is less than one and so the function must be less than one for some W in $\mathscr{S}\mathscr{F}_n$. Theorem 3 follows easily.

The second application of (2.8) is to give a quick proof of the convergence of Koecher's zeta function (the case $q = 2$ of (1.2)). Clearly (2.8) and (2.9) imply that

$$\int_{W\in\mathscr{S}\mathscr{F}_n}\sum_{\substack{A\in Z^{n\times k}/GL(k\ Z)\\ rk(A)=k\\ |W[A]|\geqq 1}}|W[A]|^{-s}dW$$

$$= c\operatorname{vol}(\mathscr{S}\mathscr{F}_k)\left(s-\frac{n}{2}\right)^{-1}\qquad\text{for}\quad \operatorname{Re}s > n/2\,.$$

The convergence proof proceeds as for Epstein's zeta function except that one requires Minkowski's reduction theory (cf. [36]) to see that there is a constant $c > 0$ such that if $I[A]\in\mathscr{F}_k$, then $|Y[A]|\leqq c|I[A]|$. Also there is a constant $c' > 0$ such that if $Y[A]\in\mathscr{F}_k$, then $|Y[A]|\geqq c'|I[A]|$. Here c, c' depend on Y, not A.

The third application of (2.8) is to the analysis of the divergent integrals arising in the attempt to use Riemann's method of analytic continuation (the method leading to formula (2.5)) on Koecher's zeta function. This requires (cf. [42, p. 13]) a study of the integrals $S_r(Q, s)$ for $Q\in\mathscr{P}_n$, $s\in C$, $1\leqq r < k\leqq n$, defined by:

$$S_r(Q,s) = \int_{\substack{|X|\geqq 1\\ X\in\mathscr{F}_k}}|X|^s\sum_{A,B}\exp(-\pi(Q[B]X[A]))dv_k\,, \tag{2.12}$$

where $B\in Z^{n\times r}$ has rank k and is summed modulo $GL(r, Z)$ and $A\in Z^{k\times r}$ can be taken as the first r columns of a matrix running through representatives of $SL(k, Z)/H(r, Z)$ with $H(r, Z)$ denoting the subgroup of matrices of the form $\begin{pmatrix} I_r & * \\ 0 & * \end{pmatrix}$. Here I_r is the $r\times r$ identity matrix. Formulas (2.8) and (2.9) show that

$$S_r(Q, s) = \operatorname{vol}(\mathcal{S}\mathcal{F}_{k-r}) \int_{t \geq 1} t^s \int_{Y \in \mathcal{P}_r} |Y|^{k/2} \times \sum_B \exp(-\pi t^{1/k}(Q[B]Y)) dv_r \frac{dt}{t}. \tag{2.13}$$

This integral diverges like the Dirichlet series $E_{P(r,n-r)}(Q, k/2)$. In order to compute the constant in the preceeding formula for $S_r(Q, s)$, one needs to know that the sum over A is obtained from summing over all N in $Z^{k\times r}$ of rank k, if one divides by the product of $\zeta(k-j)$, for $j = 0, 1, \cdots, r-1$ (cf. [42, formula (3), pape 2]). One also needs to see that

$$\operatorname{vol}(\mathcal{S}\mathcal{F}_k) \prod_{j=0}^{r-1} \zeta(k-j)^{-1} c_{k,r} = \operatorname{vol}(\mathcal{S}\mathcal{F}_{k-r}).$$

The formula for $\operatorname{vol}(\mathcal{S}\mathcal{F}_n)$ is needed here (cf. [42, formula (39)].

Next consider integral formulas which generalize (2.8) to arbitrary parabolic subgroups of $GL(n)$.

