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A NOTE ON DISCONJUGACY FOR SECOND ORDER SYSTEMS

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It is well-known that the equation

$$x''+A(t)x=0 \tag{1}$$

is disconjugate on $[a, b]$ if and only if there exists a solution which is positive on $[a, b]$, in the case that $A(t)$ is scalar-valued. In this note we generalize this simple result to the case where $A(t)=(a_{ij}(t))$ is an $n\times n$ matrix-valued function which satisfies certain generalized sign conditions. These results apply, for instance, if the off diagonal elements are nonnegative. Simple necessary and sufficient conditions are given for disconjugacy if $A(t)\equiv A$ and these are used to construct examples showing the necessity of sign conditions on $A(t)$ for the above mentioned results and other results of Sturm type for systems to be valid.

Introduction. Many authors have considered the problem of extending the well-known results on disconjugacy for the scalar equation (1) to systems. We mention the work of Morse [8] and Hartman and Wintner [5], where $A(t)$ is assumed symmetric or conditions are placed on the symmetric part of A. Recently, many new results have been obtained in the papers of Ahmad and Lazer ([1], [2], [3]) and Schmitt and the author, [9], where symmetry assumptions have generally been avoided.

Recall that (1) is said to be disconjugate on the interval $[a, b]$ if no nontrivial solution of (1) vanishes twice on $[a, b]$, otherwise (1) is conjugate on $[a, b]$. If $x\in R^n$, we write $x\geqq 0$ if $x_i\geqq 0$, $1\leqq i\leqq n$; $x>0$ if $x\geqq 0$ and $x\neq 0$; and $x\gg 0$ if $x_i>0$, $1\leqq i\leqq n$. If A is an $n\times n$ matrix we denote by $\sigma(A)$ the spectrum of A.

Below we state two corollaries of our main results and some examples to indicate the necessity of the hypotheses involved. The main results are stated in § 2 and the proofs are given in § 3.

COROLLARY 1. *Let $A(t)=(a_{ij}(t))$ be a continuous, matrix-valued function satisfying $a_{ij}(t)\geqq 0$, $i\neq j$. If (1) is disconjugate on $[a, b]$ then there is a solution $x(t)$ of (1) satisfying $x(t)>0$ on $[a, b]$.*

COROLLARY 2. *Let $A(t)$ satisfy the conditions of Corollary 1. If there exists a solution $y(t)$ of the differential inequality $y''+A(t)y\leqq 0$ satisfying $y(t)\gg 0$, $a\leqq t\leqq b$, then (1) is disconjugate on $[a, b]$.*

REMARK. Corollary 2 cannot be weakened with respect to the assumption that $y(t) \gg 0$ without additional conditions on $A(t)$ as seen by the following example: the equation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

is easily seen to be disconjugate on every interval of length less π. However, a solution is given by

$$x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} > 0$$

but $x(t) >\!\!/\!\!> 0$.

Corollary 2 generalizes Theorem 3 in [3].

We illustrate Corollary 2 by showing $x'' + \begin{pmatrix} -3t & 1 \\ 2 & -4t^2 \end{pmatrix} x = 0$ is disconjugate on $[1, \infty)$. To see this, let $y(t) = \mathrm{col}\,(t, t)$ and observe that $y(t) \gg 0$ on $1 \leqq t < \infty$ and $y'' + A(t)y \leqq 0$.

In case $A(t) \equiv A = (a_{ij})$ we have the following necessary and sufficient conditions of a particularly simple form for (1) to be disconjugate on $[a, b]$ which do not involve sign conditions on A.

LEMMA 3. *Let* $A(t) \equiv A$. *Then* (1) *is disconjugate on* $[a, b]$ *if either* $\sigma(A) \cap (0, \infty) = \phi$ *or if* $b-a < \pi/\sqrt{\lambda}$ *for all* $\lambda \in \sigma(A) \cap (0, \infty)$. (1) *is conjugate on* $[a, b]$ *if* $b-a \geqq \pi/\sqrt{\lambda}$ *for some* $\lambda \in \sigma(A) \cap (0, \infty)$.

