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CATEGORY IN FUNCTION SPACES, I

D. J. LUTZER AND R. A. MCCOY

In this paper we study Baire category in spaces of continuous, real-valued functions equipped with the topology of pointwise convergence. We show that, for normal spaces, the Baire category of $C_\pi(X)$ is determined by the Baire category of $C_\pi(Y)$ for certain small subspaces Y of X and that the category of $C_\pi(X)$ is intimately related to the existence of winning strategies in a certain topological game $\Gamma(X)$ played in the space X. We give examples of certain countable regular spaces for which $C_\pi(X)$ is a Baire space and we characterize those spaces X for which $C_\pi(X)$ has one of the stronger completeness properties, such as pseudocompleteness or Cech-completeness.

1. **Introduction.** It is well-known that the set of all continuous real-valued functions on a space X is a completely metrizable space when equipped with the topology of uniform convergence and therefore that the Baire Category Theorem is valid in this space. However, when function spaces carry other topologies, the status and rôle of the Baire category property is more mysterious, and in this paper we consider Baire category in $C_\pi(X)$, the set of all continuous real-valued functions on a completely Hausdorff space X, equipped with the topology of pointwise convergence. (See § 2 for precise definitions.) We can preview some of our less technical results as follows. Experience shows that the Baire category of $C_\pi(X)$ is determined by the kinds of limit points which countable subsets of X can have. For example, if $C_\pi(X)$ is a Baire space then X has no nontrivial convergent sequences (Corollary 3.3) and the same argument shows that the set of bounded members of $C_\pi(X)$ is never a Baire space. Furthermore, it is sometimes possible to study the Baire category of $C_\pi(X)$ by examining the continuously extendable functions defined on countable subspaces of X (3.7). Examples show that there are infinite nondiscrete spaces X for which $C_\pi(X)$ is a Baire space, e.g., any normal space in which each countable subset is closed (see Theorem 8.4). A more interesting fact is that there are even *countable* nondiscrete spaces for which $C_\pi(X)$ is a Baire space, e.g., any space $X = \omega \cup \{p\}$ where $p \in \beta\omega - \omega$, topologized as a subspace of $\beta\omega$ (see Example 7.1). It is possible for $C_\pi(X)$ to be a Baire space, where X is countable and regular, without X being embeddable in $\beta\omega$ (Example 7.2) and we can characterize those filters $\mathscr{N}$ on ω such that $C_\pi(X)$ is a Baire space when X is obtained by adjoining a single point to ω and using $\mathscr{N}$ as the neighborhood filter of the ideal point. More precisely, if X has a unique nonisolated

point whose neighborhood filter is $\mathscr{N}$, then $C_\pi(X)$ is Baire if and only if, given any sequence $\mathscr{F}_1, \mathscr{F}_2, \cdots$, where each $\mathscr{F}_n$ is an infinite family of pairwise disjoint finite sets, it is possible to choose $F_n \in \mathscr{F}_n$ in such a way that the set $(X - \cup\{F_n: n \geqq 1\})$ belongs to the filter $\mathscr{N}$ (see § 5). Other characterizations are given in terms of a certain two-person infinite game $\Gamma(X)$ in which one player attempts to construct a set with a limit point by successive choices of finite subsets of the space X (see §§ 4, 5, 6 and 7). For $C_\pi(X)$ to be Baire it is necessary that the first player *not* have a winning strategy in $\Gamma(X)$ 4.6 and there is a certain class of spaces for which that condition is also sufficient (cf. § 6). The assertion that the first player has no winning strategy in $\Gamma(X)$ can be translated into more set-theoretic terms, and that is the goal of § 5. In § 8 of our paper we investigate completeness properties which are stronger than simply being a Baire space and characterize two such properties of $C_\pi(X)$ in terms of winning strategies for the second player in $\Gamma(X)$. One result in § 8 shows that if $\boldsymbol{R}$ denotes the usual space of real numbers, the function space $C_\pi(X)$ cannot be a G_δ-subset of the Baire space $\boldsymbol{R}^X$ unless X is discrete, a fact which may be viewed as showing that the (dense) subspace $C_\pi(X)$ of $\boldsymbol{R}^X$ cannot inherit the Baire category property from $\boldsymbol{R}^X$ in the usual way. In the final section of the paper, we give a few results describing the situation in which functions are allowed to have their values in spaces other than $\boldsymbol{R}$.

2. Preliminary lemmas, special notations and definitions. The space $\boldsymbol{R}^X$, the Tychonoff product of card (X) copies of the usual space $\boldsymbol{R}$ of real numbers, is the set of all functions from X to $\boldsymbol{R}$. There are two ways to describe the topology of $\boldsymbol{R}^X$. The first is to use sets of the form

$$[F, V] = \{g \in \boldsymbol{R}^X: g[F] \subset V\},$$

where $F \subset X$ is finite and $V \subset \boldsymbol{R}$ is open, as a subbase. The second is to use sets of the form

$$\widetilde{\mathscr{N}}(f, S, \varepsilon) = \{g \in \boldsymbol{R}^X: \text{ if } x \in S \text{ then } |g(x) - f(x)| < \varepsilon\}$$

as a neighborhood base at f, where S is a finite subset of X and $\varepsilon > 0$.

The set of continuous, real-valued functions on X, topologized as a subspace of $\boldsymbol{R}^X$, is denoted by $C_\pi(X)$. If $f \in C_\pi(X)$, then we will denote basic neighborhoods of f in $C_\pi(X)$ by $\mathscr{N}(f, S, \varepsilon) = C_\pi(X) \cap \widetilde{\mathscr{N}}(f, S, \varepsilon)$.

A space X is *completely Hausdorff* if, given points $x \neq y$ in X, there is a continuous real-valued function $f: X \to \boldsymbol{R}$ having $f(x) \neq f(y)$. The function f is said to *separate the points x and y*.

LEMMA 2.1. *Let $\mathscr{S}$ and $\mathscr{T}$ be two topologies on a set X and suppose $\mathscr{S} \subset \mathscr{T}$. Let $C_\pi(X, \mathscr{S})$ and $C_\pi(X, \mathscr{T})$ denote the spaces of continuous real-valued functions on $(X, \mathscr{S})$ and $(X, \mathscr{T})$ respectively. Then*

(a) *$C_\pi(X, \mathscr{S})$ is a (topological) subspace of $C_\pi(X, \mathscr{T})$;*

(b) *$C_\pi(X, \mathscr{S})$ is a dense subspace of $C_\pi(X, \mathscr{T})$ if and only if whenever two points of X can be separated by a $\mathscr{T}$-continuous function, then they can also be separated by an $\mathscr{S}$-continuous function;*

(c) *in particular, $C_\pi(X)$ is a dense subspace of $\boldsymbol{R}^X$ if and only if X is a completely Hausdorff space.*

Convention 2.2. Henceforth, every space is at least completely Hausdorff.

The function space $C_\pi(X)$ derives much of its topological structure from the space $\boldsymbol{R}$. In particular, no matter what X is, $C_\pi(X)$ is a completely regular topological algebra.

A topological space Z is of the *second* (*Baire*) *category* if $\bigcap \{G_n: n \geqq 1\} \neq \varnothing$ whenever $G_1, G_2, \cdots$ is a sequence of dense open subsets of Z, and Z is a *Baire space* if $\bigcap \{G_n: n \geqq 1\}$ is dense in Z under the same hypotheses on the G_n's. If Z is not of the second Baire category, then Z is of the *first category*. Obviously, any Baire space is of the second Baire category, but the converse is false in general. However, if Z is a *homogeneous space* (i.e., given $x, y \in Z$, there is a homeomorphism of Z onto Z sending x to y) then "Baire space" and "second (Baire) category" are equivalent notions as our next result shows.

THEOREM 2.3. *Let Z be a homogeneous space. Then Z is a Baire space if and only if Z is of the second Baire category.*

Sketch of proof. Suppose Z is of the second Baire category. Then the Banach category theorem yields a nonempty open subspace U which, in its relative topology, is a Baire space. Fix $x \in U$. For any $y \in Z$, let $h_y: Z \to Z$ be an autohomeomorphism of Z having $h_y(x) = y$. Then $U_y = h_y[U]$ is an open neighborhood of y which is a Baire space so that, in the terminology of [1, 1.2.4e], Z is locally a Baire space. But then Z is a Baire space.

COROLLARY 2.4. *For any space X, $C_\pi(X)$ is a Baire space if and only if $C_\pi(X)$ is of the second Baire category.*

Proof. Being a topological vector space, $C_\pi(X)$ is homogeneous. Now apply 2.3.

We shall often exploit the fact that $C_\pi(X)$ is a dense subspace of $\boldsymbol{R}^X$ because of the next theorem.

THEOREM 2.5. *For any space X, $\boldsymbol{R}^X$ is a Baire space, and $C_\pi(X)$ is a Baire space if and only if given any sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of dense open subsets of $\boldsymbol{R}^X$, the set $C_\pi(X) \cap (\bigcap_{n=1}^\infty \mathscr{G}_n) \neq \emptyset$.*

Proof. That $\boldsymbol{R}^X$ is a Baire space no matter how large card (X) may be follows from, e.g., [2]. The assertion about $C_\pi(X)$ follows from 2.4.

Let $\{Y_\alpha: \alpha \in A\}$ be a family of topological spaces. We may assume that $Y_\alpha \cap Y_\beta = \emptyset$ whenever $\alpha \neq \beta$. The *topological sum* $\bigoplus\{Y_\alpha: \alpha \in A\}$, also called the "disjoint union of the Y_α's", is the set $X = \bigcup\{Y_\alpha: \alpha \in A\}$ endowed with the topology in which a set $U \subset X$ is open if and only if $U \cap Y_\alpha$ is open in Y_α for each $\alpha \in A$. At several points in the sequel we will need to exploit the natural relationship between $C_\pi(\bigoplus\{Y_\alpha: \alpha \in A\})$ and the product space $\Pi\{C_\pi(Y_\alpha): \alpha \in A\}$ which is described by

THEOREM 2.6. *The function space $C_\pi(\bigoplus\{Y_\alpha: \alpha \in A\})$ is naturally homeomorphic to the Tychonoff product space $\Pi\{C_\pi(Y_\alpha): \alpha \in A\}$.*

Conventions 2.7. The symbols $\boldsymbol{R}$, $\boldsymbol{N}$, ω and ω_1 will denote, respectively, the usual space of real numbers, the positive integers, the set of finite ordinals (i.e., $N \cup \{0\}$) and the set of countable ordinals. Finally, c will denote the cardinality of $\boldsymbol{R}$.

3. Category in $C_\pi(X)$ and subspaces. In this section we will have spaces $Y \subset X$ and, for a function $g \in C_\pi(Y)$, $\mathscr{N}_Y(g, T, \varepsilon)$ will denote $\{h \in C_\pi(Y)$: if $t \in T$ then $|h(t) - g(t)| < \varepsilon\}$, where T is a finite subset of Y.

