BOUNDS FOR THE EIGENFUNCTIONS OF A TWO-PARAMETER SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

MELVIN FAIERMAN

In previous papers the author has shown that, in contrast to the one-parameter case, the normalized eigenfunctions of two simultaneous Sturm-Liouville systems in two parameters are not necessarily uniformly bounded. Moreover, best possible bounds for the normalized eigenfunctions were also derived. However these results were only established under the assumption that the coefficients of our differential equations satisfied certain special conditions. Hence, in order to deal with problems which often arise in physical practice, it is important to extend our results to the case where the coefficients of our differential equations satisfy more general conditions then hitherto supposed. Accordingly, it is the object of this paper to derive best possible bounds for the normalized eigenfunctions of the simultaneous two-parameter systems in question under much weaker restrictions on their coefficients than was previously assumed.

The study of the behavior of the eigenvalues and eigenfunctions of multiparameter Sturm-Liouville systems was initiated by F. V. Atkinson [1, §4] who pointed out that the theory related to this subject was still far from clear. Since the appearance of Atkinson's paper, the author has obtained some relevant results, the most important of which are contained in papers [4] and [5] mentioned above. The results given in this paper therefore constitute a further stage in the development of the theory related to this subject.

Finally we mention that the methods used in this paper are quite different from those used in [4] and [5] wherein techniques from transition point theory and the method of asymptotic integration were employed. Here our results are established by utilizing the information given in [4] and [5], by making a through study of the general character of the solutions of our differential equations, and lastly, by making use of a Sobolev type inequality.

1. **The main theorem.** We shall be concerned here with the simultaneous two-parameter systems

(1)
$$y_1'' + (\lambda A_1(x_1) - \mu B_1(x_1) + q_1(x_1))y_1 = 0 ,$$
$$0 \leq x_1 \leq 1 , \qquad ' = d/dx_1 ,$$

$$(2) \quad \begin{aligned} y_1(0) \cos \alpha_1 - y_1'(0) \sin \alpha_1 &= 0 , & 0 \leq \alpha_1 < \pi , \\ y_1(1) \cos \beta_1 - y_1'(1) \sin \beta_1 &= 0 , & 0 < \beta_1 \leq \pi , \end{aligned}$$

and

$$(3) \quad \begin{aligned} & y_2'' + (-\lambda A_2(x_2) + \mu B_2(x_2) + q_2(x_2))y_2 = 0 , \\ & 0 \leq x_2 \leq 1 , \quad ' = d/dx_2 , \end{aligned}$$

$$(4) \quad \begin{aligned} y_2(0) \cos \alpha_2 - y_2'(0) \sin \alpha_2 &= 0 , & 0 \leq \alpha_2 < \pi , \\ y_2(1) \cos \beta_2 - y_2'(1) \sin \beta_2 &= 0 , & 0 < \beta_2 \leq \pi , \end{aligned}$$

where it will be supposed that, for $i = 1, 2$, q_i is real and continuous, and A_i, B_i are real and analytic in $0 \leq x_i \leq 1$. We shall also suppose that $\Delta = A_1 B_2 - A_2 B_1 \neq 0$ in I^2 (the product of the intervals $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$). Furthermore, there is no loss of generality in assuming henceforth that the A_i, B_i, and Δ are all positive for all values of x_1 and x_2 in I^2, since this can always be achieved, if necessary, by introducing a nonsingular transformation in the parameters λ an μ (see [4, Appendix A]).

Recall from [4, §2] that by an eigenvalue of the system (1–4) we mean a pair of numbers, (λ^*, μ^*), such that for $\lambda = \lambda^*$ and $\mu = \mu^*$, (1) and (3) have nontrivial solutions satisfying (2) and (4), respectively. If $y_1(x_1, \lambda^*, \mu^*)$ and $y_2(x_2, \lambda^*, \mu^*)$ denote these solutions, respectively, then the product, $\prod_{i=1}^{2} y_i(x_i, \lambda^*, \mu^*)$, is called an eigenfunction of the system (1–4) corresponding to (λ^*, μ^*). Important results pertaining to the eigenvalues and eigenfunctions of the system (1–4) were recorded in the reference just cited, and in particular it was noted that the eigenvalues were all real and could be expressed in the form $(\lambda_{j,k}, \mu_{j,k})$, $j, k = 0, 1, 2, \cdots$, where, with ϕ_i $(i = 1, 2)$ denoting the solution of $(2i - 1)$ satisfying $\phi_i(0, \lambda, \mu) = \sin \alpha_i$, $\phi_i'(0, \lambda, \mu) = \cos \alpha_i$, $\phi_1(x_1, \lambda_{j,k}, \mu_{j,k})$ has precisely j zeros in $0 < x_1 < 1$ and $\phi_2(x_2, \lambda_{j,k}, \mu_{j,k})$ has precisely k zeros in $0 < x_2 < 1$. In the sequel we shall let $\psi_{j,k}^*(x_1, x_2) = \prod_{i=1}^{2} \phi_i(x_i, \lambda_{j,k}, \mu_{j,k})$ and $\psi_{j,k} = \psi_{j,k}^*(x_1, x_2)/\|\psi_{j,k}^*\|$, where $\|f\|^2 = \iint_{I^2} \Delta(x_1, x_2)|f(x_1, x_2)|^2 dx_1 dx_2$ for any function f which is square-integrable in I^2. Then putting $\Delta_i = A_i' B_i - A_i B_i'$ $(' = d/dx_i)$ for $i = 1, 2$, we have from [4, Theorems 3.5, 4.3 and Subsection 4.4].