THEOREM 4. *Suppose that in the notation of* (1.1) *the function* $f\colon \mathcal{P}_n \to C$ *has the form*

$$f(Y) = \prod_{j=1}^{q} f_j(|Y_j|).$$

Then, if $N_j = n_1 + \cdots + n_j$ $(j = 1, 2, \cdots, q)$:

$$\int_{\mathcal{F}_n} \sum_{A \in GL(n,Z)/P(n_1,\cdots,n_q)} f(Y[A]) dv_n = \operatorname{vol}(\mathcal{S}\mathcal{F}_{n_q}) \int_{t_q > 0} f_q(t_q) t_q^{-N_{q-1}/2} \frac{dt_q}{t_q} \times \prod_{j=1}^{q-1} \left\{ \operatorname{vol}(\mathcal{S}\mathcal{F}_{n_j}) \int_{t_j > 0} f_j(t_j) t_j^{(n_j+n_{j+1})/2} \frac{dt_j}{t_j} \right\}. \tag{2.14}$$

Proof.

The case $q = 2$ is a fairly easy consequence of (2.8). One needs to use the arguments given in evaluating the constant in formula (2.13). Also one must eliminate the restriction to W of determinant one in the left-hand side of formula (2.8). This is done by replacing $f(W)$ by $g(t)f(t^{1/n}W)$ and integrating over dt/t.

To prove the general result use induction on q and write $A \in GL(n, Z)/P$ as $A = BC$ with $P = P(n_1, \cdots, n_q)$ and

$$B = (B_1 \quad *) \in GL(n, Z)/P(N_{q-1}, n_q), \quad B_1 \in Z^{n\times N_{q-1}}$$

$$C = \begin{pmatrix} D & * \\ 0 & * \end{pmatrix} \in P(N_{q-1}, n_q)/P, \quad D \in GL(N_{q-1}, Z)/P(n_1, \cdots, n_{q-1}).$$

Note that if $A = (A_1\ *)$ with $A_1 \in \boldsymbol{Z}^{n\times N_{q-1}}$, then $A_1 = B_1 D$. Thus

$$\begin{aligned}\int_{\mathscr{F}_n} &\sum_{A\in GL(n\ Z)/P} f(Y[A])dv_n \\ &= \operatorname{vol}(\mathscr{S}\mathscr{F}_{n_q}) \int_{t_q>0} f_q(t_q) t_q^{-N_{q-1}/2} \frac{dt_q}{t_q} \\ &\times \int_{\mathscr{F}_{N_{q-1}}} f_{q-1}(|Y|)|Y|^{n/2} \sum_{D\in GL(N_{q-1},Z)/P^*} \prod_{j=1}^{q-2} f_j(|(Y[D])_j|)dv_{N_{q-1}}\end{aligned}$$

where $P^* = P(n_1, \cdots, n_{q-1})$. The proof is easily completed by induction.

There is another result of the same type as Theorem 4 which does not make such restrictive hypotheses on the functions f involved. If $P(n_1, \cdots, n_q) = P$ is a given parabolic subgroup, define

$$A = \left\{ \begin{pmatrix} U_1 & & 0 \\ & \ddots & \\ 0 & & U_q \end{pmatrix} \middle| U_j \in GL(n_j, \boldsymbol{R}) \text{ for } j = 1, \cdots, q \right\},$$

$$N = \left\{ \begin{pmatrix} I_{n_1} & & * \\ & \ddots & \\ 0 & & I_{n_q} \end{pmatrix} \middle| I_{n_j} = \text{the } n_j \times n_j \text{ identity matrix} \right\}.$$

Then introduce the partial Iwasawa decomposition of $Y \in \mathscr{P}_n$ as:

$$Y = \begin{pmatrix} Q_1 & & 0 \\ & \ddots & \\ 0 & & Q_q \end{pmatrix} \begin{bmatrix} I_{n_1} & & R_{ij} \\ & \ddots & \\ 0 & & I_{n_q} \end{bmatrix} \quad \text{with } Q_j \in \mathscr{P}_{n_j} \text{ and } R_{ij} \in \boldsymbol{R}^{n_i\times n_j}. \tag{2.15}$$