LEMMA 3.1. *Let $Y \subset X$ and let $\rho: C_\pi(X) \to C_\pi(Y)$ be the restriction map. For each basic open set $\mathscr{N}(f, S, \varepsilon)$ in $C_\pi(X)$, $\rho[\mathscr{N}(f, S, \varepsilon)]$ is dense in $\mathscr{N}_Y(f|_Y, S \cap Y, \varepsilon)$.*

Proof. Let $g \in \mathscr{N}_Y(f|_Y, S \cap Y, \varepsilon)$ and let $\mathscr{N}_Y(g, T, \delta)$ be any neighborhood of g. We may assume $S \cap Y \subset T \subset Y$ and $\delta < \varepsilon$. Because X is completely Hausdorff, there is a function $h \in C_\pi(X)$ having $h|_T = g|_T$ and $h|_{(S-Y)} = f|_{(S-Y)}$. (Observe that $T \cup (S - Y)$ is finite and $T \cap (S - Y) = \emptyset$.) But then $h \in \mathscr{N}(f, S, \varepsilon)$ and $\rho(h) \in \mathscr{N}_Y(g, T, \delta)$ so that $\mathscr{N}_Y(g, T, \delta) \cap \rho[\mathscr{N}(f, S, \varepsilon)] \neq \emptyset$ as required.

THEOREM 3.2. *If $C_\pi(X)$ is a Baire space then so is $C_\pi(Y)$ for every subspace Y of X.*

Proof. Let $\rho: C_\pi(X) \to C_\pi(Y)$ be restriction. Let $\mathscr{G}_1 \supset \mathscr{G}_2, \cdots$ be dense open subsets of $C_\pi(Y)$ and define $\mathscr{H}_n = \rho^{-1}[\mathscr{G}_n]$. Since ρ is continuous, each $\mathscr{H}_n$ is open in $C_\pi(X)$. Let $\mathscr{N}(f, S, \varepsilon)$ be a basic open set in $C_\pi(X)$. Because $\rho[\mathscr{N}(f, S, \varepsilon)]$ is dense in the open set $\mathscr{N}_Y(f|_Y, S \cap Y, \varepsilon)$, $\rho[\mathscr{N}(f, S, \varepsilon)]$ must meet the dense open set $\mathscr{G}_n$. Hence $\mathscr{N}(f, S, \varepsilon) \cap \mathscr{H}_n \neq \emptyset$, showing that $\mathscr{H}_n$ is dense in $C_\pi(X)$. But then $\cap \{\mathscr{H}_n : n \geqq 1\} \neq \emptyset$ so that $\cap \{\mathscr{G}_n : n \geqq 1\} \neq \emptyset$.

COROLLARY 3.3. *If X contains an infinite pseudocompact subspace, then $C_\pi(X)$ is of first category.*

Proof. In the light of 3.2, it is enough to prove that $C_\pi(Y)$ is not a Baire space whenever Y is an infinite pseudocompact space. We define $\mathscr{G}_n = \{f \in C_\pi(Y): f[Y] \cap (n, +\infty) \neq \emptyset\}$. It is clear that each $\mathscr{G}_n$ is open, and since Y is infinite and completely Hausdorff, each $\mathscr{G}_n$ is also dense in $C_\pi(Y)$. However, any member of $\cap \{\mathscr{G}_n : n \geqq 1\}$ would be an unbounded, continuous real-valued function on Y and that is impossible since Y is pseudocompact.

A simple trick allows us to strengthen the conclusion of Theorem 3.2.

COROLLARY 3.4. *If $C_\pi(X)$ is a Baire space then so is $C_\pi(Z)$ for every space Z which can be mapped into X by a continuous $1-1$ function.*

Proof. Let $g: Z \to X$ be a continuous, $1-1$ function. Let $Y = g[X]$ and topologize Y as a subspace of X. The space Z has two topologies, viz., the given topology $\mathscr{T}$ and the topology $\mathscr{S} = \{g^{-1}[U \cap Y]: U \text{ is open in } X\}$. Clearly $\mathscr{S} \subset \mathscr{T}$ and since X is completely Hausdorff, so is $(Z, \mathscr{S})$. According to Lemma 2.1, $C_\pi(Z, \mathscr{S})$ is a dense subspace of $C_\pi(Z, \mathscr{T})$. Since $g: (Z, \mathscr{S}) \to Y$ is a homeomorphism, $C_\pi(Z, \mathscr{S})$ is a copy of $C_\pi(Y)$ which is a Baire space according to 3.2. But then $C_\pi(Z, \mathscr{T})$ has a dense, Baire, subspace, so that $C_\pi(Z, \mathscr{T})$ is itself a Baire space.

Certain partial converses of 3.2 can be obtained: they assert that the Baire category of $C_\pi(Y)$ for certain small subspaces Y of X can sometimes determine the category of $C_\pi(X)$.

DEFINITION 3.5. Let $Y \subset X$. Let $E(Y, X) = \{f \in C_\pi(Y): f$ can be extended to an element of $C_\pi(X)\}$, and topologize $E(Y, X)$ as a subspace of $C_\pi(Y)$.

THEOREM 3.6. *Suppose X is completely Hausdorff. Then $C_\pi(X)$ is Baire if and only if for each countable $Y \subset X$, the space $E(Y, X)$ is Baire.*

Proof. Suppose $C_\pi(X)$ is Baire and let $\rho: C_\pi(X) \to C_\pi(Y)$ be the restriction map. Since the range of ρ is exactly $E(Y, X)$, the argument given in the proof of 3.2 shows that $E(Y, X)$ must be Baire.

Conversely, suppose $E(Y, X)$ is a Baire space whenever Y is a countable subspace of X. Let $\mathscr{G}_1 \supset \mathscr{G}_2 \supset \cdots$ be dense open sets in $\boldsymbol{R}^X$; we show that $C_\pi(X) \cap (\cap \{\mathscr{G}_n: n \geqq 1\}) \neq \varnothing$. For each n let Ψ_n be a maximal collection of pairwise disjoint open sets in $\boldsymbol{R}^X$, say $\Psi_n = \{\tilde{\mathscr{N}}(f_{\alpha,n}, S_{\alpha,n}, \varepsilon_{\alpha,n}): \alpha \in A_n\}$, such that Ψ_{n+1} refines Ψ_n and $\cup \Psi_n \subset \mathscr{G}_n$. Then each $\cup \Psi_n$ is dense in $\boldsymbol{R}^X$. Furthermore, since the members of Ψ_n are pairwise disjoint and since $\boldsymbol{R}^X$ has countable cellularity, (i.e., each pairwise disjoint open collection is countable), each Ψ_n is countable. Let $Y = \cup \{S_{\alpha,n}: \alpha \in A_n, n \geqq 1\}$. Then, by hypothesis, $E(Y, X)$ is a Baire space. Let $f'_{\alpha,n} = f_{\alpha,n} | Y$ and let $\tilde{\mathscr{N}}_Y(f'_{\alpha,n}, S_{\alpha,n}, \varepsilon_{\alpha,n}) = \{g \in C_\pi(Y):$ if $x \in S_{\alpha,n}$ then $|f_\alpha(x) - g(x)| < \varepsilon_{\alpha,n}\}$. Define $\mathscr{H}_n = \cup \{\tilde{\mathscr{N}}_Y(f'_{\alpha,n}, S_{\alpha,n}, \varepsilon_{\alpha,n}): \alpha \in A_n\}$. Then each $\mathscr{H}_n$ is a dense open subset of $C_\pi(Y)$. Because X is completely Hausdorff, $E(Y, X)$ is dense in $C_\pi(Y)$ so that each set $\mathscr{E}_n = \mathscr{H}_n \cap E(Y, X)$ is relatively open and dense in $E(Y, X)$. Since $E(Y, X)$ is a Baire space, we may choose $h \in \cap \{\mathscr{E}_n: n \geqq 1\}$ and then a function $\hat{h} \in C_\pi(X)$ which extends h. Fix n. For some $\alpha \in A_n$, $h \in \tilde{\mathscr{N}}_Y(f'_{\alpha,n}, S_{\alpha,n}, \varepsilon_{\alpha,n})$ so that because $\hat{h}$ extends h we have $\hat{h} \in \tilde{\mathscr{N}}(f_{\alpha,n}, S_{\alpha,n}, \varepsilon_{\alpha,n})$. Hence $\hat{h} \in \mathscr{G}_n$ so that $\cap \{\mathscr{G}_n: n \geqq 1\} \neq \varnothing$. According to 2.4, $C_\pi(X)$ must be a Baire space.

In normal spaces, there is another partial converse of 3.2.

THEOREM 3.7. *Suppose X is normal and that $C_\pi(Y)$ is a Baire space whenever Y is a closed separable subspace of X. Then $C_\pi(X)$ is a Baire space.*

Proof. If $\mathscr{G}_1 \supset \mathscr{G}_2 \supset \cdots$ is a sequence of dense open sets in $\boldsymbol{R}^X$, choose maximal pairwise disjoint collections $\Psi(n)$ of basic open sets in $\boldsymbol{R}^X$ in such a way that $\cup \Psi(n) \subset \mathscr{G}_n$. Since $\boldsymbol{R}^X$ has countable cellularity, each $\Psi(n)$ is countable, say $\Psi(n) = \{\mathscr{N}(f_{n,k}, T_{n,k}, \varepsilon_{n,k}): k \geqq 1\}$ where each $T_{n,k}$ is a finite subset of X. Let $Y =$

$\mathrm{cl}_X(\bigcup\{T_{n,k}: n, k \geqq 1\})$. Then Y is a closed separable subspace of X. Let $f'_{n,k} = f_{n,k} \restriction Y$ and define $\mathscr{G}'(n) = \bigcup\{\mathscr{N}_Y(f'_{n,k}, T_{n,k}, \varepsilon_{n,k}): k \geqq 1\}$. It is easy to see that each $\mathscr{G}'(n)$ is a dense open subspace of $\boldsymbol{R}^Y$ so that, because $C_\pi(Y)$ is a Baire space, there is a function $g \in C_\pi(Y) \cap (\bigcap\{\mathscr{G}'(n): n \geqq 1\})$. Then the Tietze-Urysohn theorem yields a function $G \in C_\pi(X)$ which extends g, and one easily verifies that $G \in \bigcap\{\mathscr{G}(n): n \geqq 1\}$, as required to show that $C_\pi(X)$ is a Baire space.

REMARKS 3.8. The authors would like to thank Eric van Douwen for his helpful comments which sharpened an earlier version of 3.7 and for his observation that, since there is a pseudocompact space in which every countable set is closed, normality is an essential hypothesis in 3.7. [4, 5.1 and 5.3.]

Question 3.9. Is it true that $C_\pi(X)$ must be a Baire space given that $C_\pi(Y)$ is a Baire space for every countable subspace of X, and that X is normal?