THEOREM 1. *Suppose that $\Delta_i(x_i)$ vanishes identically in $0 \leq x_i \leq 1$ for $i = 1, 2$. Then the absolute values of the $\psi_{j,k}(x_1, x_2)$ for $(x_1, x_2) \in I^2$ remain less than some bound independent of x_1, x_2, j, and k.*

We remark that this result is also true under much weaker

conditions on the A_i and B_i than have been supposed here (see [4, §1]).

Our main concern in this paper will be the proof of the following theorem.

THEOREM 2. *Suppose that for at least one* i, $1 \leq i \leq 2$, $\Delta_i(x_i)$ *is not identically zero in* $0 \leq x_i \leq 1$. *Then the absolute values of the* $(1 + j + k)^{-1/3}\psi_{j,k}(x_1, x_2)$ *for* $(x_1, x_2) \in I^2$ *remain less than some bound independent of* x_1, x_2, j, *and* k.

It is important to observe from [5, §6] that the exponent 1/3 appearing in Theorem 2 is the best possible in the sense that if it is replaced by $1/3 - \varepsilon$, $\varepsilon > 0$, then the theorem may no longer be true.

2. **Preliminaries.** Before we can prove Theorem 2, we shall require some further information. Accordingly, let $P_1(x_1, \lambda, \mu) = \lambda A_1(x_1) - \mu B_1(x_1)$, $P_2(x_2, \lambda, \mu) = -\lambda A_2(x_2) + \mu B_2(x_2)$, and denote by b_1 and b_2 the infimum and supremum, respectively, of $A_1(x_1)/B_1(x_1)$ in $0 \leq x_1 \leq 1$ and by a_1 and a_2 the infimum and supremum, respectively, of $A_2(x_2)/B_2(x_2)$ in $0 \leq x_2 \leq 1$. Let $h_1(t) = \int_0^1 P_1^{1/2}(x_1, 1, t)dx_1$ for $-\infty < t < b_1$, $h_2(t) = \int_0^1 P_2^{1/2}(x_2, 1, t)dx_2$ for $a_2 < t < \infty$, and $g(t) = h_2(t)/h_1(t)$ for $a_2 < t < b_1$ (here and in the sequel it is always assumed that fractional powers of positive quantities have their positive values). Let $\delta = (b_1 - a_2)/100$, $t_1 = a_2 + \delta/2$, $t_2 = b_1 - \delta/2$, and $\theta_i = \tan^{-1} g(t_i)$ for $i = 1, 2$, where the principal branch of the inverse tangent is taken. Note that $0 < \theta_1 < \theta_2 < \pi/2$. Let Ω denote the sector in the (x, y)-plane defined by the inequalities $\theta_1 \leq \theta \leq \theta_2$ and Ω_1 (resp. Ω_2) the sector defined by $0 \leq \theta < \theta_1$ (resp. $\theta_2 < \theta \leq \pi/2$). Then guided by future requirements, we are now going to collect some facts concerning the behavior of the $\lambda_{j,k}$ and $\mu_{j,k}$ for $(j, k) \in \Omega_1$. To this end we first observe from [4, Eq. (4.2)] that there exists the positive numbers M_1 and M_2 such that

$$(5) \qquad\qquad M_1^2 < \lambda_{j,k}/j^2 < M_2^2$$

when $(j, k) \in \Omega_1$ and j is sufficiently large. Furthermore,

LEMMA 1. *If* $\Delta_2(x_2)$ *does not vanish identically in* $0 \leq x_2 \leq 1$, *then* $a_1 < \eta_{j,0} < \eta_{j,1} < \cdots < \eta_{j,k^*} < a_2 + \delta$ *for all* j *sufficiently large, where* $\eta_{j,k} = \mu_{j,k}/\lambda_{j,k}$ *and* $k^* = k^*(j)$ *denotes the greatest integer less than* $j \tan \theta_1$.

Before proving the lemma let us collect some of the definitions

and results given in [4, §2]. Recall that for each real λ, the totality of the values of μ for which (1) has a nontrivial solution satisfying (2) forms a countably infinite set of real numbers which we denoted by $\mu_n(\lambda)$, $n \geqq 0$, where $\mu_0(\lambda) > \mu_1(\lambda) > \cdots$, $\mu_n(\lambda) \to -\infty$ as $n \to \infty$, $\phi_1(x_1, \lambda, \mu_n(\lambda))$ has precisely n zeros in $0 < x_1 < 1$, and, for each n, $\mu_n(\lambda)$ is analytic in $-\infty < \lambda < \infty$ and satisfies here $b_1 \leqq d\mu_n(\lambda)/d\lambda \leqq b_2$. Analogous definitions and results also held for the system (3-4), the analogue of $\mu_n(\lambda)$ being denoted by $\mu_n^*(\lambda)$, and it was remarked that $a_1 \leqq d\mu_n^*(\lambda)/d\lambda \leqq a_2$ for $-\infty < \lambda < \infty$ and $n \geqq 0$. Recall also that when λ ran from $-\infty$ to ∞, the $\mu_n(\lambda)$ (resp. $\mu_n^*(\lambda)$) determined disjoint analytic curves in the (λ, μ)-plane which we denoted by C_n (resp. C_n^*), and it was noted that when j, k were any nonnegative integers, then C_j intersected C_k^* in precisely one point, namely at the eigenvalue of the system (1-4), $(\lambda_{j,k}, \mu_{j,k})$.

