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A DIFFERENCE EQUATION AND HAHN POLYNOMIALS IN TWO VARIABLES

CHARLES F. DUNKL

The space of polynomials in N variables spanned by square-free monomials of degree r and annihilated by $\sum_{i=1}^{N} \partial/\partial x_i$ furnishes an irreducible representation of S_N, the symmetric group on N objects. The elements of this space, which are invariant under permutations leaving the sets $\{x_1, \cdots, x_a\}$ and $\{x_{a+1}, \cdots, x_{a+b}\}$ $(a + b < N)$ fixed, correspond to solutions of a linear difference equation in two variables. By using ideas of representation theory, orthogonal bases for the space of solutions can be obtained. They are certain families of Hahn polynomials in two variables. When these polynomials are restricted to appropriate subsets of $\boldsymbol{R}^N$, general Hahn polynomials in two variables (defined by Karlin and McGregor for the study of populations with various types) are obtained. Further the group theory shows there are three orthogonal bases for the space of solutions of the difference equation, and the connection coefficients between different bases turn out to be balanced ${}_4F_3$-sums, related to Racah's $6-j$ symbols and Wilson's four-parameter orthogonal polynomials.

The condition for invariant polynomials to be in an irreducible representation gives rise to a difference equation for functions of two discrete variables, which is

$$\begin{aligned}(x-a)f(x+1, y) + (y-b)f(x, y+1) \\ = (c-r+1+x+y)f(x, y)\ ,\end{aligned} \tag{0.1}$$

where a, b, c, r are nonnegative integers, $a+b+c=N$, and f is defined on

$$0 \leqq x \leqq a\ ,\quad 0 \leqq y \leqq b\ ,\quad r-c \leqq x+y \leqq r\ .$$

By using two stages of decomposition of the representation one can find a basis of solutions for (0.1) which is orthogonal with respect to the weight $\binom{a}{x}\binom{b}{y}\binom{c}{r-x-y}$.

1. Outline and notation. In §2 we describe the appropriate representations of the symmetric group and the invariant vectors in them, and derive the difference equation (0.1) and some elementary solutions. The branching theorem for the symmetric groups is used to find the dimension of the space of solutions. Orthogonality relations coming from the Peter-Weyl theorem are found in §3. The relevant properties of Hahn polynomials (in one variable) are also stated therein, and then used to find orthogonal bases for the solutions of (0.1). Symmetry properties of the difference equation are used to find two other such bases.

In §4, an identity satisfied by solutions of (0.1) is used to derive a formula for the connection coefficients among these bases. They are a family of orthogonal polynomials expressed as terminating, balanced ${}_4F_3$-series. Finally, in §5, families of general Hahn polynomials in two variables are obtained as intertwining functions on the symmetric group, and the connection coefficients from §4 are shown to apply here as well.

Unless otherwise stated, all variables throughout are nonnegative integers. For integers m, n let $m \vee n$, $m \wedge n$ denote $\max(m, n)$, $\min(m, n)$ respectively. For a set ξ, let $|\xi|$ denote the cardinality of ξ. For a real number α, the shifted factorial (Pochhammer symbol) is defined by $(\alpha)_0 = 1$, $(\alpha)_{n+1} = (\alpha)_n(\alpha + n)$. For $p = 1, 2, \cdots$ real numbers $\alpha_1, \alpha_2, \cdots, \alpha_{p+1}, \beta_1, \cdots, \beta_p$, x define the generalized hypergeometric series

$$ {}_{p+1}F_p\left(\begin{matrix}\alpha_1, \alpha_2, \cdots, \alpha_{p+1} \\ \beta_1, \beta_2, \cdots, \beta_p\end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_{p+1})_n}{(\beta_1)_n \cdots (\beta_p)_n n!} x^n . $$

The series is said to be balanced if $(\sum_{i=1}^{p+1} \alpha_i) + 1 = \sum_{i=1}^{p} \beta_i$ and it terminates if one of the α_i's is a nonpositive integer.

2. The symmetric group and the difference equation. Choose nonnegative integers a, b, c, N with $N = a + b + c$. Define subsets of $\{1, 2, \cdots, N\}$ by $\eta_1 = \{1, \cdots, a\}$, $\eta_2 = \{a + 1, \cdots, a + b\}$, $\eta_3 = \{a + b + 1, \cdots, N\}$. Consider S_N, the symmetric group on N objects, as acting on elements of $\{1, \cdots, N\}$ (on the right) and also as a linear group on $\boldsymbol{R}^N$ by permuting coordinates (the ith coordinate of $\underline{x}\pi$ is defined to be the $i\pi^{-1}$ coordinate of $\underline{x}$, $\underline{x}$ (row vector) $\in \boldsymbol{R}^N$, $\pi \in S_N$).

For each r, $0 \leqq r \leqq N$, let $P_r = \operatorname{span}\{x_{i_1} \cdots x_{i_r}: 1 \leqq i_1 < i_2 \cdots < i_r \leqq N\}$, a space of polynomial functions on $\boldsymbol{R}^N$, which is an S_N-module (i.e., P_r is invariant under right translation by S_N). It is not irreducible (unless $r = 0$ or N), but $V_r = P_r \cap \ker d$ is irreducible, where $d = \sum_{i=1}^{N} \partial/\partial x_i$, an operator which commutes with S_N, (and $V_r \neq \{0\}$ for $r \leqq N/2$) see [2]. The dimension of V_r is $\binom{N}{r} - \binom{N}{r-1}$ and realizes the representation $[N - r, r]$ of S_N (corresponding to the Young tableau with $N - r$ boxes in the first row, r in the second, see Robinson [9], p. 36). We will consider elements of V_r which are invariant under Young subgroups with three factors. Specifically for $\eta \subset \{1, \cdots, N\}$ let $S(\eta)$ be the subgroup of S_N fixing each point of $\{1, \cdots, N\}\backslash\eta$ (thus $S(\eta)$ is isomorphic to the symmetric group of η), and let $H = S(\eta_1) \times S(\eta_2) \times S(\eta_3)$, isomorphic to $S_a \times S_b \times S_c$.