THEOREM 5. *Suppose that* $g\colon \mathscr{P}_n/H \to \boldsymbol{C}$ *where* $H = A_Z N_R$ *and* $g(Y) = h(Q_1, \cdots, Q_q)$ *with* $Q_j \in \mathscr{F}_{n_j}$, *using the notation of* (2.15). *Here* $g\colon \mathscr{P}_n/H \to \boldsymbol{C}$ *means that* $g(Y[V]) = g(Y)$ *for all* V *in* H. *Then*

$$\begin{aligned}\int_{\mathscr{F}_n} &\sum_{A\in GL(n,Z)/P} g(Y[A])dv_n \\ &= c\int_{Q_j\in\mathscr{F}_{n_j}} h(Q_1, \cdots, Q_q)|Q_j|^{(n-N_j-N_{j-1})/2} dv_{n_j},\end{aligned} \tag{2.16}$$

where $P = P(n_1, \cdots, n_q)$ *and*

$$c = \prod_{j=1}^{q} \operatorname{vol}(\mathscr{S}\mathscr{F}_{n_j}). \tag{2.17}$$

Proof. Argue as Weil did in [49] to see that the integral formula (2.16) follows from the computation of the Jacobian of the partial Iwasawa decomposition (cf. [24, pp. 149-150], [47, p. 293, bottom]):

$$dv_n = \prod_{j=1}^{q} |Q_j|^{(n-N_j-N_{j-1})/2} dv_{n_j} \prod_{1\leq i<j\leq q} dR_{ij} ,$$

where, as usual, $N_j = n_1 + \cdots + n_q$.

The constant c can be found by comparing the formulas resulting from Theorems 4 and 5 if the function being integrated over the fundamental domain is the Eisenstein series.

It is now easy to see that the Eisenstein series (1.2) converges when $\operatorname{Re} s_j > (n_j + n_{j+1})/2$, for all $j = 1, 2, \cdots, q-1$. The easiest proof uses (2.14) to obtain:

$$\int_{\mathscr{F}_n} \sum_{\substack{A\in GL(n,Z)/P \\ |(Y[A])_j|\geq 1,\ \text{all } j=1,\ldots,q}} p_{-s}(Y[A])dv_n = \frac{\operatorname{vol}(\mathscr{S}\mathscr{F}_{n_q})}{s_q + N_{q-1}/2} \prod_{j=1}^{q-1} \frac{\operatorname{vol}(\mathscr{S}\mathscr{F}_{n_j})}{s_j - (n_j + n_{j+1})/2} ,$$

for $\operatorname{Re} s_j > (n_j + n_{j+1})/2$, when $j = 1, \cdots, q-1$, and $\operatorname{Re} s_q > -N_{q-1}/2$. To complete the proof that the Eisenstein series (1.2) converge in this region, use the reduction theory of Minkowski, just as in the case $q = 2$, to see that it suffices to show convergence at just one point $W \in \mathscr{S}\mathscr{F}_n$. One also needs to know that the number of $A \in GL(n, Z)/P$ such that $|(Y[A])_j| \leq 1$ for some $j \in \{1, 2, 3, \cdots, q\}$ is finite.

The second application of (2.14) is to derive:

THEOREM 6. *Let $k_n = \operatorname{vol}(\mathscr{S}\mathscr{F}_n)$ and suppose that $0 < r < e^{-1/2}[k_n(n-1)/2]^{1/(n-1)}$. Then there is a number $j \in \{1, 2, 3, \cdots, n-1\}$, a matrix $A \in GL(n, Z)$ whose first j columns form the matrix A_j, and a quadratic form W in $\mathscr{S}\mathscr{F}_n$ such that $|W[A_j]| > r$.*

Proof. In (2.14) assume that $P(n_1, \cdots, n_q) = P(1, \cdots, 1)$ and set $f_j(t) = \chi_{[0,r]}(t)$, for $j = 1, 2, \cdots, n-1$, $f_n(t) = \chi_{[a,\infty]}(t)$, to obtain:

$$k_n^{-1} \int_{\mathscr{F}_n} \sum_{A\in GL(n,Z)/P} f(Y)dv_n = k_n^{-1} \frac{2}{n-1} \left(\frac{r}{\sqrt{a}}\right)^{n-1} .$$