Proof of Lemma 1. We shall henceforth assume that j is large enough so as to ensure that $\mu_j(0) < \min\{0, \mu_0^*(0)\}$. Then in light of the preceding discussion it is clear that $\lambda_{j,k+1} > \lambda_{j,k}$ for $k \geqq 0$. We now assert that if $a_1 \leqq t \leqq a_2 + \delta$, then the equation $\mu_j(\lambda) = \lambda t$ (λ real) has precisely one solution, and if we denote this solution by $\lambda(t)$ and put $\lambda^* = \lambda(a_1)$, $\lambda^\dagger = \lambda(a_2 + \delta)$, then $0 < \lambda^* \leqq \lambda(t) \leqq \lambda^\dagger$ and $\lambda(t)$ is a strictly increasing, continuous function of t. This assertion is easily proved by appealing to the implicit function theorem and to the fact that if $F(\lambda, t) = \mu_j(\lambda) - \lambda t$, then $F(0, t) < 0$ and $F(\lambda_2, t) - F(\lambda_1, t) \geqq (b_1 - t)(\lambda_2 - \lambda_1) > 0$ for $\lambda_2 > \lambda_1$. Hence it follows that $\mu_j(\lambda)/\lambda$ strictly increases as λ runs from λ^* to $\lambda^\dagger$, assuming the value a_1 at $\lambda = \lambda^*$ and $a_2 + \delta$ at $\lambda = \lambda^\dagger$.

To complete the proof of the lemma it remains only to show that $\lambda^* < \lambda_{j,0}$ and $\lambda_{j,k^*} < \lambda^\dagger$. To this end we may appeal to the results given in [6, Theorem 1] and to arguments similar to those used in [2, pp. 212-213] to establish that $a_1 < \mu_0^*(\lambda)/\lambda < a_2$ for all large λ. Hence if j is sufficiently large, then it follows from (5) and the discussion immediately preceding this proof that $a_1 < \mu_j(\lambda_{j,0})/\lambda_{j,0} < a_2$, and the first inequality follows. Turning to the second inequality, we note from [4, Theorem 3.4] that $\eta_{j,k}^\dagger = t_{j,k}^\dagger + O(j^{-2})$ as $j \to \infty$, where $k^\dagger = k^\dagger(j)$ denotes the smallest integer greater than or equal to $j \tan \theta_1$, $t_{j,k}^\dagger$ denotes the point of (a_2, b_1) at which $g(t) = (k^\dagger + \nu_2)/(j + \nu_1)$, and, for $i = 1, 2$, ν_i denotes a constant satisfying $0 \leqq \nu_i \leqq 1$. Since $(1 - j^{-1}) \tan \theta_1 < g(t_{j,k}^\dagger) < \tan \theta_1 + 2j^{-1}$ for $j > 1$, it therefore follows from the definition of g (see also [4, Subsection 3.2]) that $t_{j,k}^\dagger = t_1 + O(j^{-1})$, and hence $\eta_{j,k}^\dagger = t_1 + O(j^{-1})$ as $j \to \infty$. Thus if j is sufficiently large, then $a_2 < \mu_j(\lambda_{j,k}^\dagger)/\lambda_{j,k}^\dagger < a_2 + \delta$, and the second inequality follows.

For later use we shall also need the following result (see [7,

Theorem 3.1]).   Here we denote by D the class of continuously differentiable (complex-valued) functions on $0 \leqq s \leqq 1$.

LEMMA 2. *Given any $\varepsilon > 0$, there is a $\gamma(\varepsilon) > 0$ such that*

$$\int_0^1 |f'(s)|^2 ds + \gamma(\varepsilon) \int_0^1 |f(s)|^2 ds \geqq \varepsilon |f(0)|^2$$

for every $f(s) \in D$, where $' = d/ds$.   This result also remains valid if in the right-hand side of the above equation we replace $f(0)$ by $f(1)$.

3. **Proof of Theorem 2.** We shall firstly introduce certain assumptions which will enable us to reduce our work somewhat.

Assumption 1.   We shall henceforth suppose that $B_i(x_i)$ is constant in $0 \leqq x_i \leqq 1$ for $i = 1, 2$ and prove the theorem for this case only.

It is clear from [3, Eq. (20a), p. 292] that the proof of the theorem for the general case follows from this result.