For $\eta \subset \{1, \cdots, N\}$, $0 \leqq i \leqq |\eta|$, let $\sigma_i(\eta)$ be the elementary symmetric functions of degree i from the set of variables $\{x_j\colon j \in \eta\}$, that is, $\sigma_i(\eta)$ is the coefficient of t^i in $\prod_{j\in\eta}(1 + tx_j)$.

The H-invariant functions in P_r have the basis

$$\{\sigma_{i_1}(\eta_1)\sigma_{i_2}(\eta_2)\sigma_{i_3}(\eta_3)\colon i_1 + i_2 + i_3 = r, 0 \leqq i_1 \leqq a, 0 \leqq i_2 \leqq b, 0 \leqq i_3 \leqq c\} .$$

PROPOSITION 2.1. *Let f be a function of two integer variables, with domain $D_r(a, b, c) = \{(x, y)\colon 0 \leqq x \leqq a, 0 \leqq y \leqq b, r - c \leqq x + y \leqq r\}$, then $p = \sum \{f(i_1, i_2)\sigma_{i_1}(\eta_1)\sigma_{i_2}(\eta_2)\sigma_{r-i_1-i_2}(\eta_3)\colon (i_1, i_2) \in D_r(a, b, c)\}$ is in V_r if and only if f satisfies the difference equation*

$$(2.1)\qquad \begin{aligned}(x - a)f(x + 1, y) + (y - b)f(x, y + 1)\\ = (c - r + 1 + x + y)f(x, y)\end{aligned}$$

for $0 \leqq x \leqq a$, $0 \leqq y \leqq b$, $r - c - 1 \leqq x + y \leqq r - 1$, and $f(a + 1, y)$, $f(x, b + 1)$, $f(x, r - c - 1 - x)$ taken as zero.

Proof. It is known $d\sigma_i(\eta) = (|\eta| - i + 1)\sigma_{i-1}(\eta)$; the factor is the number of i-subsets of η containing a given $(i - 1)$-subset. Applying the product rule to dp leads to a sum of multiples of $\sigma_x(\eta_1)\sigma_y(\eta_2)\sigma_{r-1-x-y}(\eta_3)$ with (x, y) in the stated range. These functions are part of a basis for H-invariant elements of P_{r-1}, so $dp = 0$ if and only if the coefficient of each term is zero. The term for (x, y) comes from the coefficient of $\sigma_{x+1}(\eta_1)\sigma_y(\eta_2)\sigma_{r-1-x-y}(\eta_3)$, $\sigma_x(\eta_1)\sigma_{y+1}(\eta_2)\sigma_{r-1-x-y}(\eta_3)$ $\sigma_x(\eta_1)\sigma_y(\eta_2)\sigma_{r-x-y}(\eta_3)$ multiplied by $(a-x)$, $(b-y)$, $(c - (r - x - y) + 1)$ respectively.

DEFINITION 2.2. The linear space of solutions to (2.1) will be denoted by $W_r(a, b, c)$, or W_r for short.

PROPOSITION 2.3. *The dimension of $W_r(a, b, c)$ is $r \wedge a \wedge b \wedge c \wedge (a + b - r) \wedge (b + c - r) \wedge (a + c - r) \wedge (a + b + c - 2r) + 1$ if $a + b \geqq r$, $b + c \geqq r$, $a + c \geqq r$, $a + b + c \geqq 2r$ and is 0 otherwise (note that $D_r(a, b, c)$ is nonempty provided $a + b + c \geqq r$).*

Proof. The required dimension is equal to the number of times the trivial representation of H ($H \to \{1\}$) appears in the restriction of $[N - r, r]$ to H. We will restrict first to $S(\eta_1) \times S(\eta_2 \cup \eta_3)$ and iterate the known decomposition of the representation $[M - k, k]$ of S_M restricted to $S_A \times S_{M-A}$, where $0 \leqq A \leqq M$, $0 \leqq k \leqq M/2$, namely $[M - k, k]|(S_A \times S_{M-A}) \cong \sum \bigoplus\{[A - m, m] \otimes [M - A - n, n]\colon 0 \leqq m \leqq$

$A/2,\ 0 \leqq n \leqq (M-A)/2,\ m+n \leqq k,\ k-M+A \leqq m-n \leqq A-k\}$ (see [9]). Note that $[M, 0]$ is the trivial representation. The values of k allowing $[A, 0] \otimes [M-A, 0]$ to appear (exactly once) are $0 \leqq k \leqq A \wedge (M-A)$. Thus we obtain one H-invariant function in V_r for each constituent of the form $[a, 0] \otimes [b+c-k, k]$ with $0 \leqq k \leqq b \wedge c$ (decomposing $[b+c-k, k]$ into irreducible representations of $S_b \times S_c$). The set of these functions is linearly independent because the sum is direct. Applying the formula to $[N-r, r] | (S_a \times S_{b+c})$ we obtain the permitted values for k as $k \leqq r,\ r-b-c \leqq -k \leqq a-r$. The number of k satisfying every requirement is $(b \wedge c \wedge r \wedge (b+c-r)) - (0 \vee (r-a)) + 1$, provided $a+b \geqq r$, $b+c \geqq r$, $a+c \geqq r$, $a+b+c \geqq 2r$ (this was already assumed as $V_r \neq \{0\}$, $r \leqq N/2$), else there are no nonzero H-invariants in V_r. The expression for the number is equal to the original one.