Rewrite the integral on the left using the change of variables $Y = t^{1/n}W$, with $0 < t$ and $W \in \mathscr{S}\mathscr{F}_n$ as:

$$k_n^{-1} \int_{\mathscr{S}\mathscr{F}_n} \sum_{A\in GL(n,Z)/P} \int_a^\infty \prod_{j=1}^{n-1} \chi_{[0,r]}(t^{j/n} |W[A_j]|) \frac{dt}{t} dW .$$

If the preceding integral is less than one, then there must exist $W \in \mathscr{S}\mathscr{F}_n$ and A in $GL(n, Z)$ such that

$$\int_a^c t^{-1}dt < 1 \,, \quad \text{where} \quad c = \min\{(r^{-1}|\, W[A_j]|)^{-n/j} \,|\, j = 1, \cdots, n-1\} \,.$$

It follows that $c < ae$. Thus if we set $a = 1/e$, the proof of Theorem 6 is completed.

The constant $e^{-1/2}(k_n(n-1)/2)^{1/(n-1)}$ is asymptotic to $2e^{-5/8}n^{n/4} \times (2\pi e^{3/2})^{-n/4}$, as n approaches infinity. Comparing Theorem 6 with Theorem 3, one is left with the question: which m_W^j will be large? In fact, one can use the proof of Theorem 3 to see that m_W^j for $j = [n/2]$ will be the culprit.

3. Remarks on the analytic continuation of Eisenstein series. The Mellin inversion formula for $\mathscr{P}_n/GL(n, \boldsymbol{Z})$ requires the analytic continuation of the Eisenstein series $E_P(Y, s)$ beyond $\operatorname{Re} s_j > (n_j + n_{j+1})/2$. To do the analytic continuation it is natural to attempt to write the Eisenstein series as sums of higher dimensional incomplete gamma functions, imitating the method of Riemann which gives formula (2.5) (cf. [23]). The motivation for using this method rather than the Maass-Selberg relations or existential arguments from several complex variables (cf. [20] and [24, pp. 274-275]) is that one hopes to derive explicit formulas for the Eisenstein series in order to study their properties further.

It will help to write the Eisenstein series in terms of *Selberg's zeta function* defined for Y in $\mathscr{P}_n$ and s in $\boldsymbol{C}^{q-1}$ by:

$$Z_{n_1,\cdots,n_{q-1}}(Y, s) \sum_{A \in \boldsymbol{Z}^{n \times k}, rk k / P(n_1,\cdots,n_{q-1})} p_{-s}(Y[A]) \,, \tag{3.1}$$

where $k = n_1 + \cdots + n_{q-1}$, $\operatorname{Re} s_j > (n_j + n_{j+1})/2$, $(j = 1, \cdots, q-1)$. Since Selberg proved all the basic properties of these functions (cf. [34, pp. 79-81] for the special case of $SL(3)$), there is good reason for calling (3.1) Selberg's zeta function. However another function (also studied in [34, p. 75]) has already usurped that name. Maass ([24, § 17]) calls a slightly different function (with more Riemann zeta factors) Selberg's zeta function.

The relation between (3.1) and (1.2) is

$$Z_{n_1,\cdots,n_{q-1}}(Y, s) = E_{P(n_1,\cdots,n_q)}(Y, (s_1, \cdots, s_{q-1}, 0)) \prod_{i=1}^{q-1} \prod_{j=1}^{n} \zeta(r_{i,j}) \,. \tag{3.2}$$

Here $\zeta(s)$ is Riemann's zeta function and its argument is given by:

$$r_{i,j} = 2\{s_i + \cdots + s_{q-1} - (k - N_{i-1} - j)/2\} \,, \qquad \text{for} \quad N_j = n_1 + \cdots + n_j \,. \tag{3.3}$$