Now we already know from [4, Theorem 3.5] that the absolute values of the $\psi_{j,k}(x_1, x_2)$ for $(x_1, x_2) \in I^2$ and $(j, k) \in \Omega$ remain less than some bound independent of x_1, x_2, j, and k, and hence it remains only to prove the assertion of Theorem 2 for (j, k) in each of the sectors Ω_1 and Ω_2.   However, since the proof for $(j, k) \in \Omega_2$ is similar to that for $(j, k) \in \Omega_1$, we shall only prove the assertion for this latter case.   A further simplification of our work is again possible; for we know from [4, Theorem 4.3] and [5, §6] that when $\Delta_2(x_2) \equiv 0$ (resp. $\Delta_2(x_2) \neq 0$) in $0 \leqq x_2 \leqq 1$, then the absolute values of the $\psi_{j,k}(x_1, x_2)$ (resp. $j^{-1/3}\psi_{j,k}(x_1, x_2)$) for $(x_1, x_2) \in I^2$ and $(j, k) \in \Omega_1$ remain less than some bound independent of x_1, x_2, j, and k.   Thus it is clear that it remains only to verify the assertion of the theorem under the following hypotheses.

Assumption 2.   We shall assume from now on that $a_1 < a_2$, that $\Delta_2(x_2)$ vanishes in $0 \leqq x_2 \leqq 1$, and that $(j, k) \in \Omega_1$.

Let $u_{j,k}(x_1) = \phi_1(x_1, \lambda_{j,k}, \mu_{j,k})/J_1(j, k)$ and $v_{j,k}(x_2) = \phi_2(x_2, \lambda_{j,k}, \mu_{j,k})/J_2(j, k)$, where $J_i(j, k) = \left(\int_0^1 \phi_i^2(x_i, \lambda_{j,k}, \mu_{j,k}) dx_i \right)^{1/2}$ for $i = 1, 2$.   Then in light of (5) and Lemma 1, we may argue precisely as we did in [4, Theorem 4.3] to verify that the absolute values of the $u_{j,k}(x_1)$ for $0 \leqq x_1 \leqq 1$ remain less than some bound independent of x_1, j, and k.   Thus in order to complete the proof of Theorem 2 it remains only to prove the following assertion.

PROPOSITION 1.   *It is the case that the absolute values of the*

$j^{-1/3}v_{j,k}(x_2)$ *for* $0 \leqq x_2 \leqq 1$ *remain less than some bound independent of* x_2, j, *and* k.

Before passing to the proof of Proposition 1, let us simplify our notation by henceforth dropping subscripts and writing x for x_2, A for A_2, B for B_2, P for P_2, and q for q_2. Putting $r(x) = A(x)/B(x)$, let $0 \leqq x_1^* < x_2^* < \cdots < x_p^* \leqq 1$, $p \geqq 1$, denote the critical points of $r(x)$ in $0 \leqq x \leqq 1$, $m_i(\geqq 2)$ the order of the zero of $r(x) - r(x_i^*)$ at x_i^*, and let $R_i = r^{(m_i)}(x_i^*)$, where $^{(n)} = d^n/dx^n$. Choose the positive constant $d^\sharp$ small enough so that $d^\sharp < \min \{(x_{i+1}^* - x_i^*)/100\}$, $1 \leqq i \leqq p - 1$, if $p > 1$, $d^\sharp < x_1^*/100$ if $x_1^* > 0$, and $d^\sharp < (1 - x_p^*)/100$ if $x_p^* < 1$. Put $x_0^* = 0$, $x_{p+1}^* = 1$, and let c^* be a positive number chosen small enough so that if $0 \leqq m < n \leqq p + 1$ and $r(x_m^*) \neq r(x_n^*)$, then $|r(x_m^*) - r(x_n^*)| \geqq c^*$. Let c be a positive number not exceeding $c^*/100$ choosen small enough so that $c \leqq |r(x_i^* \pm d^\sharp) - r(x_i^*)|$ for $i = 0, \cdots, (p + 1)$, where, of course, only the $+$ or $-$ sign is taken if x_i^* is an end point of the interval $[0, 1]$. If $1 \leqq i \leqq p$ and $0 < x_i^* < 1$, then let E_i and E_i^* denote the intervals $x_i^* - d_i^- < x < x_i^* + d_i^+$ and $x_i^* - d_i^-/2 < x < x_i^* + d_i^+/2$, respectively, where the $x_i^* \pm d_i^\sharp$ denote the points of $[x_i^* - d^\sharp, x_i^* + d^\sharp]$ satisfying $|r(x_i^* \pm d_i^\sharp) - r(x_i^*)| = c$. If $i = 1$ (resp. $i = p$) and $x_1^* = 0$ (resp. $x_p^* = 1$), then let E_1, E_1^* (resp. E_p, E_p^*) denote the intervals $0 \leqq x < d_1^+$, $0 \leqq x < d_1^+/2$ (resp. $1 - d_p^- < x \leqq 1$, $1 - d_p^-/2 < x \leqq 1$), respectively, where d_1^+ (resp. $1 - d_p^-$) denotes the point of $[0, d^\sharp]$ (resp. $[1-d^\sharp, 1]$) satisfying $|r(d_1^+) - r(0)| = c$ (resp. $|r(1 - d_p^-) - r(1)| = c$). If $x_1^* > 0$ (resp. $x_p^* < 1$), then let E_0 (resp. E_{p+1}) denote the interval $0 \leqq x < d_0^+$ (resp. $1 - d_{p+1}^- < x \leqq 1$), where d_0^+ (resp. $1 - d_{p+1}^-$) denotes the point of $[0, d^\sharp]$ (resp. $[1 - d^\sharp, 1]$) satisfying $|r(d_0^+) - r(0)| = c$ (resp. $|r(1 - d_{p+1}^-) - r(1)| = c$). If $x_1^* = 0$ (resp. $x_p^* = 1$), then put $d_0^+ = d_1^+$ and $E_0 = E_1$ (resp. $d_{p+1}^- = d_p^-$ and $E_{p+1} = E_p$). Finally, let N denote the infimum of $|r^{(1)}(x)|$ in $[0, 1] - \bigcup_{i=1}^{p} E_i^*$, $2d$ the minimum of the $d_i^\sharp$, $0 \leqq i \leqq p + 1$, Q the supremum of $|q(x)|$ in $[0, 1]$, and put $b = B(x)$, $\varepsilon_0 = |\cot \alpha_2|$ if $\alpha_2 > \pi/2$ (see (4)), $\varepsilon_0 = 0$ if $\alpha_2 \leqq \pi/2$, $\varepsilon_1 = \cot \beta_2$ if $\beta_2 < \pi/2$, and $\varepsilon_1 = 0$ if $\beta_2 \geqq \pi/2$.