4. Necessary conditions: the game Γ.

DEFINITION 4.1. The *game* $\Gamma = \Gamma(X)$ is a game in which two players, called (I) and (II), are given an arbitrary finite starting set S_0 and then proceed to choose alternate terms in a sequence $S_0, S_1, S_2, \cdots$ of pairwise disjoint finite (possibly empty) subsets of a topological space X. The resulting sequence $(S_0, S_1, S_2, \cdots)$ is called a *play* of the game Γ and is said to result in a *win for player* (I) if and only if the set $S_1 \cup S_3 \cup S_5 \cup \cdots$ is *not* a closed discrete subspace of X, i.e., if and only if the set $S_1 \cup S_3 \cup S_5 \cup \cdots$ has a limit point in X. If player (I) does not win, then player (II) wins.

DEFINITION 4.2. A *strategy* for player (I) in the game Γ is a function σ which assigns to each pairwise disjoint sequence $(S_0, S_1, \cdots, S_{2k})$, with $k \geqq 0$, a finite set S_{2k+1} which is disjoint from $(S_0 \cup S_1 \cup \cdots \cup S_{2k})$. A strategy for player II is a function τ which assigns to each pairwise disjoint sequence $(S_0, S_1, \cdots, S_{2k+1})$ $(k \geqq 0)$ a finite set S_{2k+2} which is disjoint from $(S_0 \cup S_1 \cup \cdots \cup S_{2k+1})$.

DEFINITION 4.3. A strategy σ for player (I) in the game Γ is said to be a *winning strategy* if, whenever $(S_0, S_1, \cdots)$ is a play of the game Γ in which $S_{2k+1} = \sigma(S_0, S_1, \cdots, S_{2k})$ for each $k \geqq 0$, then player (I) wins that play, i.e., the set $S_1 \cup S_3 \cup S_5 \cup \cdots$ has a limit point in X. A strategy τ for player (II) is a winning strategy if, whenever a play $(S_0, S_1, S_2, \cdots)$ has $S_{2k+2} = \tau(S_0, S_1, \cdots, S_{2k+1})$ for every $k \geqq 0$, then $S_1 \cup S_3 \cup \cdots$ is a closed discrete subspace of X.

REMARKS 4.4. In defining strategies for player (I) we often use inductions in which $S_{2k+1} = \sigma(S_0, S_1, \cdots, S_{2k})$ is defined only for those sequences $(S_0, S_1, \cdots, S_{2k})$ which could result from an actual play of the game using σ at all earlier stages, i.e., sequences $(S_0, S_1, \cdots, S_{2k})$ in which $S_1 = \sigma(S_0)$, $S_3 = \sigma(S_0, S_1, S_2)$, etc. To be precise, we should define $\sigma(S_0, S_1, \cdots, S_{2k}) = \varnothing$ for all other sequences. An analogous remark applies to defining strategies for player (II)-cf. 4.6.

EXAMPLE 4.5. (a) Suppose X contains a nontrivial convergent sequence $x_1, x_2, x_3, \cdots$. To define a winning strategy for player (I) in $\Gamma(X)$ we let $\sigma(S_0, S_1, \cdots, S_{2k}) = \{x_n\}$, where x_n is the first point of the sequence not in $(S_0 \cup S_1 \cup \cdots S_{2k})$.

(b) Suppose X is a space in which every countable set is closed, e.g., X is an uncountable set containing a point p such that each point of $X - \{p\}$ is isolated while neighborhoods of p are co-countable sets. Then player (II) has an obvious winning strategy: for any sequence $(S_0, S_1, \cdots, S_{2k+1})$, let $\tau(S_0, S_1, \cdots, S_{2k+1}) = \varnothing$. There is a sense in which this is the only possible winning strategy for player (II)—see 8.4.

THEOREM 4.6. *If $C_\pi(X)$ is a Baire space, then player* (I) *cannot have a winning strategy in the game $\Gamma(X)$.*

Proof. Let σ be any strategy for player (I). Using the fact that $C_\pi(X)$ is a Baire space we will define a counterstrategy τ for (II) and a finite starting set S_0 such that in the play S_0, $S_1 = \sigma(S_0)$, $S_2 = \tau(S_0, S_1)$, etc., player (II) defeats player (I).

We first define an array of finite subsets $T(i_1, \cdots, i_k)$ of X, one for each finite sequence $(i_1, i_2, \cdots, i_k)$ of elements of ω, by induction on k. Let $T(0) = \varnothing$ and, if $T(0), \cdots, T(n-1)$ are defined, let $T(n) = \sigma(T(0) \cup \cdots \cup T(n-1))$. Suppose $k \geqq 1$ and that $T(i_1, \cdots, i_k)$ is defined for every k-tuple of elements of ω. Let $T(i_i, \cdots, i_k, 0) = \varnothing$ and, if $T(i_1, \cdots, i_k, j)$ is defined for $0 \leqq j \leqq n-1$, let $T(i_1, i_2, \cdots, i_k, n) = \sigma(R_0, R_1, R_2, \cdots, R_{2k-1}, T(i_1, \cdots, i_k, 0) \cup \cdots \cup T(i_1, \cdots, i_k, n-1))$ where $R_0 = T(0) \cup \cdots \cup T(i_1 - 1)$, $R_1 = T(i_1)$, $R_2 = T(i_1, 0) \cup \cdots \cup T(i_1, i_2 - 1)$, $R_3 = T(i_1, i_2), \cdots, R_{2j-1} = T(i_1, \cdots, i_j)$ and $R_{2j} = T(i_1, \cdots, i_j, 0) \cup \cdots \cup T(i_1, \cdots, i_j, i_{j+1} - 1)$ for $j \leqq k$.

We next define open subsets $\mathscr{W}(i_1, \cdots, i_k)$ of $C_\pi(X)$ inductively, one for each k-tuple of elements of ω. Let $\mathscr{W}(0) = \varnothing$ and, if $\mathscr{W}(0), \cdots, \mathscr{W}(n-1)$ are defined, let $\mathscr{W}(n) = \cap\{[\{x\}, (0, 1)]: x \in T(n)\} - \mathrm{cl}\,(\mathscr{W}(0) \cup \cdots \cup \mathscr{W}(n-1))$. (This notation is defined in § 2.) Suppose $\mathscr{W}(i_1, \cdots, i_k)$ is defined for some $k \geqq 1$. Let $\mathscr{W}(i_1, \cdots, i_k, 0) = \varnothing$. Suppose $\mathscr{W}(i_1, \cdots, i_k, j)$ is defined for every $j \in \{0, \cdots, n-1\}$. We let $\mathscr{W}(i_1, \cdots, i_k, n) = \mathscr{W}(i_1, \cdots, i_k) \cap (\cap\{[\{x\}, (k-1, k)]: x \in T(i_1, \cdots,$

$i_k, n)\}) - E$ where $E = \mathrm{cl}(\mathscr{W}(i_1, \cdots, i_k, 0) \cup \cdots \cup \mathscr{W}(i_1, \cdots, i_k, n-1))$.

Now define $\mathscr{G}(n) = \cup\{\mathscr{W}(i_1, i_2, \cdots, i_n): i_j \in \omega \text{ for } 1 \leqq j \leqq n\}$. Each $\mathscr{G}_n$ is open in $C_\pi(X)$. We claim that each $\mathscr{G}_n$ is dense. First consider $\mathscr{G}_1$. Let $\mathscr{U} = \bigcap_{j=1}^n [\{x_j\}, V_j]$ be a basic open set in $C_\pi(X)$. Let $F = \{x_1, \cdots, x_n\}$. Since F is finite and the sets $T(1), T(2), \cdots$ are pairwise disjoint, there is a least positive integer p such that $T(p) \cap F = \varnothing$. Then either there is an integer $j \in \{1, \cdots, p-1\}$ for which $\mathscr{U}$ meets $\mathscr{W}(j) \subset \mathscr{G}_1$, or else (since X is completely Hausdorff and $F \cup T(p)$ is finite) $\mathscr{U}$ meets $\mathscr{W}(p) \subset \mathscr{G}_1$. Therefore $\mathscr{G}_1$ is dense in $C_\pi(X)$. Suppose we know that $\mathscr{G}_{n-1}$ is dense in $C_\pi(X)$. To show that $\mathscr{G}_n$ is dense, it is enough to show that $\cup\{\mathscr{W}(i_1, \cdots, i_{n-1}, k): k \geqq 0\}$ is dense in $\mathscr{W}(i_1, \cdots, i_{n-1})$ for each $(n-1)$ tuple $(i_1, \cdots, i_{n-1})$. To that end, let $\mathscr{U} = \bigcap_{j=1}^k [\{x_j\}, V_j]$ be any basic open set contained in $\mathscr{W}(i_1, \cdots, i_{n-1})$. Since the set $F = \{x_j: 1 \leqq j \leqq k\}$ is finite and the sets $T(i_1, \cdots, i_k, j)$ are pairwise disjoint for $j = 1, 2, 3, \cdots$, there is a first positive integer p such that $T(i_1, \cdots, i_{n-1}, p)$ misses F. But then $U \cap (\cup\{\mathscr{W}(i_1, \cdots, i_{n-1}, j): 1 \leqq j \leqq p\}) \neq \varnothing$, so $\mathscr{G}_n$ is dense in $C_\pi(X)$.

Because $C_\pi(X)$ is a Baire space, there is a (continuous) function $f \in \cap\{\mathscr{G}_n: n \geqq 1\}$. Because the sets used to obtain each $\mathscr{G}_n$ are pairwise disjoint, there is a unique sequence $i_1, i_2, \cdots$ such that $f \in \mathscr{W}(i_1, \cdots, i_k)$ for each $k \geqq 1$. Then $f[T(i_1, i_2, \cdots, i_k)] \subset (k-1, k)$ so that, since f is continuous, the set $\cup\{T(i_1, \cdots, i_k): k \geqq 1\}$ has no limit points in X.

Now consider the play of Γ which starts with $S_0 = T(0) \cup \cdots \cup T(i_i - 1)$. Since player (I) uses strategy σ, $S_1 = \sigma(S_0) = T(i_1)$. Let player (II) respond by specifying $S_2 = T(i_1, 0) \cup \cdots \cup T(i_1, i_2 - 1)$. Then player (I) must let

$$\begin{aligned} S_3 &= \sigma(S_0, S_1, S_2) \\ &= \sigma(T(0) \cup \cdots \cup T(i_1 - 1, T(i_1), T(i_1, 0) \cup \cdots \cup T(i_1, i_2 - 1)) , \end{aligned}$$

i.e., $S_3 = T(i_1, i_2)$. Then player (II) lets $S_4 = T(i_1, i_2, 0) \cup \cdots \cup T(i_1, i_2, i_3 - 1)$, and so on. In this play of Γ, we have $S_1 \cup S_3 \cup S_5 \cup \cdots = \cup\{T(i_1, \cdots, i_k): k \geqq 1\}$ which is known to be closed and discrete. Therefore, σ could not have been a winning strategy for player (I).

5. Special results on the game Γ. As Theorem 4.6 illustrates, it is the nonexistence of winning strategies for Γ which is relevant to the study of $C_\pi(X)$. It is possible to translate this condition out of game-theoretic language, and that is the point of our first result in this section.