There is a natural isomorphism between $W_r(a, b, c)$ and $W_r(a', b', c')$ where $a'b'c'$ is a permutation of $a\ b\ c$; for example, if $g \in W_r(b, a, c)$ then the function $(x, y) \mapsto g(y, x)$ is in $W_r(a, b, c)$, and if $h \in W_r(c, b, a)$ then $(x, y) \mapsto h(r-x-y, y)$ is in $W_r(a, b, c)$.

Attempting to find a solution f of (2.1) we observe that if $c \geqq r$ then $D_r(a, b, c)$ is the union of rectangles $R_j = \{(x, y) : 0 \leqq x \leqq j, 0 \leqq y \leqq r-j\}$ with $0 \vee (r-b) \leqq j \leqq r \wedge a$, and the value of f at $(j, r-j)$ is propagated by (2.1) to every point of R_j, but no other. For given j, trying to find f which is zero off R_j and 1 at $(j, r-j)$ we look at the values at $(j-1, r-j)$, $(j, r-j-1)$, etc., guess at a solution and then verify it. Indeed we obtain the solution

$$f_j(x, y) = \binom{r-x-y}{j-x} \frac{(x-a)_{j-x}(y-b)_{r-j-y}(-1)^{r-x-y}}{(-c)_{r-x-y}}$$

$0 \vee (r-b) \leqq j \leqq r \wedge a$. It is easy to show that this satisfies (2.1) for $x+y \geqq r-c$, and indeed all solutions for (2.1) have been obtained in the form $f = \sum_j \alpha_j f_j$, with $\alpha_j = f(j, r-j)$ arbitrary. If $c < r$ then f_j is not a solution but the $(r-c)$ equations

$$(x-a)f(x+1, r-c-x-1) + (r-c-b-1-x)f(x, r-c-x) = 0\,, \quad 0 \leqq x \leqq r-c-1$$

(from (2.1)) impose conditions on the numbers α_j. Note that $(a \wedge r) - (0 \vee (r-b)) + 1 - ((r-c) \vee 0) = \dim W_r$. We will not attempt here to find a basis for W_r when $c < r$; it can be done directly. However, if $f \in W_r$ then

$$(2.2) \qquad f(x, y) = \sum_{j=(r-b)\vee x}^{a \wedge (r-y)} f(j, r-j) \binom{r-x-y}{j-x} \frac{(x-a)_{j-x}(y-b)_{r-i-y}}{(-c)_{r-x-y}(-1)^{r-x-y}}$$

holds (if the two sides agree for $r - x - y = k$, then (2.1) shows they agree for $r - x - y = k + 1$, $0 \leqq k \leqq (r \wedge c) - 1$).

3. Orthogonality. The fundamental result for orthogonality related to finite (or compact) group actions is the Peter-Weyl theorem. To apply it we need an S_N-invariant "integral" for V_r, which can be obtained by summing values over any S_N-invariant set in $\boldsymbol{R}^N$. For a subset $\Omega \subset \boldsymbol{R}^N$ define $\langle f, g\rangle_\Omega = \sum_{x \in \Omega} f(x)\overline{g(x)}$ and $\|f\|_\Omega = \langle f, f\rangle_\Omega^{1/2}$ with f, g being functions defined at least on Ω. One good choice for Ω is the hypercube $C = \{(c_1, c_2, \cdots, c_N) \in \boldsymbol{R}^N\colon c_i = \pm 1, 1 \leqq i \leqq N\}$. Then $C \cong Z(2)^N$ (a multiplicative abelian group of order 2^N) and the square-free monomials $x_{i_1} \cdots x_{i_r}$, $1 \leqq i_1 < i_2 \cdots < i_r \leqq N$ are characters, hence orthogonal with respect to $\langle\cdot, \cdot\rangle_C$. The H-invariant functions $\sigma_{i_1}(\eta_1)\sigma_{i_2}(\eta_2)\sigma_{i_3}(\eta_3)$ are mutually orthogonal and $\|\sigma_{i_1}(\eta_1)\sigma_{i_2}(\eta_2)\sigma_{i_3}(\eta_3)\|_C^2 = \binom{a}{i_1}\binom{b}{i_2}\binom{c}{i_3}$.

We recall from 2.3 that a basis for W_r can be produced by considering different representations of $S(\eta_1) \times S(\eta_2 \cup \eta_3)$. The inner product over C is invariant with respect to this group, so we expect to find an orthogonal basis for W_r, with the inner product (induced from C) defined by

$$\langle f, g\rangle = \sum_{(x,y) \in D_r} \binom{a}{x}\binom{b}{y}\binom{c}{r - x - y} f(x, y)\overline{g(x, y)}$$

and $\|f\| = \langle f, f\rangle^{1/2}$. The previously defined correspondence between $W_r(a, b, c)$ and $W_r(a', b', c')$ is isometric for this norm. The nature of the subgroup suggests solutions which treat b and c symmetrically, while the weight function suggests Hahn polynomials.