ASSUMPTION 3. We shall assume from now on that j exceeds

$$10^3[(1 + M^*)(\varepsilon_0 + \varepsilon_1 + Q^2 + Q^{1/2} + \delta^{-1} + c^{-1} + (N + 1)/Nd)]^{3/2}$$

and is large enough to satisfy the assertions of both (5) and Lemma 1, where $M^* = (bM_1^2)^{-1/2}$ and δ is defined at the beginning of §2.

Proof of Proposition 1. Let us fix (j, k) and for simplicity of notation write λ, η, $v(x)$, and $P(x)$ for $b\lambda_{j,k}$, $\eta_{j,k}$, $v_{j,k}(x)$, and $P(x, \lambda_{j,k}, \mu_{j,k})$, respectively. We are now going to prove that

(6) $$U = \int_0^1 (v'(x))^2 dx \leqq K^* \lambda \quad \text{and} \quad v^2(x) \leqq K^* \lambda$$

for $0 \leqq x \leqq 1$, where $' = d/dx$ and K^* denotes a positive constant independent of x, j, and k. To this end we first argue with (3) and (4) to show that

$$U \leqq \int_0^1 |P + q| v^2 dx + \varepsilon_0 v^2(0) + \varepsilon_1 v^2(1) ,$$

where the ε_i are defined in the preceding paragraph. If $\varepsilon_0 > 0$, then it follows from Lemma 2 (with $\varepsilon = 4\varepsilon_0$) that $\varepsilon_0 v^2(0) \leqq (U + \gamma(4\varepsilon_0))/4$, with a similar result holding for $\varepsilon_1 v^2(1)$ if $\varepsilon_1 > 0$. In light of (5) and Lemma 1, the assertion concerning U follows immediately. The second inequality in (6) also follows easily if we observe from Lemma 2 that $v^2(0) \leqq U + \gamma(1)$. Next we may argue with the above bound for U as in [3, pp. 334-335] to show that $|(v'(x))^2 + P(x)v^2(x)| \leqq K^2 \lambda$ for $0 \leqq x \leqq 1$, where K denotes a positive constant independent of x, j, and k. From these results we draw the following

CONCLUSIONS.
1. If $\eta - r(x) \geqq j^{-2/3}$, then $|v(x)| \leqq K j^{1/3}$.
2. If $\eta - r(x) \geqq -2Q/\lambda$ (i.e., if $P(x) \geqq -2Q$), then $|v'(x)| \leqq K^\sharp j$, where $K^\sharp (\geqq 1)$ denotes a constant independent of x, j, and k.
3. If $\alpha_2 > \pi/2$ and $r(0) - \eta \geqq j^{-2/3}$, then $0 < v(0) \leqq K^\sharp j^{1/3}$.
4. If $\beta_2 < \pi/2$ and $r(1) - \eta \geqq j^{-2/3}$, then $|v(1)| \leqq K^\sharp j^{1/3}$.
5. If $0 \leqq x_1 < x_2 \leqq 1$, if $J = \{x | x_1 \leqq x \leqq x_2\}$, and if $P(x) + q(x) \geqq 0$ for $x \in J$, then $|v(x)| < |v(x_2)| + 3K^\sharp j^{1/3}$ for $x \in J$.
6. If $0 \leqq x_1 < x_2 \leqq 1$, if $J = \{x | x_1 \leqq x \leqq x_2\}$, and if $P(x) \geqq -2Q$ for $x \in J$, then there is an $\xi \in J$ such that $|v'(\xi)| < 3K^\sharp j^{1/3}(x_2 - x_1)^{-1}$.
7. If $0 \leqq x_1 \leqq 1 - j^{-2/3}$, if $J = \{x | x_1 \leqq x \leqq x_1 + j^{-2/3}\}$, if $P(x_1) \geqq -2Q$, and if $P(x) + q(x) \leqq 0$ for $x \in J$, then $|v(x_1)| < 3K^\sharp j^{1/3}$.