The proof of (3.2) involves the decomposition which Koecher proved in [19] to do the special case $q = 2$ and $n_q = 0$. The decomposition says that $A \in \boldsymbol{Z}^{n\times k}$ of rank k modulo $P(n_1, \cdots, n_{q-1})$ can be written uniquely as a product BC with $B \in \boldsymbol{Z}^{n\times k}$, $(B \ \ *) \in GL(n, Z)/P(n_1, \cdots, n_q)$ and

$$C = \begin{pmatrix} d_1 & & & d_{ij} \\ & d_2 & & \\ & & \ddots & \\ & 0 & & d_k \end{pmatrix}, \quad \text{with} \quad d_i > 0 \quad \text{and} \quad d_{ij} (\operatorname{mod} d_i) .$$

One would like to expand the Selberg zeta functions or Eisenstein series in higher dimensional incomplete gamma functions. Let us discuss the appropriate gamma function first. For $s \in \boldsymbol{C}^{q-1}$ with $\operatorname{Re} s_j$ sufficiently large, define *the higher dimensional gamma function* by:

$$(3.4) \qquad \Gamma_{n_1,\cdots,n_{q-1}}(s) = \int_{\mathscr{P}_k} p_{-s}(X^{-1}) \exp\left[-\pi \operatorname{Tr}(X)\right] dv_k .$$

Here $k = n_1 + \cdots + n_{q-1}$. Then

$$(3.5) \qquad \Gamma_{n_1,\cdots,n_{q-1}}(s) = \prod_{i=1}^{q-1} \prod_{j=1}^{n_i} \pi^{-r_{i,j}/2} \Gamma(r_{i,j}/2) ,$$

with $r_{i,j}$ as in (3.3). Special cases of this higher dimensional gamma function appear in work of Siegel (cf. [35, Vol. 1, p. 384]) and in multivariate statistics (cf. [10, p. 312]).

To prove (3.5), write $X = {}^tTT$ for T an upper triangular matrix with $t_j > 0$ as its jth diagonal element and note that $|\partial X/\partial T| = 2^n t_1 t_2^2 \cdots t_n^n$. The result follows easily.

Motivated by (3.5), define the *higher dimensional incomplete gamma function* for $s \in \boldsymbol{C}^{q-1}$ and $Y \in \mathscr{P}_n$, $k = n_1 + \cdots + n_{q-1}$, by:

$$(3.6) \qquad I_{n_1,\cdots,n_{q-1}}(s, Y) = \int_{X \in \mathscr{P}_k, |X| \geq 1} p_{-s}(X^{-1}) \exp\left[-\pi \operatorname{Tr}(XY)\right] dv_k .$$

Set $Y = {}^tTT$ with T upper triangular and make the change of variables $V = X[{}^tT]$ in (3.6) to obtain:

$$I_{n_1,\cdots,n_{q-1}}(s, Y) = p_{-s}(Y) \int_{V \in \mathscr{P}_k \ |V| \geq |Y|} p_{-s}(V^{-1}) \exp\left[-\pi \operatorname{Tr}(V)\right] dv_k .$$

The change of variables that proved (3.5) suggests that we define yet another incomplete gamma function:

$$(3.7) \qquad \Gamma_k(s, a) = \int_{\Pi v_j \geq a^k \ v_j > 0} \prod_{j=1}^{k} v_j^{s_j} e^{-v_j} \frac{dv_j}{v_j} , \qquad \text{for} \quad s \in \boldsymbol{C}^k , \quad a > 0 .$$

The change of variables proving (3.5) shows that:

$$(3.8)\qquad I_{n_1,\cdots,n_{q-1}}(s,\ Y) = p_{-s}(Y)\Gamma_k\left(\frac{r}{2},\ \pi\,|\,Y\,|^{1/k}\right)\pi^{-b}\ ,\quad \text{with}\quad b = \frac{1}{2}\sum_{i=1}^{q-1}\sum_{j=1}^{n_i} r_{i,j}\ ,$$

where $r_{i,j}$ is as in (3.3) and r denotes the vector whose N_{i-1+j}th component is $r_{i,j}$.