Lemma 3 may be employed to construct some interesting examples. For instance, let

$$A(\varepsilon) = \begin{pmatrix} 6 & 16 + \varepsilon^2 \\ -1 & -2 \end{pmatrix}.$$

Then $\sigma(A(\varepsilon)) = \{2 + \varepsilon i, 2 - \varepsilon i\}$. According to Lemma 3,

$$x'' + A(1)x = 0$$

is disconjugate on $[0, 4]$ while

$$x'' + A(0)x = 0$$

is conjugate on $[0, 4]$ since $4 \geqq \pi/\sqrt{2}$. Thus the Sturm comparison test does not hold, in general, for systems since $A(1) \geqq A(0)$ (in the usual sense). In [9] it was shown that the Sturm test does hold if, for instance, both matrices are nonnegative (they need not be constant; see [9] for a more precise result). It is easy to construct examples showing that the sign conditions on $A(t)$ in Corollary 1 are not superfluous.

2. Main results. Let K be a cone in R^n with nonempty interior. We write $x \geqq 0$ if $x \in K$, $x > 0$ if $x \in K - \{0\}$, and $x \gg 0$ if $x \in \operatorname{int} K$ where $\operatorname{int} K$ denotes the interior of K. Let $A(t)$ be a continuous matrix-valued function defined on $[a, b]$ satisfying:

(H) There exists $\lambda \geqq 0$ such that $(A(t) + \lambda I)(K) \subseteqq K$ for all $t \in [a, b]$ where I denotes the identity matrix.

Where required, we assume $A(t)$ is defined on all of R satisfying condition (H). Simply let $A(t) = A(b)$ for $t > b$ and similarly for $t < a$.

THEOREM 1. *Assume that* (H) *holds and that* (1) *is disconjugate on* $[a, b]$. *Then there is a solution* $y(t)$ *of* (1) *satisfying* $y(t)>0$, $a \leqq t \leqq b$.

THEOREM 2. *If* (H) *holds and if* $y(t)$ *is twice differentiable, satisfies the differential inequality*

$$y'' + A(t)y \leqq 0 \ ,$$

and if $y(t) \gg 0$ *on* $a \leqq t \leqq b$, *then* (1) *is disconjugate on* $[a, b]$.

Finally, we point out that Vandergraft [10] has given sufficient conditions for a matrix A to leave a cone with nonempty interior invariant involving only the spectral properties of A. In particular, every strictly triangular matrix has an invariant cone and if A is symmetric then either A or $-A$ leaves some cone invariant.

3. Proofs. First, we show that it suffices to prove Theorems 1 and 2 with the condition (H) replaced by the following: (H'): For each t, $A(t)(K) \subseteqq (K)$, i.e., $A(t)$ is a positive operator.

To see this make the change in dependent variable by letting $t(s) = a + 1/2k \operatorname{Log}(1/1 - s)$ and change the independent variable by letting $v(s) = e^{-kt(s)}x(t(s))$. Then (1) is equivalent to

$$v''(s) + (t'(s))^2[k^2 I + A(t(s))]v(s) = 0 \ . \tag{2}$$

It is assumed that $k^2 = \lambda$ where λ is as in assumption (H). Clearly (1) is disconjugate on $[a, b]$ if and only if (2) is disconjugate on the appropriate interval. Thus, if Theorem 1 holds under assumption (H'), then the assumption that (1) is disconjugate on $[a, b]$ implies the existence of a solution $v(s) > 0$ of (2) on the interval $t^{-1}([a, b])$ and hence a solution $x(t)$ of (1) on $[a, b]$ with $x(t) > 0$ on $[a, b]$. Similar reasoning shows that it suffices to prove Theorem 2 under the assumption (H′). In all that follows we assume (H′) holds.