We recall some facts about the renormalized Hahn polynomials $E_m(a, b, c, x)$ used in [4] (temporarily a, b, c, r, x are any nonnegative integers). They are defined by

$$E_m(a, b, c, x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (b-m+1)_j (a-m+1)_{m-j} (-x)_j (x-c)_{m-j} \tag{3.1}$$

and are polynomial of degree m in the two variables x, c. They are related to the usual Hahn polynomials (see Karlin and McGregor [6]) by

$$\begin{aligned} E_m(a, b, c, x) &= (-1)^m(-a)_m(-c)_m Q_m(x; -a - 1, -b - 1, c) \\ &= (-1)^m(-a)_m(-c)_m\, {}_3F_2\left(\begin{matrix} -m,\ m - a - b - 1,\ -x \\ -a,\ -c \end{matrix}; 1\right). \end{aligned} \tag{3.2}$$

The domain for orthogonality is $(c - b) \vee 0 \leqq x \leqq a \wedge c$, and $E_m = 0$ on this set unless $m \leqq a \wedge b \wedge c \wedge (a + b - c)$. The orthogonality relation is

$$\begin{aligned}(3.3)\qquad &\sum_{x=0\vee(c-b)}^{a\wedge c}\binom{a}{x}\binom{b}{c-x}E_m(a,b,c,x)E_n(a,b,c,x)\\ &=\delta_{mn}\binom{a+b-2m}{c-m}(-1)^m(m-a-b-1)_m(-a)_m(-b)_m m!\;.\end{aligned}$$

There are several symmetry relations and difference equations (see §3, [4]); among these:

$$E_m(a,b,c,x)=(-1)^m E_m(b,a,c,c-x) \tag{3.4}$$

$$\begin{aligned}(3.5)\qquad &(a-x)E_m(a,b,c+1,x+1)+(b-c+x)E_m(a,b,c+1,x)\\ &=(a+b-c-m)E_m(a,b,c,x)\;.\end{aligned}$$

$$E_m(a,b,c,0)=(-1)^m(-a)_m(-c)_m \tag{3.6}$$

and

$$E_m(a,b,c,c)=(-b)_m(-c)_m\;. \tag{3.7}$$

Here is a transformation formula

$$\sum_{x=0\vee(c-d+y)}^{c\wedge y}\binom{y}{x}\binom{d-y}{c-x}E_m(a,b,c,x)=\binom{d-m}{c-m}E_m(a,b,d,y) \tag{3.8}$$

(which can be proved from (3.1) since

$$\binom{y}{x}\binom{d-y}{c-x}(-x)_j(x-c)_{m-j}=(-y)_j(y-d)_{m-j}\binom{y-j}{x-j}\binom{d-y-m+j}{c-x-m+j}\,,$$

and summing over x yields $\binom{d-m}{c-m}(-y)_j(y-d)_{m-j}$ by the Chu-Vandermonde sum) (the formula is due to Gasper [5], p. 180, (2.5)).

We guess at a solution to (2.1) of the form $f(x, y) = g(x)E_m(b, c, r - x, y)$ with m, g to be determined (suggested by the $\binom{b}{y}\binom{c}{r-x-y}$ part of the weight function). Substituting in (2.1) and using (3.5) we obtain

$$\begin{aligned}(3.9)\qquad &g(x)(b+c+1+x-r-m)E_m(b,c,r-x-1,y)\\ &=g(x+1)(x-a)E_m(b,c,r-x-1,y)\;,\end{aligned}$$

with $(r-c-1)\vee 0\leqq x+y\leqq r-1$, $0\leqq x\leqq a$, $0\leqq y\leqq b$.

We claim nonzero solutions are possible exactly for $0\vee(r-a)\leqq m\leqq b\wedge c\wedge r\wedge(b+c-r)$. The E_m term in f is zero unless $m+r-b-c\leqq x\leqq r-m$ and $m\leqq b\wedge c\wedge r$. The E_m term in (3.9) is nontrivial for $m-b-c+r-1\leqq x\leqq r-m-1$ (and $m\leqq b\wedge c\wedge r$). This and the rest of (3.9) show that the values of $g(x)$, $(m+r-b-c)\vee 0\leqq x\leqq a\wedge(r-m)$, are either all zero or all nonzero. But if $b + c - r < b \wedge c \wedge r$ and $m > b + c - r$, then put $x = m - b - c + r - 1 \geqq 0$ in (3.9), leading to $(m + r - 1 - a - b - c)g(m - b - c + r)E_m(b, c, b + c - m, y) = 0$ forcing $g \equiv 0$ (the first factor satisfies $m + r - 1 - a - b - c \leqq m - r - 1 \leqq -1$). Thus $m \leqq b + c - r$ is necessary. Similarly if $r - m > a$, put $x = a$ in (3.9) leading to

$$(a + b + c + 1 - r - m)g(a)E_m(b, c, r - a - 1, y) = 0$$

again forcing $g \equiv 0$ (note $m \leqq b \wedge c \wedge (r-a-1) \wedge (a+b+c-r+1)$ so the E_m term is nonzero).

In the permitted range of m, the solution for g is given by

$$g(x) = \frac{(b + c - r - m + 1)_x}{(-a)_x} g(0) , \qquad 0 \leqq x \leqq r - m \leqq a .$$

We recall that for arbitrary integers a, b, c, $E_c(a, b, c, x) = ((b - c + 1)_x/(-a)_x)c!\ (-a)_c$, for $x = 0, 1, \cdots, a \wedge c$; thus we choose $g(0) = (r - m)!\ (-a)_{r-m}$ and obtain a polynomial solution $g(x) = E_{r-m}(a, b + c - 2m, r - m, x)$. The explicit values for $0 \leqq x \leqq r - m$ are $g(x) = (r - m)!\ (-1)^{r-m-x}(a - r + m + 1)_{r-m-x}(b + c + m - r + 1)_x$. Thus we have obtained solutions to (2.1):

$$\text{(3.10)} \qquad \phi_m(x, y) = E_{r-m}(a, b + c - 2m, r - m, x)E_m(b, c, r - x, y) ,$$

$0 \vee (r - a) \leqq m \leqq b \wedge c \wedge r \wedge (b + c - r)$ and $\phi_m(x, y) = 0$ for $x > r - m$.

We describe the other families of solutions obtained by permuting elements of $\{(a, x), (b, y), (c, r - x - y)\}$. To show the dependence, we write $\phi_m(x, y; a, b, c)$ for ϕ_m as in (3.10), (throughout when parameters are omitted, they are understood to be (a, b, c)). We list the possible permutations of $a\ b\ c$, followed by the resulting elements of $W_r(a, b, c)$.