THEOREM 5.1. *The following are equivalent for any space X:*

(a) *Player* (I) *has no winning strategy in the game* Γ;

(b) *given infinite, pairwise disjoint collections* $\mathscr{F}_1, \mathscr{F}_2, \cdots$ *of finite sets, it is possible to choose infinite subcollections* $\mathscr{F}_n' \subset \mathscr{F}_n$ *in such a way that* $\cup \{F: F \in \mathscr{F}_n'$ *for some* $n \geqq 1\}$ *is closed and discrete*;

(c) *given infinite, pairwise disjoint collections* $\mathscr{F}_1, \mathscr{F}_2, \cdots$ *of finite sets, it is possible to choose one member* $F_n \in \mathscr{F}_n$ *such that* $\cup \{F_n: n \geqq 1\}$ *is closed and discrete.*

Proof. To prove that (a) implies (b), suppose no winning strategy for player (I) in Γ exists, and that the collections $\mathscr{F}_1, \mathscr{F}_2, \cdots$ are given. Write $N = \cup \{N_k: k \geqq 1\}$ where the sets N_k are infinite, pairwise disjoint subsets of the set N of natural numbers. Now define a strategy σ for player (I) as follows. Let S_0 be any finite starting set. Choose the unique index k having $1 \in N_k$. Define $\sigma(S_0)$ to be the first member of $\mathscr{F}_k$ which is disjoint from S_0. Suppose $(S_0, S_1, \cdots, S_{2n})$ is a pairwise disjoint sequence of finite sets. Find the unique index k such that $n \in N_k$ and let $\sigma(S_0, S_1, \cdots, S_{2n})$ be the first member of $\mathscr{F}_k$ which is disjoint from $(S_0 \cup \cdots \cup S_{2n})$. By hypothesis, this σ cannot be a winning strategy for player (I) so that there must be a play of the game Γ, say $(S_0, S_1, S_2, \cdots)$, such that $S_{2k+1} = \sigma(S_0, \cdots, S_{2k})$ for each $k \geqq 1$, and yet $\cup \{S_{2k+1}: k \geqq 0\}$ is closed and discrete. Then if we let $\mathscr{F}_n' = \{S_{2k+1}: S_{2k+1} \in \mathscr{F}_n\}$ we obtain an infinite subcollection of $\mathscr{F}_n$, and $\cup \{F: F \in \mathscr{F}_n'$ for some $n \geqq 1\} = S_1 \cup S_3 \cup S_5 \cup \cdots$ is closed and discrete.

That (b) implies (c) is obvious.

To prove that (c) implies (a), suppose that σ is a strategy for player (I) in game Γ. We will show how to find a finite starting set S_0 and a play $(S_0, S_1, S_2, \cdots)$ of Γ in which $S_{2k+1} = \sigma(S_0, \cdots, S_{2k})$ and yet $(S_1 \cup S_3 \cup \cdots)$ is a closed discrete set. Using the strategy σ, define finite sets $T(i_1, \cdots, i_k)$ for each k-tuple of elements of ω, exactly as in the proof of Theorem 4.6. Let $\mathscr{F} = \{T(j): j \geqq 1\}$ and for each k-tuple $(i_1, i_2, \cdots, i_k)$ of elements of N, let $\mathscr{F}(i_1, \cdots, i_k) = \{T(i_1, \cdots, i_k, j): j \geqq 1\}$. Applying our hypothesis about disjoint collections, we may choose one member from each of these collections in such a way that the union of the chosen sets is closed and discrete. To be more specific, let the chosen sets be $T(m) \in \mathscr{F}$ and, for each k-tuple $(i_1, \cdots, i_k)$, let $T(i_1, \cdots, i_k, m(i_1, i_2, \cdots, i_k))$ be the chosen member of $\mathscr{F}(i_1, i_2, \cdots, i_k)$. Now let $j_1 = m$, $j_2 = m(j_1), \cdots, j_{n+1} = m(j_1, j_2, \cdots, j_n)$. Define the starting set $S_0 = T(0) \cup \cdots \cup T(j_1 - 1)$. Using the strategy σ, player (I) must specify the set $S_1 = \sigma(S_0) = \sigma(T(0) \cup \cdots \cup T(j_1 - 1)) = T(j_1)$. Then let player (II) respond by letting $S_2 = T(j_1, 0) \cup \cdots \cup T(j_1, j_2 - 1)$ so that, following strategy σ, Player (I) must let $S_3 = \sigma(S_0, S_1, S_2) = \sigma(T(0) \cup \cdots \cup T(j_1 - 1), T(j_1),$

$T(j_i, 0) \cup \cdots \cup T(j_1, j_2 - 1)) = T(j_1, j_2)$. And when player (II) takes his or her nth turn, he or she lets $S_{2n} = T(i_1, \cdots, i_n, 0) \cup \cdots \cup T(i_1, \cdots, i_n, i_{n+1} - 1)$ so that player (I) is forced to let $S_{2n+1} = \sigma(S_0, \cdots, S_{2n}) = T(i_1, \cdots, i_{n+1})$. But then, because $\cup \{T(i_1, \cdots, i_k): k \geqq 1\}$ is closed and discrete, Player (I) has lost in this play of the game and so σ could not have been a winning strategy.

REMARK 5.2. It is sometimes convenient to have an indexed version of the property described in 5.1 (b). Obviously 5.1 (b) equivalent to: (b)′ given collections $\mathscr{F}_n = \{F(n, \alpha): \alpha \in A_n\}$ of pairwise disjoint finite sets where each A_n is an infinite index set, there are infinite subsets $A'_n \subset A_n$ such that $\cup \{F(n, \alpha): \alpha \in A'_n \text{ for some } n \geqq 1\}$ is closed and discrete. The version given in (b)′ covers the case where many of the sets $F(n, \alpha)$ belonging to $\mathscr{F}_n$ are empty, and that will simplify the proof of 6.5 below.

6. Sufficient conditions involving the game Γ. In this section, let Σ be the class of all collectionwise Hausdorff [3] spaces X in which the set $X^d = \{x \in X: x \text{ is not isolated}\}$ is a discrete subspace of X. Other characterizations of members are given in our next lemma, whose proof is elementary.

LEMMA 6.1. *The following properties of a space X are equivalent:*

(a) $X \in \Sigma$;

(b) *X is paracompact and X^d is discrete;*

(c) *X is a topological sum $X = \bigoplus \{Y_\alpha: \alpha \in A\}$ of Hausdorff spaces Y_α each having at most one nonisolated point.*

The notation $\tilde{\mathscr{N}}(f, S, \varepsilon)$ was defined in § 2. We need the following constructive lemma.

LEMMA 6.2. *Suppose that, in the product space $\boldsymbol{R}^X$, we have a sequence $\tilde{\mathscr{N}}(f_n, S_n, \varepsilon_n)$ of basic open sets with $\tilde{\mathscr{N}}(f_n, S_n, \varepsilon_n) \supset \tilde{\mathscr{N}}(f_{n+1}, S_{n+1}, 2\varepsilon_{n+1})$ for each $n \geqq 1$. Suppose $\lim_n \varepsilon_n = 0$ and let $T = \cup \{S_n: n \geqq 1\}$. Then the sequence $\langle f_n \rangle$ converges pointwise to some function g on the set T and $\hat{g} \in \cap \{\tilde{\mathscr{N}}(f_n, S_n, \varepsilon_n): n \geqq 1\}$ whenever $\hat{g} \in \boldsymbol{R}^X$ has $\hat{g}|_T = g$. Furthermore, if $p \in T$ and if, for each n, E_n is a (possibly empty) subset of S_n for which $f_n[E_n] \subset [f_{n-1}(p) - \varepsilon_{n-1}, f_{n-1}(p) + \varepsilon_{n-1}]$, then for every $\varepsilon > 0$ there is an integer $N = N(\varepsilon)$ such that $\hat{g}[\cup \{E_n: n \geqq N\}] \subset (\hat{g}(p) - \varepsilon, \hat{g}(p) + \varepsilon)$.*

Proof. Fix any $t \in T$. Choose any n such that $t \in S_n$. If $i, j \geqq n$

then $f_i, f_j \in \widetilde{\mathscr{N}}(f_n, S_n, \varepsilon_n)$ so that $|f_i(t) - f_j(t)| < 2\varepsilon_n$. Therefore the sequence $\langle f_k(t)\rangle$ is a Cauchy sequence in $\boldsymbol{R}$ and has a limit, say $g(t)$. Let $\hat{g} \in \boldsymbol{R}^X$ agree with g on T. With n fixed and $t \in S_n$, for each $k \geqq n+1$, $f_k \in \widetilde{\mathscr{N}}(f_{n+1}, S_{n+1}, \varepsilon_{n+1})$ and so $f_k(t) \in (f_{n+1}(t) - \varepsilon_{n+1}, f_{n+1}(t) + \varepsilon_{n+1})$. Therefore $\hat{g}(t) = \lim_k f_k(t) \in [f_{n+1}(t) - \varepsilon_{n+1}, f_{n+1}(t) + \varepsilon_{n+1}] \subset (f_n(t) - \varepsilon_n, f_n(t) + \varepsilon_n)$. Hence $\hat{g} \in \bigcap \{\widetilde{\mathscr{N}}(f_n, S_n, \varepsilon_n): n \geqq 1\}$.

To prove the final assertion, suppose $p \in S_{N_1}$. Fix $\varepsilon > 0$ and choose N_2 so large that whenever $n \geqq N_2$, both $\varepsilon_n < \varepsilon/3$ and $|f_n(p) - \hat{g}(p)| < \varepsilon/3$. Let $N = \max(N_1, N_2)$ and consider $\hat{g}[E_n]$ where $n \geqq N + 1$. Fix $t \in E_n$. Then

(a) $\hat{g}(t) \in [f_n(t) - \varepsilon_n, f_n(t) + \varepsilon_n]$;

(b) $f_n(t) \in [f_{n-1}(p) - \varepsilon_{n-1}, f_{n-1}(p) + \varepsilon_{n-1}]$;

(c) from (a) and (b),

$$\hat{g}(t) \in [f_{n-1}(p) - \varepsilon_{n-1} - \varepsilon_n, f_{n-1}(0) + \varepsilon_{n-1} + \varepsilon_N] .$$

(d) Because $n - 1 \geqq N$,

$$f_{n-1}(p) \in [\hat{g}(p) - \varepsilon/3, g(p) + \varepsilon/3] \quad \text{and so}$$

(e) $\hat{g}(p) - \varepsilon < \hat{g}(p) - \varepsilon/3 - \varepsilon_{n-1} - \varepsilon_n \leqq f_{n-1}(p) - \varepsilon_{n-1} - \varepsilon_n \leqq \hat{g}(t) \leqq f_{n-1}(p) + \varepsilon_{n-1} + \varepsilon_n < \hat{g}(p) + \varepsilon/3 + \varepsilon_{n-1} + \varepsilon_n < \hat{g}(p) + \varepsilon$.

Therefore, $n \geqq N + 1$ implies $\hat{g}[E_n] \subset (\hat{g}(p) - \varepsilon, \hat{g}(p) + \varepsilon)$.