Note that $I_1(s,\ y) = G(s,\ y) = y^{-s}\Gamma_1(s,\ y)$, where $G(s,\ y)$ was the function (2.6) which appeared in the incomplete gamma expansion of Epstein's zeta function.

It is also useful to define yet another analogue of $G(s,\ a)$ in higher dimensions by:

$$(3.9)\qquad G_k(r,\ a) = \int_{\Pi y_j\geqq 1,\, y_j>0}\prod_{j=1}^{k} y_j^{r_j}\exp\left(-a_jy_j\right)\frac{dy_j}{y_j}\ ,$$

for $a,\ r\in \boldsymbol{C}^k$ and $\operatorname{Re} a_j > 0$. Then one has

$$(3.10)\qquad \begin{cases} G_k(r,\ a) = \displaystyle\prod_{j=1}^{k} a_j^{-r_j}\Gamma_k\left(r,\ \prod_{j=1}^{k} a_j^{1/k}\right)\ , \\ I_{n_1,\cdots,n_{q-1}}(s,\ Y) = p_{-s}(Y)\,|\,Y\,|^{b/k}G_k\left(\dfrac{r}{2},\ \pi\,|\,Y\,|^{1/k}(1,\ \cdots,\ 1)\right)\ , \end{cases}$$

with b and r as defined in (3.8).

The most important property of these incomplete gamma functions is their exponential decay in the second argument as it becomes large. It is shown in [13] that

$$\Gamma_k(r,\ x) \sim \sqrt{(2\pi)^{k-1}/k}\,e^{-kx}x^c \quad \text{as}\quad x\to\infty\ ,\quad \text{if}\quad c = \sum_{j=1}^{k} r_j - \frac{k+1}{2}\ .$$

It follows that:

$$(3.11)\qquad I_{n_1,\cdots,n_{q-1}}(s,\ Y) \sim \exp\left(-k\pi\,|\,Y\,|^{1/k}\right)p_{-s}(Y)\,|\,Y\,|^{c}\pi^{-1}\sqrt{2^{k-1}/k}\ ,\quad |\,Y\,|\to\infty\ ,$$

where c is as in the preceding sentence. This is the result needed for the analytic continuation of Eisenstein series as sums of such incomplete gamma functions.

Lavrik has obtained in [23] expansions of Dirichlet series with functional equations involving multiple gamma factors of one complex variable, assuming that the analytic continuation and functional equation have already been proved by some other method. Lavrik writes his functions as inverse Mellin transforms. The following theorem connects the two expressions.

THEOREM 7. *If* $\tilde{s} = (s, \cdots, s)$, *then*

$$G_k(\tilde{s}, (a, \cdots, a)) = (2\pi i)^{-1} \int_{\mathrm{Re}\, w=c} a^{-w} \Gamma(w/k)^k (w - ks)^{-1} dw .$$

Proof. The theorem is equivalent to the formula:

$$\int_{a>0} a^{s-1} \int_{\Pi t_j \geqq 1} \prod t_j^{z-1} e^{-at_j} dt_j da = \Gamma(s/k)^k (s - kz)^{-1} .$$

To prove this make the change of variables $y_k = \prod_{j=1}^k t_j$, $y_{k-1} = at_{k-1}, \cdots, y_1 = at_1$, $b = a^k$. Integrate first over b.

The analytic continuation of the Eisenstein series for $GL(n)$ starts from the higher dimensional Mellin transform formula:

$$\begin{aligned} (3.12) \qquad 2\Lambda_{n_1,\cdots,n_{q-1}}(Y, s) &= 2\Gamma_{n_1,\cdots,n_{q-1}}(s) Z_{n_1,\cdots,n_{q-1}}(Y, s) \\ &= \int_{\mathscr{F}_{tP}} p_{-s}(X^{-1}) \theta_k(Y, X) dv_k . \end{aligned}$$