(3.11) (i) bac: $(x, y) \longmapsto \phi_m(y, x; b, a, c)$
$$= E_{r-m}(b, a + c - 2m, r - m, y)E_m(a, c, r - y, x)$$

denoted by $\psi_m(x, y; a, b, c)$ (or $\psi_m(x, y)$),

$$0 \vee (r - b) \leqq m \leqq a \wedge c \wedge r \wedge (a + c - r) ;$$

(ii) cba: $(x, y) \longmapsto \phi_m(r - x - y, y; c, b, a)$
$$= E_{r-m}(c, b + a - 2m, r - m, r - x - y)E_m(b, a, x + y, y)$$

denoted by $\theta_m(x, y; a, b, c)$ (or $\theta_m(x, y)$),

$$0 \vee (r - c) \leqq m \leqq a \wedge b \wedge r \wedge (a + b - r) ;$$

(iii) acb: $(x, y) \longmapsto \phi_m(x, r - x - y; a, c, b)$
$$= E_{r-m}(a, c + b - 2m, r - m, x)E_m(c, b, r - x, r - x - y)$$
$$= (-1)^m \phi_m(x, y; a, b, c) \text{ by (3.4);}$$

(iv) $bca\colon (x, y) \longmapsto \phi_m(y, r-x-y; b, c, a)$
$$= E_{r-m}(b, a+c-2m, r-m, y)E_m(c, a, r-y, r-x-y)$$
$$= (-1)^m \psi_m(x, y; a, b, c) \text{ by } (3.4);$$

(v) $cab\colon (x, y) \longmapsto \phi_m(r-x-y, x; c, a, b)$
$$= (-1)^m \theta_m(x, y; a, b, c) \text{ by } (3.4)\ .$$

Note that only two new families of solutions have been obtained.

THEOREM 3.1. *The set* $\{\phi_m\colon 0 \vee (r-a) \leqq m \leqq b \wedge c \wedge r \wedge (b+c-r)\}$ *is an orthogonal basis for* $W_r(a, b, c)$ *and*

$$\sum_{(x,y)\in D_r} \binom{a}{x}\binom{b}{y}\binom{c}{r-x-y}\phi_m(x, y)\phi_n(x, y) = \delta_{mn}$$

$$(3.12) \qquad \left(\frac{b+c-m+1}{b+c-2m+1}\right)(m-b-c)_r(-a)_{r-m}(-b)_m(-c)_m m! \times (r-m)!\,(a+b+c-2r+2)_{r-m}(-1)^m\ .$$

Proof. The orthogonality relation follows from

$$\sum_y \binom{b}{y}\binom{c}{r-x-y}E_m(b, c, r-x, y)E_n(b, c, r-x, y) = 0\ , \qquad m \neq n\ .$$

The calculation of $\|\phi_m\|^2$ is routine, summing over y first, then x, using (3.3) twice. The set $\{\phi_m\}$ is a basis because it is linearly independent with cardinality $=\dim W_r$ (see Proposition 2.3).

COROLLARY 3.2. *Each of* $\{\psi_m\}$, $\{\theta_m\}$ *is an orthogonal basis for* W_r. *The expressions for* $\|\psi_m\|^2$, $\|\theta_m\|^2$ *are obtained from* (3.12) *by interchanging* a *and* b, a *and* c *respectively.*

4. Expansions and connection coefficients. Suppose $f \in W_r$ and it is desired to express $f = \sum_m \alpha_m\phi_m$. The orthogonality relations holding along lines $x =$ constant immediately show that there is a formula of the sort

$$\alpha_m = \sum_y \binom{b}{y}\binom{c}{r-x-y}f(x, y)E_m(b, c, r-x, y)$$

times an expression in (m, x, a, b, c, r). Note that $x \leqq r-m$ is necessary. The sum is actually a double sum (one more for E_m). A single sum expression can be obtained by using (2.2).

THEOREM 4.1. *If* $f \in W_r$, *then* $f = \sum_m \alpha_m\phi_m$ *with*

$$(4.1)\qquad \alpha_m = \sum_{j=0\vee(r-b)}^{a\wedge r} f(j, r-j) \times \frac{(-1)^j(-a)_j(-m)_{r-j}(m-b-c-1)_{r-j}(b+c-2m+1)}{(-a)_{r-m}(m-b-c)_r(-c)_m m!\,(r-m)!\,(r-j)!\,(b+c-m+1)} .$$

Proof. Choose $x \leqq r - m$, and form $S_{xm} = \sum_{y=0\vee(r-c-x)}^{b\wedge(r-x)} \binom{b}{y} \cdot \binom{c}{r-x-y} f(x, y)E_m(b, c, r-x, y)$, then $S_{xm} = \alpha_m$.

$$\times \binom{b+c-2m}{r-m-x}(-1)^m(m-b-c-1)_m(-b)_m(-c)_m m! \times E_{r-m}(a, b+c-2m, r-m, x) .$$

In S_{xm} replace $f(x, y)$ by formula (2.2) and obtain

$$S_{xm} = \sum_j f(j, r-j)\frac{(x-a)_{j-x}}{(j-x)!}(-b)_{r-j} \times \sum_{y=0\vee(r-c-x)}^{r-j} \frac{E_m(b, c, r-x, y)}{y!\,(r-j-y)!}(-1)^y .$$

The sum over y can be done if $r - c - x \leqq 0$; by finite differences

$$\sum_{y=0}^{n}\binom{n}{y}(-1)^y\sum_{i=0}^{m}\beta_i(-y)_i = n!\,\beta_n ,$$

for m, n integers, $0 \leqq n \leqq m$ and any numbers β_i; and we have such an expression for E_m in (3.2). Note $m \leqq c$, so choose $x = 0 \vee (r-c)$, then the sum over y is

$$(-b)_m(x-r)_m(-1)^m\frac{(m-b-c-1)_{r-j}(-m)_{r-j}}{(-b)_{r-j}(x-r)_{r-j}(r-j)!} .$$

Now collect all the terms, and put α_m on one side of the equation to get the stated formula.