Conclusions 1-4 follow immediately from the foregoing discussion and (5). Conclusions 5-7 are not obvious. However in order not to impair the continuity of the proof, we shall not elaborate upon them until the final stage of the proof.

We are now going to utilize the above information to investigate the behaviour of $v(x)$ and $P(x) + q(x)$ in various subintervals of $[0, 1]$. The proof of Proposition 1 will then follow from these results.

Case 1. Suppose that $x_0 \in [0, 1] - \bigcup_{n=0}^{p+1} E_n$ and $\eta = r(x_0)$. We note that for this case $x_i^* < x_0 < x_{i+1}^*$ for some i, $0 \leqq i \leqq p$. Now let x_1 and x_2 denote the points of the interval $(x_0 - d, x_0 + d)$ at

which $r(x) = \eta + Q/\lambda$ and $r(x) = \eta - j^{-2/3}$, respectively, and assume firstly that $r'(x_0) < 0$. Then $(x_2 - x_1) < 2N^{-1}j^{-2/3}$ and from conclusions 1 and 2 above we also have $|v(x_2)| \leqq Kj^{1/3}$ and $|v'(x)| \leqq K^{\sharp}j$ in $x_1 \leqq x \leqq x_2$. Hence $|v(x)| < K_1 j^{1/3}$ for $x_1 \leqq x \leqq x_{i+1}^*$ and $P(x) + q(x) < 0$ for $x_i^* \leqq x < x_1$, where $K_1 = K + 2N^{-1}K^{\sharp}$. Similarly, if $r'(x_0) > 0$, then $|v(x)| < K_1 j^{1/3}$ for $x_i^* \leqq x \leqq x_1$ and $P(x) + q(x) < 0$ for $x_1 < x \leqq x_{i+1}^*$.

Case 2. Suppose that for some i, $1 \leqq i \leqq p$, $0 < x_i^* < 1$, m_i is even, and $R_i > 0$ (see the paragraph following the statement of Proposition 1 for terminology). For simplicity of notation let us write x^* for x_i^* and assume firstly that $2Q/\lambda \leqq \eta - r(x^*) < c$. Let x_1 and x_2 denote the points of the interval $(x^*, x^* + d_i^+ + d)$ at which $r(x) = \eta - Q/\lambda$ and $r(x) = \eta + Q/\lambda$, respectively. Then $P(x) + q(x)$ is positive in $[x^*, x_1)$, negative in $(x_2, x_{i+1}^*]$, and $-Q \leqq P(x) \leqq Q$ for $x_1 \leqq x \leqq x_2$. Hence it follows from Conclusions 5–7 that $|v(x)| < |v(x_1)| + K_2 j^{1/3}$ for $x^* \leqq x \leqq x_1$, $|v(x_2)| < K_2 j^{1/3}$, and there is an ξ, $x_1 \leqq \xi \leqq x_2$, such that

$$(7) \qquad\qquad |v'(\xi)| < K_2 j^{1/3}/(x_2 - x_1) \,,$$

where $K_2 = K + 3K^{\sharp}$. Since we may argue with (3) as in [8, p. 167] to show that

$$(8) \qquad |v(x) - v(\xi) - v'(\xi)(x - \xi)| \leqq 4Q(x_2 - x_1)^{3/2}/3 \,, \qquad x_1 \leqq x \leqq x_2 \,,$$

we therefore conclude that $|v(x)| < 6K_2 j^{1/3}$ for $x^* \leqq x \leqq x_2$ and $P(x) + q(x) < 0$ for $x_2 < x \leqq x_{i+1}^*$. Similarly we can show that $|v(x)| < 6K_2 j^{1/3}$ for $x_2^{\dagger} \leqq x \leqq x^*$ and $P(x) + q(x) < 0$ for $x_{i-1}^* \leqq x < x_2^{\dagger}$, where $x_2^{\dagger}$ denotes the point of the interval $(x^* - d_i^- - d, x^*)$ at which $r(x) = \eta + Q/\lambda$.

Assume next that $-Q/\lambda < \eta - r(x^*) < 2Q/\lambda$ and denote by x_1 the point of the interval $(x^*, x^* + d_i^+)$ at which $r(x) = \eta + Q/\lambda$. Then $-Q \leqq P(x) < 2Q$ for $x^* \leqq x \leqq x_1$ and $P(x) + q(x) < 0$ for $x_1 < x \leqq x_{i+1}^*$. From Conclusions 6 and 7 it follows that $|v(x_1)| < K_2 j^{1/3}$ and that there is an ξ, $x^* \leqq \xi \leqq x_1$, such that

$$(9) \qquad\qquad |v'(\xi)| < K_2 j^{1/3}/(x_1 - x^*) \,.$$

Since

$$(10) \qquad |v(x) - v(\xi) - v'(\xi)(x - \xi)| \leqq 2Q(x_1 - x^*)^{3/2} \,, \qquad x^* \leqq x \leqq x_1 \,,$$

we therefore conclude that $|v(x)| < 5K_2 j^{1/3}$ for $x^* \leqq x \leqq x_1$. Similarly we can show that $|v(x)| < 5K_2 j^{1/3}$ for $x_1^{\dagger} \leqq x \leqq x^*$ and $P(x) + q(x) < 0$ for $x_{i-1}^* \leqq x < x_1^{\dagger}$, where $x_1^{\dagger}$ denotes the point of the interval $(x^* - d_i^-, x^*)$ at which $r(x) = \eta + Q/\lambda$.

