PACIFIC JOURNAL OF MATHEMATICS
Vol. 91, No. 2, 1980

GROUPS OF SQUARE-FREE ORDER ARE SCARCE

MICHAEL E. MAYS

We devise an upper bound for $B(n)$, the number of nonisomorphic groups whose orders are square-free and no larger than n, and a lower bound for $T(n)$, the number of nonisomorphic groups whose orders are no larger than n. It is then noted that $B(n) = o(T(n))$.

An open problem is to find a formula for $N(n)$, the number of nonisomorphic groups of order n. Balash [1] discovered such a formula in the special case where n is square-free, and Higman [4] and Sims [6] developed an asymptotic formula for the number of groups of order a power of a prime. In this paper we use Balash's result to determine an upper bound for $B(n)$, where

$$B(n) = \sum_{\substack{k \leqq n \\ k \text{ square-free}}} N(k) ,$$

and the work of Higman and Sims to bound $T(n)$, given by

$$T(n) = \sum_{k \leqq n} N(k) ,$$

from below.

Higman's result, as refined by Sims, is

LEMMA 1. *Let $A = A(n, p)$ be defined by $\log_p (N(p^n)) = An^3$. Then $A = 2/27 + O(n^{-1/3})$.*

Higman originally offered 2/27 as the function in the lower bound for A with error term $O(n^{-1})$ and 2/15 in the upper bound. Sims reduced the upper bound to $2/27 + O(n^{-1/3})$. The lower bound is all we need, and the constant is not important as long as it is positive.

THEOREM 1. *There exists a positive constant c such that*

$$T(n) \gg n^{c \log^2 n} .$$

Proof. Let $2^m < n \leqq 2^{m+1}$. Then for $n > 1$,

$$T(n) \geqq T(2^m) \gg 2^{km^3} \geqq n^{c \log^2 n} .$$

Murty and Murty [5] show that $T(n) \gg n \log\log\log n$, which is enough to conclude, with a result of Erdös and Szekeres [2], that abelian groups are scarce. They then ask about nilpotent groups. Their lower bound grows more slowly than n^2, whereas the bound in Theorem 1 grows faster than any polynomial in n.

An upper bound for $T(n)$ of

$$n^{an^{2/3}\log n},$$

with a explicitly given, was provided by Gallagher [3]. Every group counted in the proof of Theorem 1 is a p-group and hence nilpotent. For $\mathscr{N}(n)$ the number of nonisomorphic nilpotent groups of order no greater than n, then,

$$n^{c\log^2 n} \ll \mathscr{N}(n) \leqq n^{an^{2/3}\log n},$$

and if the lower bound were the correct order of magnitude of $\mathscr{N}(n)$ then we could say that almost all groups are nilpotent.

If n is square-free, the number of groups of order n is determined by the unitary congruences among the prime divisors of n. Such a congruence exists if, for p and q prime factors of n, $p \equiv 1 (\mathrm{mod}\, q)$. If none exist, then $(n, \phi(n)) = 1$, where $\phi(n)$ is the totient function of Euler, and in that case it was shown by Szele [7] that there is exactly one group of order n. Roughly, the more congruences there are the more nonisomorphic groups of order n there can be. Balash's formula below gives the number of groups of a square-free order n in terms of the unitary congruences among n's prime factors.

LEMMA 2. *For k square-free and $m|k$, let $\mathfrak{l}(k/m, p)$ be the number of prime divisors q of k/m for which $q \equiv 1 (\mathrm{mod}\, p)$. Then*

$$N(k) = \sum_{m|k} \prod_{p|m} \frac{p^{\mathfrak{l}(k/m,p)} - 1}{p-1}.$$

Thus, for instance, $N(6) =$ (summand for 1) + (summand for 2) + (summand for 3) + (summand for 6) $= 1 + 1 + 0 + 0 = 2$.

This is used in

THEOREM 2. $B(n) \ll n^2 \log\log n$.

Proof. For k square-free, Lemma 2 gives

$$N(k) \leqq \sum_{m|k} \prod_{p|m} p^{\mathfrak{l}(k/m,p)}.$$

But

$$\begin{aligned}\prod_{p|m} p^{\mathfrak{l}(k/m,p)} &= \prod_{q|k/m} \prod_{p|(q-1,m)} p \\ &= \prod_{q|k/m} (q-1, m) \leqq \prod_{q|k/m} q = k/m.\end{aligned}$$

So

$$N(k) \leqq \sum_{m \mid k} k/m = \sigma(k) \ll k \log \log k$$

and

$$B(n) = \sum_{\substack{k \leq n \\ k \text{ square-free}}} N(k) \ll n^2 \log \log n \ .$$

Now we have

THEOREM 3. $B(n) = o(T(n))$.

The author would like to thank the referee for several helpful suggestions. Some of these results appeared in the author's Ph. D. thesis, written under the guidance of Professor Raymond Ayoub at the Pennsylvania State University.

REFERENCES

1. E. E. Balash, *On the number of nonisomorphic groups whose order is not divisible by the square of a prime,* Izv. Vyss. Uchebn. Zav. Math., **53** (1966), 1-8.
2. P. Erdös and G. Szekeres, *Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwendtes zahlentheoretisches Problem,* Acta Scient. Math., **7** (1934-35), 95-102.
3. P. X. Gallagher, *Counting finite groups of given order*, Math. Zeitschrift, **102** (1967), 236-237.
4. G. Higman, *Enumerating p-groups. I: Inequalities,* Proc. London Math. Soc., **10** (1960), 24-30.
5. M. Ram Murty and V. Kumar Murty, *The number of groups of a given order*, (preprint).
6. C. Sims, *Enumerating p-groups,* Proc. London Math. Soc., **15** (1965), 151-166.
7. T. Szele, *Über die endlichen ordnungszahlen zu denen nur eine Gruppe gehirt,* Commenj. Math. Helv., **20** (1947), 265-67.

Received February 27, 1978 and in revised form August 27, 1979.

WEST VIRGINIA UNIVERSITY
MORGANTOWN, WV 26506