This formula makes it easy to find the connection coefficients among ϕ_m, ψ_m, θ_m.

THEOREM 4.2. *For* $0 \vee (r-b) \leqq k \leqq a \wedge c \wedge r \wedge (a+c-r)$

$$\psi_k = \sum_m \alpha_{km}(a, b, c)\phi_m , \quad (\textit{sum over } 0\vee(r-a)\leqq m\leqq b\wedge c\wedge r\wedge(b+c-r)) ,$$

where

$$\alpha_{km}(a, b, c) = (-1)^{r+k}\frac{(-r)_m(-b)_{r-k}(a-r+1)_m(-c)_k(b+c-2m+1)}{m!\,(m-b-c)_r(-c)_m(b+c-m+1)}$$

$$\times {}_4F_3\left(\begin{matrix} k-r,\, a+c+1-k-r,\, -m,\, m-b-c-1 \\ -r,\, -b,\, a-r+1 \end{matrix};1\right) \quad if\ a \geqq r\,,$$

$$= \frac{(-1)^{r+k}(-r)_m(-a)_k(r-a-b)_{a-k}(m-b-c-1)_{r-a}(-c)_{r-a+k}}{(r-a)!\,(-r)_k(m-b-c)_r(-c)_m}$$

$$\times \left(\frac{b+c-2m+1}{b+c-m+1}\right)$$

$$\times {}_4F_3\left(\begin{matrix} k-a,\, c+1-k,\, r-m-a,\, m-a-b-c-1+r \\ -a,\, -a-b+r,\, r-a+1 \end{matrix};1\right) \quad if\ a \leqq r\,.$$

Proof. The formula (3.7) shows

$$\begin{aligned} &\psi_k(j, r-j) \\ &\quad = E_{r-k}(b, a+c-2k, r-k, r-j)E_k(a, c, j, j) \\ &\quad = (-1)^{j-k}(r-k)!\,(b-r+k+1)_{j-k}(a+c-k-r+1)_{r-j}(-c)_k(-j)_k \end{aligned}$$

(zero for $j < k$). Now substitute this in (4.1). The sum extends over $(r-b) \vee k \leqq j \leqq a \wedge r$. Change the variable in the sum, letting $i = (a \wedge r) - j$, doing the two cases $a \leqq r$, $a \geqq r$ separately.

COROLLARY 4.3. *If* $b \geqq r$ *then*

$$\alpha_{0m}(a, b, c) = \frac{(-1)^r(b-r+1)_{r-m}(r-a-b-c-1)_m(b+c-2m+1)}{(m-b-c)_r m!\,(r-m)!\,(b+c-m+1)}\,,$$

and if $b \leqq r$ *then*

$$\alpha_{r-b,m}(a, b, c) = \frac{(r-a-b-c-1)_m(b+c-2m+1)}{m!\,(c-m+1)_b(b+c-m+1)}\,.$$

Proof. For both $k = 0$, $r-b$ the ${}_4F_3$-sums reduce to balanced ${}_3F_2$-sums which are done by the Pfaff-Saalschütz formula (see Bailey, [1], p. 9). The results are independent of the sign of $r-a$.

The ${}_4F_3$-sums in 4.2 are balanced, and are examples of Wilson's version of Racah's $6-j$ symbols (see Racah [8]). Wilson ([10], 2.13) defined

$$R_n(\lambda(x); \alpha, \beta, \gamma, \delta) = {}_4F_3\left(\begin{matrix} -n,\, n+\alpha+\beta+1,\, -x,\, x+\gamma+\delta+1 \\ \alpha+1,\, \beta+\delta+1,\, \gamma+1 \end{matrix};1\right)$$

with $\lambda(x) = x(x+\gamma+\delta+1)$ and one of $\alpha+1$, $\beta+\delta+1$, $\gamma+1$ being a negative integer. Then R_n is a polynomial of degree n in $\lambda(x)$, and the set $\{R_n\}$ (with suitably bounded n) is a family of orthogonal polynomials with respect to a finitely supported weight function—(Wilson found nondiscrete orthogonality relations for the case when none of the denominator parameters is a negative integer). The Racah $6-j$ symbols can be expressed in terms of R_n (see Wilson [10], Ch. II). The ${}_4F_3$-sums in 4.2 can be written as

$$R_{r-k}(\lambda(m); a-r, c-r, -b-1, -c-1) \quad \text{for} \quad r \leqq a\ ,$$

or

$$R_{a-k}(\lambda(m-r+a); r-a, c-r, r-a-b-1, r-a-c-1) \quad \text{for} \quad r \geqq a\ .$$

(λ is different in the two expressions, depending on the parameter values.)

We get the orthogonality from $\{\phi_m\}$, $\{\psi_k\}$ both being orthogonal bases for W_r.

PROPOSITION 4.4. *$\sum_m \|\phi_m\|^2 \alpha_{km}(a, b, c)\alpha_{lm}(a, b, c) = \delta_{kl}\|\psi_k\|^2$, where $0 \vee (r-a) \leqq m \leqq b \wedge c \wedge r \wedge (b+c-r)$ and $0 \vee (r-b) \leqq k$, $l \leqq a \wedge c \wedge r \wedge (a+c-r)$ (values of $\|\phi_m\|^2$, $\|\psi_k\|^2$ as in* (3.12)).