We are now in a position to characterize those members X of Σ for which $C_\pi(X)$ is a Baire space.

THEOREM 6.3. *Let $X \in \Sigma$. Then $C_\pi(X)$ is a Baire space if and only if player* (I) *has no winning strategy in $\Gamma(X)$.*

Proof. In the light of Theorem 4.6, it remains only to prove that if player (I) has no winning strategy in $\Gamma(X)$, then $C_\pi(X)$ is a Baire space. We establish the contrapositive, namely (cf. Lemma 2.4) that if $C_\pi(X)$ is a first category space, then there is a winning strategy for player (I) in Γ.

Since X^d is a discrete subspace of X, there is a partition $\{V_p: p \in X^d\}$ of the space X into open subspaces for which $\{p\} = X^d \cap V_p$. Since $C_\pi(X)$ is the first category there is a sequence $\mathscr{G}_1 \supset \mathscr{G}_2 \supset \cdots$ of dense open subsets of $\boldsymbol{R}^X$ having $(\bigcap \{\mathscr{G}_n: n \geqq 1\}) \cap C_\pi(X) = \varnothing$ (cf. Lemma 2.5).

Now suppose S_0 is any finite starting set for the game Γ. Player (I) should let $f_0 \equiv 0$ and $\varepsilon_0 = 1$. Then $\widetilde{\mathscr{N}}(f_0, S_0, \varepsilon_0)$ must intersect the dense open set $\mathscr{G}_1$. Player (I) chooses a basic open set $\widetilde{\mathscr{N}}(f_1, T_1, 2\varepsilon_1) \subset \mathscr{G}_1 \cap \widetilde{\mathscr{N}}(f_0, S_0, \varepsilon_0)$. Without loss of generality, $\varepsilon_1 \in (0, 2^{-1}]$. Furthermore, enlarging T_1 if necessary, we may assume that if $V_p \cap S_0 \neq \emptyset$ then $p \in T_1$. Player (I) should now define $S_1 = \sigma(S_0) = T_1 - S_0$. For induction hypothesis, suppose sets $S_0, S_1, \cdots, S_{2n-1}$, numbers $\varepsilon_0, \cdots, \varepsilon_{2n-1}$ and functions $f_0, f_1, \cdots, f_{2n-1}$ are defined in such a way that

(a) $\varepsilon_i \in (0, 2^{-i}]$;

(b) if $V_p \cap (\cup\{S_i: 0 \leqq i \leqq 2n-2\}) \neq \emptyset$ then $p \in S_0 \cup S_1 \cup \cdots \cup S_{2n-1}$;

(c) $\tilde{\mathscr{N}}(f_{2j-1}, T_{2j-1}, 2\varepsilon_{2j-1}) \subset \mathscr{G}_{2j-1} \cap \tilde{\mathscr{N}}(f_{2j-2}, T_{2j-2}, \varepsilon_{2j-2})$ if $j \leqq n$, where $T_k = S_0 \cup \cdots S_k$;

(d) $f_{2j}(x) = \begin{cases} f_{2j-1}(p) & \text{if } x \in S_{2j} \cap V_p \text{ for some } p \in X^d; \\ f_{2j-1}(x) & \text{otherwise.} \end{cases}$

Now suppose S_{2n} is a finite set which is disjoint from $(S_0 \cup S_1 \cup \cdots \cup S_{2n-1})$. Define a function f_{2n} by

$$f_{2n}(x) = \begin{cases} f_{2n-1}(p) & \text{if } x \in S_{2n} \cap V_p \text{ for some } p \in X^d \\ f_{2n-1}(x) & \text{otherwise.} \end{cases}$$

Let $\varepsilon_{2n} = (1/2)\varepsilon_{2n-1}$. Let $T_{2n} = S_0 \cup S_1 \cup \cdots \cup S_{2n-1} \cup S_{2n}$. Then the basic open set $\tilde{\mathscr{N}}(f_{2n}, T_{2n}, \varepsilon_{2n})$ must intersect the dense open set $\mathscr{G}_{2n+1}$ so that there is a basic open set $\tilde{\mathscr{N}}(f_{2n+1}, T_{2n+1}, 2\varepsilon_{2n+1}) \subset \mathscr{G}_{2n+1} \cap \tilde{\mathscr{N}}(f_{2n}, T_{2n}, \varepsilon_{2n})$. Enlarging T_{2n+1} and shrinking ε_{2n+1} if necessary, we may assume $\varepsilon_{2n+1} \in (0, 2^{-(2n+1)}]$ and that if $V_p \cap T_{2n} \neq \emptyset$, then $p \in T_{2n+1}$. Player (I) should now define $S_{2n+1} = T_{2n+1} - T_{2n}$. Thus the strategy σ is defined.

We claim that in any play $(S_0, S_1, S_2, \cdots)$ of Γ in which $S_{2n+1} = \sigma(S_0, S_1, \cdots, S_{2n})$ is defined as above, the set $(S_1 \cup S_3 \cup S_5 \cup \cdots)$ must fail to be closed and discrete. For suppose the set $T_0 = S_1 \cup S_2 \cup S_3 \cup \cdots$ is closed and discrete. Let $T_E = S_0 \cup S_2 \cup S_4 \cdots$ and let $T = T_0 \cup T_E$. Because part (b) of the induction step, T is a closed set. Let $T' = \{p \in T: p \text{ is a limit point of } T_E\}$. Then $T_E \cup T' \subset T$. According to Lemma 6.2, the sequence $\langle f_n \rangle$ converges pointwise to some function $g: T \to \boldsymbol{R}$. We claim that g is continuous on T. Certainly g is continuous when restricted to the closed discrete space T_0. To see that g is also continuous on $T_E \cup T'$, let $p \in T'$ and let $\varepsilon > 0$. We apply the final conclusion of Lemma 6.2 with $E_n = \emptyset$ if n is odd, and $E_n = S_n \cap V_p$ if n is even. Then $f_{n+1}(E_{n+1}) \subset [f_n(p) - \varepsilon_n, f_n(p) + \varepsilon_n]$ for each $n \geqq 1$ so that, corresponding to the given ε, there is an integer N having $g[\{p\} \cup (\cup \{S_n \cap V_p: n \text{ is even, } n \geqq N\})] \subset (g(p) - \varepsilon, g(p) + \varepsilon)$. Now $V_p \cap (T_E \cup T')$ is a relative neighborhood of p in $T_E \cup T'$, and $V_p \cap (T_E \cup T') = \{p\} \cup (\cup \{S_n \cap V_p | n \text{ is even}\})$. Because X is Hausdorff and $\cup \{S_n: n < N\}$ is finite, the set $\{p\} \cup (\cup \{S_n \cap V_p: n \text{ is even and } n \geqq N\})$ is a relative neighborhood of p. But that establishes continuity of g at p. Since $p \in T'$ was arbitrary, g is continuous on $(T_E \cup T')$. Since $(T_E \cup T')$ is closed in T, g is continuous on T.

But X, being in Σ, is normal (cf. 6.1) so that because T is a closed subspace of X, there must be a continuous $\hat{g} \in C_\pi(X)$ which extends g. But then

$$\hat{g} \in C_\pi(X) \cap (\cap \{\mathscr{G}_n : n \geqq 1\}) = \varnothing$$

which is impossible.

Perhaps the most important open question raised by our results asks whether the game $\Gamma(X)$ characterizes category in $C_\pi(X)$ for every space X (or for every normal space X). In view of the known pathological behavior of the class of Baire spaces under the formation of products [6], the following result leads us to conjecture a negative answer, and may point the way to the anticipated counterexample.

THEOREM 6.4. *Suppose X_1 and X_2 are spaces for which $C_\pi(X_1)$ and $C_\pi(X_2)$ are Baire spaces. Then either*

(a) *$C_\pi(X_1) \times C_\pi(X_2)$ is a Baire space; or else*

(b) *$C_\pi(X_1 \oplus X_2)$ is a first category space even though player* (I) *has no winning strategy in the game $\Gamma(X_1 \oplus X_2)$.*

Proof. Suppose $C_\pi(X_1) \times C_\pi(X_2)$ is not a Baire space. According to 2.6, $C_\pi(X_1) \times C_\pi(X_2)$ is homeomorphic to $C_\pi(X_1 \oplus X_2)$ so that, by 2.4, $C_\pi(X_1 \oplus X_2)$ is of first category. Therefore, to complete the proof, it is enough to show that player (I) cannot have a winning strategy in the game Γ played in $X_1 \oplus X_2$. That is the point of our next lemma which uses the indexed version of 5.1 (b)—cf. Remark 5.2.

LEMMA 6.5. *Let $\{X_\alpha : \alpha \in A\}$ be any family of spaces such that for each $\alpha \in A$, player* (I) *does not have a winning strategy in $\Gamma(X_\alpha)$. Then player* (I) *cannot have a winning strategy in $\Gamma(\oplus\{X_\alpha : \alpha \in A\})$.*

Proof. For each n, let $\mathscr{F}_n = \{F(n, \beta) : \beta \in B_n\}$ be an infinite pairwise disjoint family of subsets of $\oplus\{X_\alpha : \alpha \in A\}$. We may assume that each B_n is countably infinite. Then there is a countable $A_0 \subset A$ such that $\cup\{F(n, \beta) : \beta \in B_n, n \geqq 1\} \subset \oplus\{X_\alpha : \alpha \in A_0\}$, say $A_0' = \{\alpha_1, \alpha_2, \cdots\}$. Applying 5.2 to the space X_{α_1}, we find infinite sets $B_n(1) \subset B_n$ such that $\cup\{X_{\alpha_1} \cap F(n, \beta) : \beta \in B_n(1), n \geqq 1\}$ is a closed discrete subset of X_{α_1}. We inductively choose infinite sets $B_n(1) \supset B_n(2) \supset B_n(3) \supset \cdots$ such that for each $k \geqq 1$, $\cup\{X_{\alpha_k} \cap F(n, \beta) : \beta \in B_n(k)$ for some $n \geqq 1\}$ is closed and discrete in X_{α_k}. Now choose $\beta_n \in B_n(n) \subset B_n$. Fix $k \geqq 1$. Then $X_{\alpha_k} \cap (\cup\{F(n, \beta_n) : n \geqq 1\}) = [F((1, \beta_1) \cup \cdots \cup F(k-1, \beta_{k-1})) \cap X_{\alpha_k}] \cup [\cup\{F(n, \beta_n) : n \geqq k\} \cap X_{\alpha_k}]$ which is the union of a finite set and a closed discrete set. Hence $\cup\{F(n, \beta_n): n \geqq 1\}$ is closed and discrete in $\bigoplus\{X_\alpha: \alpha \in A\}$.