At this point we require some notation. Let $X = BC(R, R^n)$, the Banach space of bounded continuous functions of R into R^n with supremum norm. Let $\mathscr{K} = \{x \in X : x(t) \in K \text{ for all } t \in R\}$. Then $\mathscr{K}$ is a cone in X which is total, i.e., $\overline{K-K}=X$. If $a, b \in R$, $a<b$, define the compact linear operator $A_{a,b}: X \to X$ by

$$(A_{a,b}x)(t) = \begin{cases} 0 & t > b \\ \int_a^b G(a, b; t, s)A(s)x(s)ds & \\ 0 & t < a \end{cases}$$

where $G(a, b; t, s)$ is the nonnegative Green's function for $-d^2x/dt^2 = f(t)$, $x(a) = x(b) = 0$. Notice, see [9], that (we assume (H′) holds) $A_{a,b}$ is a positive operator, i.e., $A_{a,b}\mathscr{K} \subseteqq \mathscr{K}$. If $a < b$ define $r(a, b) = \rho(A_{a,b})$, the spectral radius of $A_{a,b}$. We require the following lemma which is a trivial modification of lemmas 3.1 and 3.4 and the proof of Theorem 3.5 in [9].

LEMMA 1. *The function $r(a, b)$ defined for $a < b$ is continuous in a for fixed b and continuous in b for fixed a. Moreover, $r(a, b)$ is nondecreasing in b (for fixed a) and nonincreasing in a (for fixed b), and $r(a, b) \to 0+$ as $b - a \to 0+$. In addition,* (1) *is disconjugate on $[a, b]$ if and only if $r(a, b) < 1$.*

Proof of Theorem 1. If (1) is disconjugate on $[a, b]$ then $r(a, b) < 1$ by Lemma 1. Also by Lemma 1, we can choose $a_1 < a$ and $b_1 > b$ such that $r(a_1, b_1) < 1$. Now either (i) $r(a_1, b_2) < 1$ for all $b_2 \geqq b_1$ or (ii) there exists $b_2 > b_1$ such that $r(a_1, b_2) = 1$. In case (ii) we may conclude (by the Krein-Rutman theorem as applied in [9]) the existence of a solution $y(t)$ of (1) satisfying $y(a_1) = y(b_2) = 0$ and $y(t) > 0$, $a_1 < t < b_2$. Thus Theorem 1 is proved in this case. In case (i), (1) is disconjugate on $[a_1, \infty)$ and Theorem 3.11 of [9] completes the proof of this case.

Proof of Theorem 2. For this argument let $X = C([a, b]R^n)$ and $\mathscr{K}$ the corresponding cone. If $y(t) \gg 0$ on $a \leqq t \leqq b$ is a solution of the differential inequality $y'' + A(t)y \leqq 0$, then we observe that $y \in \operatorname{int} \mathscr{K} (y \gg 0)$. Let $z = A_{a,b}y$ so $z(t)$ satisfies

$$z'' + A(t)y = 0,\ z(a) = z(b) = 0,\ z(t) \geqq 0 \quad a \leqq t \leqq b\,.$$

Then $y(t) - z(t)$ satisfies

$$(y - z)'' \leqq 0 \text{ and } (y - z)(a) \gg 0,\ (y - z)(b) \gg 0\,.$$

Hence, if φ is a positive linear functional with respect to $K \subseteqq \boldsymbol{R}^n$ and $v(t) = \varphi(y(t) - z(t))$ then $v'' \leqq 0$ and $v(a) > 0$, $v(b) > 0$. Thus $v(t) > 0$ on $a \leqq t \leqq b$. Since φ was an arbitrary positive linear functional we conclude that $y(t) - z(t) \gg 0$ on $a \leqq t \leqq b$, i.e., $y \gg z$ in $\mathscr{K}$.