Proof. The sum

$$\sum_{(x\ y) \in D_r} \binom{a}{x}\binom{b}{y}\binom{c}{r-x-y}\psi_k(x, y)\psi_l(x, y) = \delta_{kl}\|\psi_k\|^2\ ,$$

but also equals

$$\sum_{(x, y) \in D_r} \binom{a}{x}\binom{b}{y}\binom{c}{r-x-y}\Sigma_m \alpha_{km}\phi_m(x, y)\Sigma_n \alpha_{ln}\phi_n(x, y)$$
$$= \Sigma_m \alpha_{km}\alpha_{lm}\|\phi_m\|^2\ .$$

The $6-j$ symbols arise as the relations among the three ways of decomposing a tensor product of three irreducible (continuous, unitary) representations by using the product-of-two result twice (the $3-j$ symbols). In the present situation we restrict an irreducible representation of S_N to $S_a \times S_b \times S_c$ by two steps, passing through $S_a \times S_{b+c}$, etc., and the coefficients α_{km} express the relations between the $S_a \times S_{b+c}$ and $S_b \times S_{a+c}$ methods. Further there is a relation between representations of S_N and the N-fold tensor product of the principal representation of $SU(2)$ (actually, any $U(n)$ or $SU(2)$).

Another relation among the ${}_4F_3$-functions can be found by considering the remaining connection coefficients among $\{\theta_n\}$, $\{\phi_m\}$, $\{\psi_k\}$ which can be obtained by permuting a b c in 4.2 and using the formulas (3.11).

PROPOSITION 4.5. *The ranges of m, k are those appropriate to* *the domains of definition, and the parameters* (a, b, c) *are understood in* θ, ϕ, ψ:

(i) $\quad \theta_k = \sum_m (-1)^{k+m}\alpha_{km}(a, c, b)\phi_m$

(ii) $\quad \theta_k = \sum_m (-1)^m \alpha_{km}(b, c, a)\psi_m$

(iii) $\quad \psi_k = \sum_m (-1)^k \alpha_{km}(c, b, a)\theta_m$

(iv) $\quad \phi_k = \sum_m \alpha_{km}(b, a, c)\psi_m$

(v) $\quad \phi_k = \sum_m (-1)^{k+m}\alpha_{km}(c, a, b)\theta_m$.

COROLLARY 4.6. $(-1)^{k+m}\alpha_{km}(a,c,b)=\sum_n (-1)^n \alpha_{kn}(b,c,a)\alpha_{nm}(a,b,c)$, *sum over* $0 \vee (r-b) \leqq n \leqq a \wedge c \wedge r \wedge (a+c-r)$.

Proof. Express θ_n in terms of $\{\psi_n\}$, and these in $\{\phi_m\}$.

5. Intertwining functions and general Hahn polynomials in two variables. Recall from §2 that V_r is a space of functions on $\boldsymbol{R}^N$. Choose any point $\underline{x} \in \boldsymbol{R}^N$ and consider the space $\widetilde{V}_r$ of functions on S_N given by $\widetilde{f}(\pi) = f(\underline{x}\pi)$, $(f \in V_r, \pi \in S_N)$. Then $\widetilde{V}_r$ is closed under right translation, and by Schur's lemma is either $\{0\}$ or isomorphic to V_r. Further the functions in $\widetilde{V}_r$ are left-invariant for $G(\underline{x}) = \{\pi \in S_N: \underline{x}\pi = \underline{x}\}$, the stabilizer of $\underline{x}$. Thus $H \cong S_a \times S_b \times S_c$ invariant functions in V_r correspond to $G(\underline{x}) - H$-invariant functions on S_N (so-called intertwining functions, see [3]). Since we considered only the S_N-modules V_r, $0 \leqq r \leqq N/2$, we are able to find all $G(\underline{x}) - H$ invariant functions only if $G(\underline{x})$ is isomorphic to $S_M \times S_{N-M}$ for some M. It is known (see [4] or [9]) that the set of functions on the coset space $(S_M \times S_{N-M})\backslash S_N$ (that is, the representation of S_N induced by the trivial one of $S_M \times S_{N-M}$) is isomorphic to $\sum_{r=0}^{M\wedge(N-M)} \bigoplus V_r$.

We choose $M \leqq N$ and let $x = (1, 1, \cdots, 1, 0, \cdots, 0)$, M 1's followed by $N-M$ 0's. Thus $G(\underline{x}) = S(\{1, \cdots, M\}) \times S(\{M+1, \cdots, N\})$, denoted by H_M. The $H_M - H$ invariant functions in $\widetilde{V}_r$ will be found by evaluating the H-invariant functions in V_r at $\underline{x}\pi$.

DEFINITION 5.1. For $\pi \in S_N$, let $u_i(\pi) = |\{1, \cdots, M\}\pi \cap \eta_i|$ $i = 1, 2, 3$ (recall η_i from §2; note $u_1 + u_2 + u_3 = M$).

LEMMA 5.2. *The value of* $\sigma_{i_1}(\eta_1)\sigma_{i_2}(\eta_2)\sigma_{i_3}(\eta_3)$ *at* $\underline{x}\pi$ *is*

$$\binom{u_1(\pi)}{i_1}\binom{u_2(\pi)}{i_2}\binom{u_3(\pi)}{i_3} .$$

Proof. The required value is the coefficient of

$$t_1^{i_1}t_2^{i_2}t_3^{i_3}$$

in the expansion of

$$\prod_{i=1}^{3}\prod_{j\in\eta_i}(1+t_i(\underline{x}\pi)_j)\ ,$$

which product equals $(1+t_1)^{u_1}(1+t_2)^{u_2}(1+t_3)^{u_3}$.