Using the idea in the proof of 6.4 we can obtain a positive result about products of certain Baire spaces, namely,

THEOREM 6.6. *Let $\{X_\alpha: \alpha \in A\}$ be any subfamily of Σ. If each space $C_\pi(X_\alpha)$ is a Baire space, then so is $\Pi\{C_\pi(X_\alpha): \alpha \in A\}$.*

Proof. As in the proof of 6.4, $\Pi\{C_\pi(X_\alpha): \alpha \in A\}$ is homeomorphic to $C_\pi(\bigoplus\{X_\alpha: \alpha \in A\})$. Because each $C_\pi(X_\alpha)$ is a Baire space, player (I) cannot have a winning strategy in any of the games $\Gamma(X_\alpha)$ so that, in the light of 6.5, player (I) has no winning strategy in $\Gamma(\bigoplus\{X_\alpha: \alpha \in A\})$. But $\bigoplus\{X_\alpha: \alpha \in A\}$ is a member of Σ so that, by 6.3, $C_\pi(\bigoplus\{X_\alpha: \alpha \in A\})$ is a Baire space.

REMARK 6.7. Theorem 6.6 might lead the reader to conjecture that for $X \in \Sigma$, if $C_\pi(X)$ is Baire, then $C_\pi(X)$ must have one of the stronger completeness properties (e.g., subcompactness, pseudocompleteness) which were designed to make product spaces have the Baire property. That is not the case, as a combination of Example 7.1 and Theorem 8.4 shows.

7. Applications of the game Γ.

In this section we present some countable regular spaces whose function spaces are Baire.

EXAMPLE 7.1. Let X be a countable subspace of $\beta\omega$ for which X^d is discrete (X^d is defined in § 6). Then $C_\pi(X)$ is a Baire space.

Proof. There is a countable collection $\{V_n: n \geqq 1\}$ of pairwise disjoint open subsets of X having $X = \cup\{V_n: n \geqq 1\}$ and such that each V_n contains at most one limit point of X. Therefore X is the topological sum $\bigoplus\{V_n | n \geqq 1\}$ so that $X \in \Sigma$. Because $C_\pi(X) \cong \Pi\{C_\pi(V_n): n \geqq 1\}$ and because each $C_\pi(V_n)$ is a separable metric space, a theorem of Oxtoby [6] shows that $C_\pi(X)$ is a Baire space provided each $C_\pi(V_n)$ is Baire. But that is an easy consequence of 6.3 and 5.1. For suppose $\mathscr{F}_1, \mathscr{F}_2, \cdots$ is a sequence of infinite, pairwise disjoint subcollections of finite subsets of V_n. Let F_1 and F'_1 be distinct members of $\mathscr{F}_1$. Inductively choose distinct members F_n, F'_n of $\mathscr{F}_n$ which are disjoint from the finite set $F_1 \cup F'_1 \cup \cdots \cup F_{n-1} \cup F'_{n-1}$. If the set $F_1 \cup F_2 \cup \cdots$ is closed and discrete, we are done. And otherwise, p is a limit point of $(F_1 \cup F_2 \cup \cdots) - \{p\}$. Since disjoint subsets of ω cannot have a common limit point in $\beta\omega$, it now follows that $\cup\{F'_n: n \geqq 1\}$ is closed and discrete.

EXAMPLE 7.2. There is a countable regular space X having exactly one nonisolated point such that $C_\pi(X)$ is Baire and yet X cannot be embedded in $\beta\omega$.

Proof. Let O and E be the odd and even positive integers respectively. Let $\tilde{p}$ be an ultrafilter of subsets of O and let $\tilde{q}$ be an ultrafilter of subsets of E. Let $\mathscr{F} = \{S \cup T \mid S \in \tilde{p},\ T \in \tilde{q}\}$.

Topologize ω by isolating each $n \neq 0$ and by letting neighborhoods of 0 have the form $\{0\} \cup F$ where $F \in \mathscr{F}$. If X denotes the resulting space, then X cannot be embedded in $\beta\omega$ since not every function on $X - \{0\}$ can be continuously extended over all of X. However, $C_\pi(X)$ is a Baire space, again in the light of 6.3 and 5.1. For let $\mathscr{F}_1, \mathscr{F}_2, \cdots$ be infinite collections of pairwise disjoint finite subsets of X. Inductively choose distinct sets F_n, F'_n, F''_n from $\mathscr{F}_n$ in such a way that $F_{n+1} \cup F'_{n+1} \cup F''_{n+1}$ is disjoint from $(F_1 \cup F'_1 \cup F''_1 \cup \cdots \cup F_n \cup F'_n \cup F''_n)$. Suppose that neither $(F_1 \cup F_2 \cup \cdots)$ nor $(F'_1 \cup F'_2 \cup \cdots)$ is closed and discrete. Then each neighborhood of 0 meets $F_1 \cup F_2 \cup \cdots$ so that $F_1 \cup F_2 \cup F_3 \cup \cdots$ contains a member of $\tilde{p}$ or a member of $\tilde{q}$. We may assume that some member $S \in \tilde{p}$ is a subset of $F = F_1 \cup F_2 \cup \cdots$. Similarly, $F' = F'_1 \cup F'_2 \cup \cdots$ contains either a member of $\tilde{p}$ or a member of $\tilde{q}$. Since $F' \cap F = \varnothing$, F' must contain a member of $\tilde{q}$. But then $F'' = F''_1 \cup F''_2 \cup \cdots$ cannot contain a member of $\tilde{p}$ or of $\tilde{q}$ so that F'' must be closed and discrete.

REMARK 7.3. It would be of interest to have a filter $\mathscr{F}$ on N so that

(a) $\mathscr{F}$ is constructed without invoking the axiom of choice; and

(b) if ω is topologized by using $\{\{0\} \cup F : F \in \mathscr{F}\}$ as the filter of neighborhoods of 0, then $C_\pi(\omega)$ is Baire.

8. Completeness properties stronger than Baire. It is well-known that the family of Baire spaces is badly behaved under the formation of Cartesian products and this fact has led researchers to formulate completeness properties which are well-behaved under the formation of products (see [1] and [5] for surveys). In this section we concentrate on three such properties: weak α-favorability [7], pseudocompleteness [6] and Čech-completeness.

DEFINITION 3.1. [6]. A *pseudocomplete sequence* for a space Z is a sequence $\Phi_1, \Phi_2, \cdots$ of collections of nonempty open subsets of Z such that

(a) for each $n \geqq 1$, if $U \neq \varnothing$ is open then some $P \in \Phi_n$ has $\varnothing \neq P \subset U$

(b) if a sequence $P_n \in \Phi_n$ has $\mathrm{cl}(P_{n+1}) \subset P_n$, for each $n \geqq 1$, then $\bigcap \{P_n: n \geqq 1\} \neq \varnothing$.

The space Z is called *pseudocomplete* if Z has a pseudocomplete sequence and has the property that each nonvoid open set contains the closure of some nonvoid open set.

DEFINITION 8.2. [7]. The space $(X, \mathscr{T})$ is said to be *weakly α-favorable* if there is a sequence $\langle \phi_n \rangle$ of functions satisfying

(a) $\phi_n: \mathscr{T}^* \to \mathscr{T}^*$, where $\mathscr{T}^* = \{U \in \mathscr{T}: U \neq \varnothing\}$, with $\phi_n[U] \subset U$;

(b) the domain of ϕ_n is $\Psi_n = \{(U_1, U_2, \cdots, U_n) \in (\mathscr{T}^*)^n: U_{j+1} \subset \phi_j(U_1, \cdots, U_j)$ for each $j = 1, 2, \cdots, n-1\}$ and $\phi_n: \Psi_n \to \mathscr{T}^*$ satisfies $\phi_n(U_1, \cdots, U_n) \subset U_n$ whenever $n \geqq 2$;

(c) if $U_1, U_2, \cdots$ is a sequence having $(U_1, \cdots, U_n) \in \Psi_n$ then $\bigcap \{U_n: n \geqq 1\} \neq \varnothing$.

The first major result in this section requires that the space X be *pseudonormal*, i.e., that if H and K are disjoint closed sets, one of which is countable, then there are disjoint open sets U and V with $H \subset U$ and $K \subset V$. The crucial property of pseudonormal spaces required by the theorem is given in the next lemma.

LEMMA 8.3. *If Y is a countable closed discrete subspace of a completely regular, pseudonormal space X, then every real-valued function on Y can be extended to a continuous real-valued function on X.*

THEOREM 8.4. *Let X be pseudonormal and completely regular. Then the following are equivalent:*

(a) *$C_\pi(X)$ is pseudo-complete;*

(b) *$C_\pi(X)$ is weakly α-favorable;*

(c) *player* (II) *has a winning strategy in the game $\Gamma(X)$;*

(d) *every countable subset of X is closed;*

(e) *$C_\pi(X)$ intersects every nonvoid G_δ-subset of $\boldsymbol{R}^X$.*

Proof. The implication (a) $\Rightarrow$ (b) is always true [7]: We prove (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and (d) $\Leftrightarrow$ (e).

(b) $\Rightarrow$ (c). Let $\langle \phi_n \rangle$ be the sequence of functions given by weak α-favorability of $C_\pi(X)$. We define a strategy τ for player (II) in the game $\Gamma(X)$ as follows. Suppose disjoint finite sets S_0 and S_1 are given. Player (II) should let $T_1 = S_0 \cup S_1$, $\varepsilon_1 = 2^{-1}$, and should define $f_1 \in \boldsymbol{R}^X$ by the rule

$$f_1(x) = \begin{cases} 0 & \text{if } x \in X - S_1 , \\ 1 & \text{if } x \in S_1 . \end{cases}$$

Because $C_\pi(X)$ is dense in $\boldsymbol{R}^X$, the set $\mathscr{N}_1 = \tilde{\mathscr{N}}(f_1, T_1, \varepsilon_1) \cap C_\pi(X)$ is a nonvoid open set in $C_\pi(X)$ so that $\phi_1(\mathscr{N}_1)$ is defined. Then there is a basic open set $\mathscr{N}_2 = \mathscr{N}(f_2, T_2, \varepsilon_2)$ having $\mathscr{N}(f_2, T_2, 2\varepsilon_2) \subset \phi_1(\mathscr{N}_1)$. Without loss of generality we may assume $\varepsilon_2 \leqq 2^{-2}$. Player (II) should define $S_2 = \tau(S_0, S_1) = T_2 - T_1$.

For induction hypothesis, suppose $n \geqq 1$ and that pairwise disjoint finite sets $S_0, S_1, \cdots, S_{2n}$, numbers $\varepsilon_1, \cdots, \varepsilon_{2n}$ and functions $f_1, \cdots, f_{2n} \in \boldsymbol{R}^X$ are defined in such a way that

(a) $\varepsilon_i \in (0, 2^{-i}]$ for $1 \leqq i \leqq 2n$;

(b) if $\mathscr{N}_k = C_\pi(X) \cap \tilde{\mathscr{N}}(f_k, T_k, \varepsilon_k)$, where $T_k = S_0 \cup S_1 \cup \cdots \cup S_k$, then $(\mathscr{N}_1, \cdots, \mathscr{N}_{2j+1})$ is in the domain of ϕ_j for $1 \leqq j \leqq n$;

(c) $\mathscr{N}_{2j} \subset C_\pi(X) \cap \tilde{\mathscr{N}}(f_{2j}, T_{2j}, 2\varepsilon_{2j}) \subset \phi_j(\mathscr{N}_1, \cdots, \mathscr{N}_{2j-1})$ for $i \leqq j \leqq n$.