If (1) were not disconjugate on $[a, b]$, then $r(a, b) \geqq 1$ and thus there exists $b' \leqq b$ with $r(a, b') = 1$ and hence (Theorem 3.5 in [9]) a solution $u(t)$ of (1) satisfying $u(a) = u(b') = 0$ and $u(t) > 0$ on $[a, b']$. Define $u(t) = 0$ on $(b', b]$ so $u \in \mathscr{K}$. Since $y \in \operatorname{int} \mathscr{K}$ we may choose $\alpha > 0$ maximal such that $\alpha u \leqq y$ (i.e., if $\beta u \leqq y$ then $\beta \leqq \alpha$). Then we have

$$\alpha u = \alpha A_{a,b'}(u) \leqq \alpha A_{a,b}(u) \leqq A_{a,b} y = z \ll y \ .$$

But $\alpha u \ll y$ implies we may choose $\eta > \alpha$ such that $\eta u \ll y$, a contradiction to the maximality of α. This contradiction proves the theorem. Notice that we used the easily established fact that if $a \leqq a' < b' \leqq b$ then $A_{a',b'} x \leqq A_{a,b} x$ for all $x \in \mathscr{K}$.

Proof of Lemma 3. The lemma follows immediately from the following assertion: Equation (1) has a nontrivial solution satisfying $x(0) = x(T) = 0$ if and only if there exists $\lambda \in \sigma(A) \cap (0, \infty)$ such that $\sqrt{\lambda}\, T = k\pi$ for some positive integer k. To prove the assertion, first assume that $0 \notin \sigma(A)$ so that there exists a complex matrix B satisfying $B^2 = A$. A $\boldsymbol{C}^n$-valued function $x(t)$ satisfies (1) and $x(0) = 0$ if and only if there exists $x_0 \in \boldsymbol{C}^n$ such that $x(t) = (\sin Bt) x_0$. Thus (1) has a nontrivial solution satisfying $x(0) = x(T) = 0$ if and only if $\det [\sin BT] = 0$. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A. Then by the spectral mapping theorem and elementary properties of the determinant,

$$\det [\sin BT] = \prod_{i=1}^{n} \sin \sqrt{\lambda_i} T \ .$$

Thus $\det [\sin BT] = 0$ if and only if $\sqrt{\lambda_j} T = k\pi$ for some j, $i \leqq j \leqq n$ and some integer k. This last holds only if $\sqrt{\lambda_j}$ is real, in particular λ_j must be positive and k must be positive. Hence a necessary and sufficient condition for there to be a nontrivial $\boldsymbol{C}^n$-valued solution of (1) satisfying $x(0) = x(T) = 0$ is for $\sqrt{\lambda}\, T = k\pi$ for some $\lambda \in \sigma(A) \cap (0, \infty)$ and some positive integer k. Such a solution will be of the form $x(t) = (\sin Bt) x_0$ where $x_0 \neq 0$ is in the null space of $\sin BT$. The real and imaginary parts of x_0, at least one of which is nonzero, will also be solutions of (1) satisfying $x(0) = x(T) = 0$. This completes the proof of the assertion in case $0 \notin \sigma(A)$. In case $0 \in \sigma(A)$ write $\boldsymbol{R}^n = M \oplus N$ where M is the generalized nullspace of

A, ($M = \bigcup_{n=1}^{\infty} \operatorname{Ker} A^n = \operatorname{Ker} A^p$, p some positive integer which we may assume is the smallest such) and $N = \operatorname{Range} A^p$. The complementary subspaces M and N reduce A and A/M is nilpotent on M. Write $A/M = B$, $A/N = C$. Then (1) becomes

$$y'' + By = 0 \tag{2}$$

$$\begin{aligned} z'' + Cz &= 0 \\ x &= y + z\,. \end{aligned} \tag{3}$$

The previous analysis applies to (3) since $\sigma(C) = \sigma(A) - \{0\}$. Since B is nilpotent it is easy to see that the only solution of (2) satisfying $y(0) = y(T) = 0$ is the trivial solution (multiply (2) by B^{p-1} where $B^p = 0$). This completes the proof in this case.

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