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PLANAR CONTINUA WITH RESTRICTED LIMIT DIRECTIONS

C. L. BELNA, M. J. EVANS, AND P. D. HUMKE

An affirmative answer is given to a question of D. M. Campbell and J. Lamoreaux concerning minimal conditions on the set of limit directions of a planar continuum that guarantee it is a line segment.

Throughout we let E denote a planar continuum. The set E is said to have a *limit direction* $e^{i\alpha}$ at the point z in E if there is a sequence of points z_n in $E - \{z\}$ with $z_n \to z$ and $(z_n - z)/|z_n - z| \to e^{i\alpha}$; this limit direction is called a *right limit direction* if we also have $\mathrm{Re}\,(z_n) \geqq \mathrm{Re}\,(z)$ for each z_n. The set of all [right] limit directions of E at z is denoted by $\mathscr{D}(z)[\mathscr{D}_R(z)]$ and is called the contingent of E at z in the older terminology of Saks [2].

D. M. Campbell and J. Lamoreaux [1] proved: *Let K be a subset of E such that $\mathscr{D}(z) \cap \{e^{i\theta}: 0 < |\theta| \leqq \pi/2\} = \emptyset$ for each z in $E - K$. If the projection of K on the y-axis has measure zero, then E is a horizontal line segment.* Then they asked whether this theorem remains true when the condition on $\mathscr{D}(z)$ is replaced by the conition $\mathscr{D}_R(z) \subseteqq \{1\}$. We now show this to be the case.

THEOREM. *Let K be a subset of E such that $\mathscr{D}_R(z) \subseteqq \{1\}$ for each z in $E - K$. If the projection of K on the y-axis has measure zero, then E is a horizontal line segment.*

Proof. To prove this theorem we show that the projection of E on the y-axis is of measure zero.

One observes $\mathscr{D}_R(z) \subseteqq \{1\}$ implies $\mathscr{D}(z) \cap \{e^{i\theta}: 0 < |\theta| < \pi/2\} = \emptyset$ and therefore for every point of $E - K$ the set $\mathscr{D}(z)$ cannot be the entire circle $\{e^{i\theta}: 0 \leqq \theta \leqq 2\pi\}$. By the first fundamental theorem on contingents of plane sets ([2], p. 266), at every point of $E - K$, except those of a set L of linear measure zero, the set $\mathscr{D}(z)$ is either a doubleton $\{e^{i\alpha}, -e^{i\alpha}\}$ or a semicircle $\{e^{i\theta}: \alpha \leqq \theta \leqq \alpha + \pi\}$. Since $\mathscr{D}_R(z) \subseteqq \{1\}$ on $E - K$, it follows that for each z in $E - (K \cup L)$, the set $\mathscr{D}(z)$ is either the doubleton $\{i, -i\}$, the doubleton $\{1, -1\}$, or the arc $\{e^{i\theta}: \pi/2 \leqq \theta \leqq 3\pi/2\}$.

The second fundamental theorem on contingents of plane sets ([2], p. 267) asserts that $M \equiv \{z \in E - (K \cup L): \mathscr{D}(z) = \{1, -1\}\}$ has a projection on the y-axis of measure zero. Thus, to complete the proof we now show that the set $N \equiv E - (K \cup L \cup M)$ is countable.

For each $z \in N$, $\mathscr{D}_R(z) = \varnothing$ and hence there is a rational number $r(z)$ and a corresponding closed half-disk

$$D(z, r(z)) \equiv \{\zeta: -\pi/2 \leqq \arg(\zeta - z) \leqq \pi/2 \text{ and } |\zeta - z| \leqq r(z)\}$$

such that $D(z, r(z)) \cap E = \{z\}$. Also, for each rational number r the set $N_r \equiv \{z \in N: r(z) = r\}$ is an isolated set, and the countability of N is established.

In closing we note that in view of its proof, the theorem above remains true when the hypothesis $\mathscr{D}_R(z) \subseteqq \{1\}$ is replaced by any condition which guarantees that if $z \in E - K$, then either (i) $\mathscr{D}_R(z) = \varnothing$ or (ii) $1 \in \mathscr{D}(z)$ and $\mathscr{D}(z)$ is a subset of either $\{e^{i\theta}: 0 \leqq \theta \leqq \pi\}$ or $\{e^{i\theta}: \pi \leqq \theta \leqq 2\pi\}$.

REFERENCES

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PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PA 16802
WESTERN ILLINOIS UNIVERSITY
MACOMB, IL 61455
AND
WESTERN ILLINOIS UNIVERSITY
MACOMB, IL 61455
Current address (C. L. Belna)
SYRACUSE UNIVERSITY
SYRACUSE, NY 13210

Current address: (P. D. Humke)
St. Olaf College
Northfield, MN 55057