Finally, if $\eta \leqq r(x^*) - Q/\lambda$, then $P(x) + q(x) \leqq 0$ for $x_{i-1}^* \leqq x \leqq x_{i+1}^*$.

Case 3. Suppose that for some i, $1 \leq i \leq p$, $0 < x_i^* < 1$, m_i is even, and $R_i < 0$. Writing x^* for x_i^*, let us assume firstly that $-c < \eta - r(x^*) \leq -2Q/\lambda$. Let x_1, x_2, and x_3 denote the points of the interval $(x^*, x^* + d_i^+ + d)$ at which $r(x) = \eta + Q/\lambda$, $r(x) = \eta - Q/\lambda$, and $r(x) = \eta - j^{-2/3}$, respectively. Then $-Q \leq P(x) \leq Q$ for $x_1 \leq x \leq x_2$ and it is easy to see from Conclusions 1, 5, and 6 that $|v(x)| < K_2 j^{1/3}$ for $x_2 \leq x \leq x_3$ and that (7) is valid for some ξ in $[x_1, x_2]$. In light of (8) we conclude that $|v(x)| < 5K_2 j^{1/3}$ for $x_1 \leq x \leq x_{i+1}^*$ and $P(x) + q(x) < 0$ for $x^* \leq x < x_1$. Similarly we can show that $|v(x)| < 5K_2 j^{1/3}$ for $x_{i-1}^* \leq x \leq x_1^\dagger$ and $P(x) + q(x) < 0$ for $x_1^\dagger < x \leq x^*$, where $x_1^\dagger$ denotes the point of the interval $(x^* - d_i^-, x^*)$ at which $r(x) = \eta + Q/\lambda$.

Assume next $-2Q/\lambda < \eta - r(x^*) < Q/\lambda$ and denote by x_1 and x_2 the points of the interval $(x^*, x^* + d_i^+)$ at which $r(x) = \eta - Q/\lambda$ and $r(x) = \eta - j^{-2/3}$, respectively. Then $-2Q < P(x) \leq Q$ for $x^* \leq x \leq x_1$ and it is easy to see from Conclusions 1, 5, and 6 that $|v(x)| < K_2 j^{1/3}$ in $x_1 \leq x \leq x_2$ and that (9) is valid for some ξ in $[x^*, x_1]$. Hence we conclude from (10) that $|v(x)| < 5K_2 j^{1/3}$ for $x^* \leq x \leq x_{i+1}^*$. Similarly we can show that $|v(x)| < 5K_2 j^{1/3}$ for $x_{i-1}^* \leq x \leq x^*$.

Finally if $\eta \geq r(x^*) + Q/\lambda$, then we may argue with Conclusions 1 and 5 above to show that $|v(x)| < K_2 j^{1/3}$ for $x_{i-1}^* \leq x \leq x_{i+1}^*$.

Case 4. Suppose that $x_1^* = 0$ and $R_1 > 0$. Then it follows from Case 2 above that if $-Q/\lambda < \eta - r(0) < c$, then $|v(x)| < 6K_2 j^{1/3}$ for $0 \leq x \leq x_1$ and $P(x) + q(x) < 0$ for $x_1 < x \leq x_2^*$, where x_1 denotes the point of the interval $(0, d_1^+ + d)$ at which $r(x) = \eta + Q/\lambda$. If $\eta \leq r(0) - Q/\lambda$, then $P(x) + q(x) \leq 0$ for $0 \leq x \leq x_2^*$ and we see from Conclusion 3 above that the only further problem which requires investigation here occurs when $-j^{-2/3} < \eta - r(0) \leq -Q/\lambda$ and $\alpha_2 > \pi/2$. Accordingly, fixing our attention upon this situation and observing that $v(0) > 0$, let us firstly suppose that $v(x) > 0$ in $0 \leq x \leq j^{-2/3}$. Then it follows from (3), (4), and Assumption 3 that in this interval $v'(x) \geq v(0) \cot \alpha_2$, $v(x) > v(0)/2$, and hence we conclude from the equation

$$(11) \qquad\qquad \int_0^1 v^2(x)dx = 1$$

that $v(0) < 2j^{1/3}$. Similar arguments also show that $v(x)$ cannot vanish in $[0, j^{-2/3}]$.