Here for $1 \leqq k < n$, $Y \in \mathscr{P}_n$ and $V \in \mathscr{P}_k$, the *theta function* is defined by:

$$(3.13) \qquad \theta_r(Y, V) = \sum_{\substack{A \in Z^{n \times k} \\ rkr}} \exp\left[-\pi \operatorname{Tr}(Y[A]V)\right] , \qquad \theta = \sum_{r=0}^k \theta_r .$$

And in (3.12), $\mathscr{F}_{t_P}$ denotes a fundamental domain for $\mathscr{P}_k$ modulo *the transposed parabolic subgroup* ${}^tP = \{V \mid {}^tV \in P = P(n_1, \cdots, n_{q-1})\}$. If $q = 2$, then $\mathscr{F}_{t_P} = \mathscr{F}_P = \mathscr{F}_k = \mathscr{F}_k/GL(k, \boldsymbol{Z})$.

When using Riemann's method to obtain the analytic continuation of the Eisenstein series, the singular terms θ_r cause divergent integrals to arise as we noted in formulas (2.12) and (2.13) in the case $q = 2$. In the 1960's Selberg had the idea of using $GL(k, \boldsymbol{R})$-invariant differential operators on $X \in \mathscr{P}_k$ to annihilate the $\theta_r(Y, X)$ for $0 \leqq r \leqq k - 1$. The details are to be found in [24, § 16] and [39]. Set

$$(3.14) \qquad D_X = \det\left(\frac{1}{2}(1 + \delta_{ij}) \frac{\partial}{\partial x_{ij}}\right) , \qquad \text{for} \quad X = (x_{ij}) \in \mathscr{P}_k ,$$

where δ_{ij} is Kronecker's delta. Then $D_X \exp[\operatorname{Tr}(XY)] = |Y| \exp[\operatorname{Tr}(XY)]$. So D_X will kill the singular terms in $\theta(Y, X)$. But D_X is not $GL(k, \boldsymbol{R})$-invariant. To obtain such an operator, look at

$$(3.15) \qquad D_a = |X|^a D_X |X|^{1-a} , \qquad \text{for} \quad a \in \boldsymbol{R} .$$

Note that the algebra of all $GL(k, \boldsymbol{R})$-invariant differential operators on $\mathscr{P}_k$ is commutative (cf. [15, p. 396], [24, p. 56], [34, p. 57], [44, Ch. 5]). One should also know how these operators transform under the map $\alpha(X) = X^{-1}$. Set $L^{\alpha}f = L(f \circ \alpha) \circ \alpha^{-1}$, for any differential operator L acting on functions $f: \mathscr{P}_k \to \boldsymbol{C}$. Now if L is a $GL(k, \boldsymbol{R})$-invariant differential operator, then L^{α} turns out to be the conjugate of the *formal adjoint* L^* defined by:

$$\int_{\mathscr{P}_k} (Lf)\bar{g}\,dv_k = \int_{\mathscr{P}_k} f\overline{(L^*g)}\,dv_k \,.$$

And it is easy to compute L^* by integration by parts. Therefore

$$D_a = D_a^* = (-1)^k D_{a^*} \qquad \text{where} \quad a^* = 1 - a + \frac{k+1}{2} \,.$$

Then Selberg defines

$$L = D_a D_1 \qquad \text{for} \quad a = (k - n + 1)/2 \,, \quad k = n_1 + \cdots + n_{q-1} \,. \tag{3.16}$$

It follows that:

$$L_X\theta(Y, X) = |\,Y\,|^{-k/2}|\,X\,|^{-n/2} L_X\theta(Y^{-1}, X^{-1}) \,. \tag{3.17}$$

It is proved in [24, § 5], [44, Ch. 5] that there is a polynomial $\lambda_L(s)$ such that if $f(X) = p_{-s}(X^{-1})$, then $Lf = \lambda_L(s)f$, for every $GL(k, \boldsymbol{R})$-invariant differential operator L on $\mathscr{P}_k$.