Suppose that player (I) specifies some finite set S_{2n+1} which is disjoint from T_{2n}. Then player (II) should let $T_{2n+1} = T_{2n} \cup S_{2n+1}$, $\varepsilon_{2n+1} = (1/2)\varepsilon_{2n}$, and should define $f_{2n+1} \in \boldsymbol{R}^X$ by the rule that

$$f_{2n+1}(x) = \begin{cases} f_{2n}(x) & \text{if } \ x \in X - S_{2n+1} \, , \\ 2n+1 & \text{if } \ x \in S_{2n+1} \, . \end{cases}$$

Because $C_\pi(X)$ is dense in $\boldsymbol{R}^X$, the set $\mathscr{N}_{2n+1} = \tilde{\mathscr{N}}(f_{2n+1}, T_{2n+1}, \varepsilon_{2n+1}) \cap C_\pi(X)$ is nonempty, open, and has $\mathscr{N}_{2n+1} \subset \mathscr{N}_{2n} \subset \phi_n(\mathscr{N}_1, \mathscr{N}_3, \cdots, \mathscr{N}_{2n-1})$ so that $(\mathscr{N}_1, \cdots, \mathscr{N}_{2n-1}, \mathscr{N}_{2n+1})$ belongs to the domain of ϕ_{n+1}. Then the nonempty open set $\phi_{n+1}(\mathscr{N}_1, \cdots, \mathscr{N}_{2n+1})$ must contain a basic open set $C_\pi(X) \cap \tilde{\mathscr{N}}(f_{2n+2}, T_{2n+2}, 2\varepsilon_{2n+2})$ where we may assume $\varepsilon_{2n+2} \leqq (1/2)\varepsilon_{2n+1}$. Let $\mathscr{N}_{2n+2} = C_\pi(X) \cap \tilde{\mathscr{N}}(f_{2n+2}, T_{2n+2}, \varepsilon_{2n+2})$ and define $S_{2n+2} = \tau(S_0, S_1, \cdots, S_{2n+1}) = T_{2n+2} - T_{2n+1}$. Thus the strategy τ is defined.

To show that τ is a winning strategy for player (II) we note that weak α-favorability of $C_\pi(X)$ yields a $g \in C_\pi(X)$ having $g \in \mathscr{N}_1 \cap \mathscr{N}_3 \cap \mathscr{N}_5 \cap \cdots$. As in Lemma 6.2 the sequence $\langle f_{2n+1} \rangle$ converges pointwise to the limit function g on the set $T = T_1 \cup T_3 \cup T_5 \cup \cdots$ and for $t \in T_{2n+1}$, $g(t) \in [f_{2n+1}(t) - \varepsilon_{2n+1}, f_{2n+1}(t) + \varepsilon_{2n+1}]$. In particular, for $t \in S_{2n+1}$,

$$g(t) \in [(2n+1) - \varepsilon_{2n+1}, (2n+1) + \varepsilon_{2n+1}] \subset [2n, 2n+2]$$

so that the set $g[S_1 \cup S_3 \cup S_5 \cup \cdots]$ has no limit point in $\boldsymbol{R}$. Because $g: X \to R$ is continuous, the set $S_1 \cup S_3 \cup S_5 \cup \cdots$ has no limit point in X, i.e., player (II) wins the game Γ.

(c) $\Rightarrow$ (d). Suppose player (II) has a winning strategy in $\Gamma(X)$. Then player (II) also has a winning strategy in each game $\Gamma(Z)$ for every subspace Z of X. We begin by examining the case where $Z = \{x_n : n \geqq 0\}$ is a countable subspace of X; we claim that player (II) has a winning strategy τ^* for $\Gamma(Z)$ such that if $(S_0, S_1, S_2, \cdots)$ is a play of $\Gamma(Z)$ in which $S_{2n} = \tau^*(S_0, S_1, S_2, \cdots)$ for each n, then

$\cup\{S_{2n+1}: n \geqq 0\}$ is a closed discrete subspace of Z and $\cup\{S_n: n \geqq 0\} = Z$. To define τ^* we begin with τ, a winning strategy for player (II) in $\Gamma(Z)$. Let S_0, S_1 be disjoint finite sets in Z. Compute $S_2' = \tau(S_0, S_1)$ and let $E_2 = \{x_i: 0 \leqq i \leqq 2\} - (S_0 \cup S_1 \cup S_2')$. Define $S_2 = \tau^*(S_0, S_1)$ to be $S_2' \cup E_2$. Suppose (S_0, S_1, S_2, S_3) is a pairwise disjoint sequence of finite subsets of z with $S_2 = \tau^*(S_0, S_1)$. Then $S_2' = \tau(S_0, S_1) \subset S_2$ so that (S_0, S_1, S_2', S_3) belongs to the domain of τ. Compute $S_4' = \tau(S_0, S_1, S_2', S_3)$ and $E_4 = \{x_i | 0 \leqq i \leqq 4\} - (S_0 \cup S_1 \cup S_2 \cup S_3)$. Define $S_4 = \tau^*(S_0, S_1, S_2, S_3) = S_4' \cup E_4$. Continuing in this fashion, we recursively define τ^* in such a way that if $S_0, S_1, \cdots, S_{2n+1}$ is a pairwise disjoint sequence of finite sets in Z having $S_{2j} = \tau^*(S_0, S_1, S_2, \cdots, S_{2j-1})$ whenever $1 \leqq j \leqq n$ then, writing $S_2' = \tau(S_0, S_1)$, $S_4' = \tau(S_0, S_1, S_2, S_3)$, etc., we have $S_{2i}' \subset S_{2i}$ for each $i \leqq n$, and $\{x_i | 0 \leqq i \leqq 2n\} \subset \bigcup_{i=1}^{2n} S_i$. It follows that whenever $(S_0, S_1, S_2, \cdots)$ is a play of $\Gamma(Z)$ in which player (II) has used strategy τ^*, then $(S_0, S_1, S_2', S_3, S_4', \cdots)$ is also a play of $\Gamma(Z)$ in which player (II) has used strategy τ, so that $S_1 \cup S_3 \cup S_5 \cup \cdots$ must be closed and discrete in Z.

Now suppose Y is a countable subspace of X and fix $p \in X - Y$. Let $Z = Y \cup \{p\}$ and let τ^* be a winning strategy for $\Gamma(Z)$ as described above. Then if $(S_0, S_1, S_2, \cdots)$ is any play of the game Γ in which player (II) uses strategy τ^*, the set $\{p\} \cup S_2 \cup S_4 \cup \cdots$ is guaranteed to be a relative neighborhood of p in the space Z. Now consider two plays of the game $\Gamma(Z)$ in which player (I) experiments with two different strategies and player (II) uses strategy τ^*. In the first play of the game, the starting set is $R_0 = \{p\}$. Player (I) specifies the set $R_1 = \tau^*(\phi, R_0)$. Player (II) must respond by choosing $R_2 = \tau^*(R_0, R_1)$. In general, player (I) specifies $R_{2n+1} = \tau^*(\phi, R_0, R_1, \cdots, R_{2n})$ and player (II) is forced to respond with the set $R_{2n+2} = \tau^*(R_0, R_1, \cdots, R_{2n+1})$. Because of the special properties of τ^*, we are guaranteed that the set $\{p\} \cup R_2 \cup R_4 \cup \cdots$ is a neighborhood of p in Z. In the second play of $\Gamma(Z)$, the starting set is $T_0 = \varnothing$. Player (I) chooses to let $T_1 = R_0$. Then Player (II) is compelled to let $T_2 = \tau^*(T_0, T_1)$. For $n \geqq 1$, player (I) defines $T_{2n+1} = \tau^*(T_1, T_2, \cdots, T_{2n})$ and player (II) responds with $T_{2n+2} = \tau^*(T_0, T_1, \cdots, T_{2n+1})$. The result of this play is that the set $\{p\} \cup T_2 \cup T_4 \cup T_6 \cup \cdots$ is known to be a neighborhood of p in Z. But any easy induction shows that $T_k = R_{k-1}$ whenever $k \geqq 1$ so that $\{p\} \cup T_2 \cup T_4 \cup T_5 \cup \cdots = \{p\} \cup R_1 \cup R_3 \cup R_5 \cup \cdots$. Because $(R_2 \cup R_4 \cup R_6 \cup \cdots) \cap (R_1 \cup R_3 \cup R_5 \cup \cdots) = \varnothing$, it cannot be that both $\{p\} \cup R_2 \cup R_4 \cup \cdots$ and $\{p\} \cup R_1 \cup R_3 \cup R_5 \cup \cdots$ are neighborhoods of p in Z unless p is an isolated point of the space Z. Therefore p cannot be a limit point of Y. Since p was an arbitrary point of $X - Y$, Y must be closed. Hence (d) is established.

(d) $\Rightarrow$ (a). Suppose each countable subspace of X is closed. For each n, let Ψ_n be the collection of all basic neighborhoods of the form $\mathscr{N}(f, S, \varepsilon)$ where $f \in C_\pi(X)$, S is finite, and $\varepsilon < 2^{-n}$. Suppose, for each $n \geqq 1$, $\mathscr{N}(f_n, S_n, \varepsilon_n) \in \Psi_n$ with $\mathscr{N}(f_n, S_n, \varepsilon_n) \supset \mathrm{cl}(\mathscr{N}(f_{n+1}, S_{n+1}, \varepsilon_{n+1}))$. Let $T = \bigcup\{S_n: n \geqq 1\}$. Then T is countable, and hence a closed and discrete, subspace of X, and on the set T the sequence $\langle f_n \rangle$ converges pointwise to some function g on T. Because T is a discrete space, $g \in C_\pi(T)$. Because X is pseudonormal and T is countable and discrete, it is possible to find a function $\hat{g} \in C_\pi(X)$ which extends g (cf. 8.3). But then, as in Lemma 6.2, $\hat{g} \in \bigcap\{\mathscr{N}(f_n, S_n, \varepsilon_n): n \geqq 1\}$, as required to establish (a).

(d) $\Rightarrow$ (e). Let $\mathscr{S} = \bigcap\{\mathscr{G}_n: n \geqq 1\}$ be any nonvoid G_δ subset of $\boldsymbol{R}^X$ where each $\mathscr{G}_n$ is open in $\boldsymbol{R}^X$. Fix $f \in \mathscr{S}$ and choose basic neighborhoods $\mathscr{N}(f, S_n, \varepsilon_n)$ of f having $\mathscr{N}(f, S_n, \varepsilon_n) \subset \mathscr{G}_n$. We may assume ε_n is so small that $\mathscr{N}(f, S_n, 2\varepsilon_n) \subset \mathscr{N}(f, S_{n-1}, \varepsilon_{n-1})$ and $\varepsilon_n \in (0, 2^{-n})$. Let $T = \bigcup\{S_n: n \geqq 1\}$. Then T is a countable closed discrete subspace of X. Hence $f|_T$ is a continuous function on T. As above, pseudonormality of X allows us to find a function $\hat{f} \in C_\pi(X)$ which extends $f|_T$. But then $\hat{f} \in C_\pi(X) \cap \mathscr{S}$ as required.