Case 5. Suppose that $x_1^* = 0$ and $R_1 < 0$. Then it follows from Case 3 above that $|v(x)| < 5K_2 j^{1/3}$ for $0 \leq x \leq x_2^*$ if $\eta > r(0) - 2Q/\lambda$. If $-c < \eta - r(0) \leq -2Q/\lambda$, then $|v(x)| < 5K_2 j^{1/3}$ for $x_1 \leq x \leq x_2^*$, $P(x) + q(x) < 0$ for $0 \leq x < x_1$ (where x_1 denotes the point of the interval

$(0, d_1^+)$ at which $r(x) = \eta + Q/\lambda$), and the only further problem which requires investigation here occurs when $-j^{-2/3} < \eta - r(0) \leqq -2Q/\lambda$ and $\alpha_2 > \pi/2$ (see Conclusion 3 above). However, for this situation we may argue as in Case 4 above to show that $v(0) < 2j^{1/3}$ if $x_1 \geqq j^{-2/3}$ and $v(0) < 2v(x_1) < 10K_2 j^{1/3}$ otherwise (recall that $v(0) > 0$).

Case 6. Suppose that $x_1^* > 0$. Then the results of Case 4 (resp. Case 5) above, with d_1^+ and x_2^* replaced by d_0^+ and x_1^*, respectively, remain valid for this case if $r'(0) > 0$ (resp. $r'(0) < 0$).

Finally, results analogous to those above hold for $v(x)$ and $P(x) + q(x)$ in the intervals $[x_{p-1}^*, 1]$ if $x_p^* = 1$, $[x_p^*, 1]$ if $x_p^* < 1$, and $[x_{i-1}^*, x_{i+1}^*]$ if $1 \leqq i \leqq p$, $0 < x_i^* < 1$, and m_i is odd. The proof of Proposition 1 is then completed by appealing to Lemma 1 and to the fact that if $0 \leqq x_1 < x_2 \leqq 1$ and $P(x) + q(x) \leqq 0$ in $[x_1, x_2]$, then in this interval $v(x)v'(x)$ is monotonic increasing and $|v(x)| \leqq \max \{|v(x_1)|, |v(x_2)|\}$.

We now fulfill a commitment made earlier to explain in greater detail how Conclusions 5–7 were arrived at. Accordingly, let us firstly fix our attention upon Conclusion 5, assume that $|v(x)|$ attains its absolute maximum in J at the point x_0, and put $v_0 = v(x_0)$, $v_2 = v(x_2)$. It is clear that we need only consider the case $|v_0| > |v_2|$. Suppose firstly that $v(x) > 0$ for $x \in J$. Then

$$v(x) \geqq v_2 + (v_0 - v_2)(x_2 - x)/(x_2 - x_0)$$

for $x_0 \leqq x \leqq x_2$, and hence it follows from (11) that $(v_0 - v_2)^2(x_2 - x_0) < 3$. On the other hand we see from Conclusion 2 that $v_0 - v_2 \leqq K^\# j(x_2 - x_0)$, and so Conclusion 5 follows for this case. If $v(x)$ vanishes in J or if $v(x) < 0$ for $x \in J$, then Conclusion 5 is arrived at by using similar arguments.

Turning next to Conclusion 6, assume that $|v'(x)| \geqq \gamma = 3K^\# j^{1/3}/(x_2 - x_1)$ for $x \in J$. Then a simple integration shows that for $x \in J$, $\gamma^{-1}|v(x)|$ exceeds $(x - x_1)$ if $v(x)v'(x) > 0$ for all x in J, exceeds $(x_2 - x)$ if $v(x)v'(x) < 0$ for all x in J, and is not less than $|x - x_0|$ if $x_0 \in J$ and $v(x_0) = 0$. Hence it follows from (11) that $(3K^\# j^{1/3})^2(x_2 - x_1) < 24$, and since $\gamma \leqq K^\# j$ (see Conclusion 2), we arrive at the contradiction that $27(K^\#)^2 < 24$.

Fixing our attention lastly upon Conclusion 7, assume that $v_1 = v(x_1) \geqq 3K^\# j^{1/3}$. If $v'(x_1) \geqq 0$, then it follows from (3) that $v(x) \geqq v_1$ for $x \in J$, and hence we conclude from (11) that $(3K^\#)^2 < 1$, which is a contradiction. Suppose next that $v'(x_1) < 0$ and $v(x) > 0$ for $x \in J$. Then $v'(x) \geqq -K^\# j$ (see Conclusion 2) and $v(x) \geqq 2K^\# j^{1/3}$ for $x \in J$, and hence it follows from (11) that $(2K^\#)^2 < 1$, which again is a

contradiction. If $v'(x_1) < 0$, if $x_1 < x_0 \leqq x_1 + j^{-2/3}$, if $v(x) > 0$ for $x_1 \leqq x < x_0$, and if $v(x_0) = 0$, then $v'(x) \geqq -K^* j$ for $x_1 \leqq x \leqq x_0$, and hence we arrive at the contradiction that $v_1 \leqq K^* j^{1/3}$. Similarly we can show that the supposition $v_1 \leqq -3K^* j^{1/3}$ is untenable.

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UNIVERSITY OF THE WITWATERSRAND,
JOHANNESBURG, SOUTH AFRICA