Suppose that U_k is the $k \times k$ matrix defined by

$$U_k = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & 0 \end{pmatrix} . \tag{3.18}$$

Let $P^* = P(n_{q-1}, \cdots, n_2, n_1)$ if $P = P(n_1, \cdots, n_{q-1})$. Suppose that p_s denotes the power function for P and p_s^* denotes the power function for P^*. Then it is easy to see that

$$p_s(X^{-1}[U_k]) = p_{s^*}^*(X) \,, \qquad \text{if} \quad s^* = \left(s_{q-2}, \cdots, s_2, s_1, -\sum_{j=1}^{q-1} s_j\right) . \tag{3.19}$$

Riemann's proof of the functional equation and analytic continuation of the Riemann zeta function can then be modified according to Selberg's ideas to prove:

THEOREM 8. *Suppose that L is the differential operator given by* (3.16), *λ_L is the polynomial defined just after* (3.17), *the matrix U_k is defined by* (3.18), *and $p_{s^*}^*$ is as in* (3.19). *Then the following formula provides an analytic continuation of Selberg's zeta function* *to a meromorphic function defined for* s_{s-1} *in* C *and the rest of the variables restricted as in* (3.1):

$$(3.2)\quad 2\lambda_{L^*}(s)\Lambda_{n_1,\cdots,n_{q-1}}(Y, s)$$
$$= \int_{X\in\mathscr{F}_{t_P},|X|\geqq 1} p_{-s}(X^{-1}) \sum_{A\in Z^{n\times k}\ rkk} L_X \exp[-\mathrm{Tr}\,(Y[A]X)]dv_k$$
$$+|Y|^{-k/2}\int_{X\in\mathscr{F}_{t_{P^*}}|X|\geqq 1} p^*_{-\tilde{s}}(X^{-1}) \sum_{A\in Z^{n\times k}\ rkk} L_X \exp[-\mathrm{Tr}(Y^{-1}[A]X)]dv_k$$

where $\tilde{s} = (s_{q-2}, \cdots, s_2, s_1, n/2 - \sum_{j=1}^{q-1} s_j)$.

Proof. Use formula (3.12) and the definition of L^* to see that

$$2\lambda_{L^*}(s)\Lambda_{n_1,\cdots,n_{q-1}}(Y, s) = \int_{\mathscr{F}_{t_P}} p_{-s}(X^{-1})L_X\theta(Y, X)dv_k\,.$$

Now try Riemann's trick and write the integral on the right above as a sum of two integrals:

$$\int_{X\in\mathscr{F}_{t_P}\ |X|\geqq 1} + \int_{X\in\mathscr{F}_{t_P}\ |X|\leqq 1}\,.$$

In the second integral send X to $X^{-1}[U_k]$ with U_k as in (3.18). This map takes $\mathscr{F}_{t_P}$ to $\mathscr{F}_{t_{P^*}}$ with P^* as in (3.19). Then use (3.17) to write:

$$2\lambda_{L^*}(s)\Lambda_{n_1,\cdots,n_{q-1}}(Y, s)$$
$$= \int_{X\in\mathscr{F}_{t_P}\ |X|\geqq 1} p_{-s}(X^{-1})L_X\theta(Y, X)dv_k$$
$$+ |Y|^{-k/2}\int_{X\in\mathscr{F}_{t_{P^*}}\ |X|\geqq 1} p_{-s}(X[U_k])L_X\theta(Y^{-1}, X)|X|^{n/2}dv_k\,.$$

Formula (3.19) completes the proof of (3.20).

The fact that (3.20) only provides an analytic continuation of the Eisenstein series or Selberg zeta function $Z_{n_1,\cdots,n_{q-1}}(Y, s)$ for the last variable s can easily been using formulas (2.14) and (3.11).

The analytic continuation of $Z_{n_1,\cdots,n_{q-1}}(Y, s)$ in the other s-variables has only been obtained by rather indirect methods it seems (cf. [22], [24, pp. 274-275], [39], and [44]).

Formula (3.20) does not relate the Eisenstein series for P with itself but instead with that for P^*. However, one can apply (3.19) to see that $E_{P^*}(Y^{-1}, s^*) = E_P(Y, s)$.

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