(e) $\Rightarrow$ (d). Let Y be a countable subspace of X and suppose $p \in X - Y$. Let $Z = Y \cup \{p\}$ and define a function $f \in \boldsymbol{R}^X$ by the rule that $f(x) = 1$ if $x \in Y$ and $f(x) = 0$ if $x \in X - Y$. Index Z as $Z = \{x_n: n \geqq 1\}$. Let $S_n = \{x_i: 0 \leqq i \leqq n\}$ and let $\varepsilon_n = 2^{-n}$. Then the set

$$\mathscr{S} = \bigcap\{\mathscr{N}(f, S_n, \varepsilon_n): n \geqq 1\}$$

is a nonempty G_δ-set in $\boldsymbol{R}^X$ so that (e) yields a continuous function $g \in \mathscr{S}$. But then $g(p) = 0$ while $g(x) = 1$ for each $x \in Y$, so that p cannot be a limit point of Y. Therefore (d) is established.

REMARKS 8.5. It is easy to see that, in 8.4, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) for any space; no additional separation hypotheses are needed. Furthermore, the pseudonormality hypothesis in 8.4 can be replaced by any hypothesis which enables (continuous) functions on countable closed discrete subspaces to be extended to continuous functions on all of X. One such hypothesis is that X be strongly collectionwise Hausdorff [3] and completely regular. Examples show that some hypothesis beyond complete regularity is needed to prove 8.4 since there is a completely regular pseudocompact space X in which each countable set is closed [4, 5.1 and 5.3] and for such an X, $C_\pi(X)$ cannot even be a Baire space (cf. 3.3). Finally, assertion (c) of 8.4 is equivalent to the formally stronger statement "every countable subset of X is closed and discrete".

Students of Baire category theory recognize pseudocompleteness as being a very weak completeness property. One very strong completeness property is called Čech-completeness: a space Z is *Čech-complete* if Z is completely regular and is a G_δ-subset of its Stone-Čech compactification βZ. It is known that Z is Čech-complete if and only if Z is a G_δ-subspace of any completely regular space in which Z is densely embedded. Our next result shows that, important as Čech-completeness may be in the general theory of Baire category, there is little point in studying Čech-completeness in $C_\pi(X)$.

THEOREM 8.6. *The following properties of a space X are equivalent*:

(a) $C_\pi(X)$ *is Čech-complete;*
(b) X *is countable and discrete;*
(c) $C_\pi(X)$ *is a completely metrizable space.*

The proof of 8.6 requires a lemma which may be of some interest in its own right.

LEMMA 8.7. *Suppose that $C_\pi(X)$ contains a nonempty G_δ-subset of $\boldsymbol{R}^X$. Then there is a countable closed and open subset T of X such that each point of $Y = X - T$ is isolated, and then $C_\pi(X) \cong C_\pi(T) \times \boldsymbol{R}^Y$.*

Proof of 8.7. Let $\mathscr{S} = \cap\{\mathscr{G}_n : n \geqq 1\}$ be a nonempty G_δ-subset of $\boldsymbol{R}^X$ with $\mathscr{S} \subset C_\pi(X)$ and each $\mathscr{G}_n$ being open in $\boldsymbol{R}^X$. Choose $f \in \mathscr{S}$ and for each n find a basic open set $\tilde{\mathscr{N}}(f, S_n, \varepsilon_n) \subset \mathscr{G}_n$. Let $T = \cup\{S_n : n \geqq 1\}$. Then T is countable and whenever $g \in \boldsymbol{R}^X$ has $g|_T = f|_T$, then $g \in C_\pi(X)$. We claim that X^d, the set of nonisolated points of X, is a subset of T. For if not, choose $p \in X^d - T$ and define a function $g: X \to R$ by $g(x) = f(x)$ if $x \neq p$, $g(p) = f(p) + 1$. Since $g|_T = f|_T$, $g \in C_\pi(X)$. But f and g agree on the dense set $X - \{p\}$ so that continuity forces $f = g$, which is impossible. Hence $X^d \subset T$, so that T is closed. Next we claim that the set T is also open. For otherwise there is a point $q \in T$ which is a limit point of $X - T$. Define a new function $h: X \to R$ by $h(x) = f(x)$ if $x \in T$ and $h(x) = f(x) + 1$ if $x \in X - T$. As above, $h \in C_\pi(X)$. But then, because q is a limit point of $X - T$, $h(q) = f(q) + 1$, contrary to $h(q) = f(q)$. Therefore T is also open. Let $Y = X - T$. Since $X^d \subset T$, each point of Y is isolated and X is the topological sum $X = T \oplus Y$. Therefore (see 2.6) $C_\pi(X) \cong C_\pi(T) \times C_\pi(Y)$. But Y is a discrete space so that $C_\pi(Y) \cong \boldsymbol{R}^Y$, as required.

Proof of 8.6. Obviously (c) implies (a). To see that (b) implies (c), note that if X is countable and discrete then $C_\pi(X) \cong R^\omega$ which is completely metrizable. It remains only to prove that (a) implies (b). Assuming $C_\pi(X)$ is Čech-complete, $C_\pi(X)$ is a G_δ-subset of $\boldsymbol{R}^X$ since $C_\pi(X)$ is dense in $\boldsymbol{R}^X$. According to 8.7 there is a countable set T which is both closed and open, such that if $Y = X - T$ then $C_\pi(X) \cong C_\pi(T) \times \boldsymbol{R}^Y$. Since any Čech-complete space is pseudocomplete [1] Theorom 8.4 forces every countable subset of X to be closed. (No additional separation hypotheses are needed; cf. 8.5.) But then every countable subset of X is closed and discrete. It is known that if card $(Y) \geqq \omega_1$, then $\boldsymbol{R}^Y$ cannot be Čech-complete. But, being a factor of the Čech-complete space $C_\pi(X)$, $\boldsymbol{R}^Y$ is Čech-complete, so Y is also countable. Therefore $X \cong T \oplus Y$ is a countable discrete space.

REMARKS 8.8. (a) The proof of 8.6 yields a related result which may be of some interest, namely:

THEOREM. *The following properties of a space X are equivalent*:
(a) $C_\pi(X)$ *is a* G_δ*-subset of* $\boldsymbol{R}^X$;
(b) X *is a discrete space*;
(c) $C_\pi(X) = \boldsymbol{R}^X$.

Proof. Obviously (b) $\Rightarrow$ (c) $\Rightarrow$ (a). To prove that (a) $\Rightarrow$ (b), use 8.7 to obtain $X \cong T \oplus (X - T)$ where T is a countable closed and open subset of X and $(X - T)$ is discrete. Because $C_\pi(X)$ is a dense G_δ-subset of the weakly α-favorable space $\boldsymbol{R}^X$, $C_\pi(X)$ is weakly α-favorable [7, 2]. (We remark that $\boldsymbol{R}^X$ is also pseudocomplete; however it is not yet known whether a dense G_δ-subset of a pseudocomplete space must be pseudocomplete.) But then the proof of 8.4, (b) $\Rightarrow$ (d) applies to show that every countable subset of X is closed and discrete (even though we are not assuming that X is completely regular and pseudonormal; cf. 8.5) so that X is a discrete space as required.

It might be interesting to know more about the descriptive theory of $C_\pi(X)$ in $\boldsymbol{R}^X$.

9. The function spaces $C_\pi(X, Y)$. The results concerning $C_\pi(X)$ in earlier sections can be generalized to spaces of functions which are not real-valued and in this section $C_\pi(X, Y)$ will be the set of continuous functions from X into the space Y, endowed with the topology of pointwise convergence, i.e., topologized as a subspace of the product Y^X. We will assume that members of $C_\pi(X, Y)$ separate points of X or, equivalently that $C_\pi(X, Y)$ is dense in Y^X. That

would be the case, for example, if X were completely Hausdorff and Y contained an arc.

The following result is an immediate generalization of 4.6.

THEOREM 9.1. *Suppose Y contains an infinite discrete family of open sets. If $C_\pi(X, Y)$ is a Baire space, then player* (I) *cannot have a winning strategy in $\Gamma(X)$.*

The hypothesis in 9.1 that Y contains an infinite discrete collection of open sets is equivalent (for Y completely regular) to the assertion that Y is not pseudocompact. If the domain space X is countable, then 9.1 can be strengthened considerably.

THEOREM 9.2. *Let X be countable and let Y be any infinite regular space. If $C_\pi(X, Y)$ is a Baire space, then player* (I) *cannot have a winning strategy in $\Gamma(X)$.*

Proof. We construct a subspace Z of Y such that $C_\pi(X, Z)$ is Baire and Z contains an infinite discrete collection of open sets. Given such a Z, Theorem 9.1 would apply.

Since Y is an infinite regular space, Y contains an infinite pairwise disjoint collection V of open sets whose union is dense in Y. We let $Z = \cup \mathscr{V}$. The inclusion $j\colon Z \to Y$ induces an embedding $j_*\colon C_\pi(X, Z) \to C_\pi(X, Y)$ and we let $\mathscr{Z}$ denote the image of j_*. Since $C_\pi(X, Y)$ is Baire, $C_\pi(X, Z)$ will be Baire provided we show that $C_\pi(X, Y) - \mathscr{Z}$ is a first category subset of $C_\pi(X, Y)$. For each $x \in X$, let $\mathscr{F}_x = \{f \in C_\pi(X, Y)\colon f(x) \in Y - Z\}$. Each $\mathscr{F}_x$ is a closed, nowhere dense subset of $C_\pi(X, Y)$ and $\cup \{\mathscr{F}_x\colon x \in X\} = C_\pi(X, Y) - \mathscr{Z}$, as required.

The sufficiency proved in 6.3 can be generalized to $C_\pi(X, Y)$ provided Y is a complete metric space.

THEOREM 9.3. *Let $X \in \Sigma$ and let Y be a complete metric space. If player* (I) *has no winning strategy in $\Gamma(X)$, then $C_\pi(X, Y)$ is a Baire space.*

Combining 9.2 and 9.3 we obtain:

THEOREM 9.4. *Let $X \in \Sigma$ and suppose Y is a complete metric space. If either* (i) *Y is not compact or* (ii) *Y is infinite and X is countable, then $C_\pi(X, Y)$ is a Baire space if and only if player* (I) *has no winning strategy in $\Gamma(X)$.*

We also point out that Theorems 8.4, 8.6 and 8.8 can be generalized to $C_\pi(X, Y)$ in the case where Y is a noncompact complete metric space such that every continuous function from a countable closed discrete subspace of X into Y admits a continuous extension over all of X.

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TEXAS TECH UNIVERSITY
LUBBOCK, TX 79409
AND
VPI & SU
BLACKSBURG, VA 24061