The sum over S_N of an $H_M - H$ invariant function is given by

$$\text{(5.1)}\qquad \frac{1}{N!}\sum_{\pi\in S_N} f(u_1(\pi),\, u_2(\pi),\, u_3(\pi)) = \sum_{u_1,u_2}\binom{a}{u_1}\binom{b}{u_2}\binom{c}{M-u_1-u_2} \times f(u_1,\, u_2,\, M-u_1-u_2)\ ,$$

where

$$0 \leqq u_1 \leqq a\ ,\qquad 0 \leqq u_2 \leqq b\ ,\qquad c-M \leqq u_1+u_2 \leqq M\ .$$

THEOREM 5.3. *The restriction of the H-invariant function in V_r corresponding to ϕ_m, namely $\sum_{i_1,i_2}\varphi_m(i_1,\, i_2)\sigma_{i_1}(\eta_1)\sigma_{i_2}(\eta_2)\sigma_{r-i_1-i_2}(\eta_3)$, is $\pi\mapsto \tilde{\phi}_m(u_1(\pi),\, u_2(\pi),\, u_3(\pi)) = E_{r-m}(a,\, b+c-2m,\, u_1+u_2+u_3-m,\, u_1)E_m(b,\, c,\, u_2+u_3,\, u_2)$, $0 \vee (r-a) \leqq m \leqq b \wedge c \wedge r \wedge (b+c-r)$. The functions $\{\tilde{\phi}_m\}$ are orthogonal with respect to the weight from* (5.1). *Furthermore $\tilde{\phi}_m$ is in the $S_a \times S_{b+c}$-submodule* $[a,\, 0]\otimes[b+c-m,\, m]$.

Proof. By 5.2,

$$\begin{aligned}\tilde{\phi}_m(u_1,\, u_2,\, u_3) &= \sum_{i_1,i_2} E_{r-m}(a,\, b+c-2m,\, r-m,\, i_1)E_m(b,\, c,\, r-i_1,\, i_2)\\ &\qquad\times\binom{u_1}{i_1}\binom{u_2}{i_2}\binom{u_3}{r-i_1-i_2}\\ &= \sum_{i_1}\binom{u_1}{i_1}\binom{u_2+u_3-m}{r-m-i_1}E_{r-m}(a,\, b+c-2m,\, r-m,\, i_1)\\ &\qquad\times E_m(b,\, c,\, u_2+u_3,\, u_2)\ \text{(by (3.8))}\\ &= E_{r-m}(a,\, b+c-2m,\, u_1+u_2+u_3-m,\, u_1)\\ &\qquad\times E_m(b,\, c,\, u_2+u_3,\, u_2)\ .\end{aligned}$$

The orthogonality relations follow from the properties of E_m. Theorem 4.2 of [4] shows that $\tilde{\phi}_m$ is in the $S(\eta_1)\times S(\eta_2\cup\eta_3)$-submodule corresponding to $[a,\, 0]\otimes[b+c-m,\, m]$.

Similarly we can produce orthogonal bases for the intertwining functions from $\{\psi_m\}$, $\{\theta_m\}$. Indeed we obtain

$\tilde{\phi}_m(u_1, u_2, u_3) = E_{r-m}(b, a+c-2m, u_1+u_2+u_3-m, u_2)\cdot E_m(a, c, u_1+u_3, u_1)$, $0\vee(r-b)\leqq m\leqq a\wedge c\wedge r\wedge(a+c-r)$, (belonging to the $[a+c-m, m]\otimes[b, 0]$ module for $S(\eta_1\cup\eta_3)\times S(\eta_2)$);

$$\tilde{\theta}_m(u_1, u_2, u_3) = E_{r-m}(c, a+b-2m, u_1+u_2+u_3-m, u_3) \times E_m(b, a, u_1+u_2, u_2) ,$$

$0\vee(r-c)\leqq m\leqq a\wedge b\wedge r(a+b-r)$, (belonging to the $[a+b-m, m]\otimes[c, 0]$ module for $S(\eta_1\cup\eta_2)\times S(\eta_3)$).

The restriction map carries over the connection coefficients, and a typical formula is $E_{r-k}(b, a+c-2k, u_1+u_2+u_3-k, u_2)E_k(a, c, u_1+u_3, u_1) = \Sigma_m\alpha_{km}(a, b, c)E_{r-m}(a, b+c-2m, u_1+u_2+u_3-m, u_1)\cdot E_m(b, c, u_2+u_3, u_2)$. Recall that $\alpha_{0m}(a, b, c)$, $\alpha_{r-b,m}(a, b, c)$ are in closed form, see 4.3.

Karlin and McGregor [7] studied Hahn polynomials in several variables for use in models for the growth of populations with several types. The functions $\tilde{\phi}_m, \tilde{\psi}_m, \tilde{\theta}_m$ can be expressed in their terms, for example,

$$\tilde{\theta}_m(u_1, u_2, u_3; a, b, c) = \frac{(2m-a-b)_{r-m}(-a)_m(-1)^m}{(-u_1-u_2-u_3)_r} \times \phi\left(\begin{matrix} u_1, u_2, u_3 \\ -a-1, -b-1, -c-1 \end{matrix}\,\middle|\, m, r-m\right) \text{(formula 5.6, p. 277 [7])} .$$

To extend the connection coefficients to real parameters, we first note that if $a, b, c\geqq r$ then the summations in 4.2, 4.3, 4.5 extend over $0\leqq m\leqq r$, and the relations are rational in a, b, c with poles at most at $0, 1, 2, \cdots, r-1$. Thus the formulas hold for any complex a, b, c with none having values in $\{0, 1, \cdots, r-1\}$. We state 4.2 for the Karlin-McGregor ϕ functions in three variables, with $\alpha_1, \alpha_2, \alpha_3$ replacing $-a-1, -b-1, -c-1$ respectively.

PROPOSITION 5.4. *For $k=0, 1, \cdots, r$ and none of $\alpha_1, \alpha_2, \alpha_3$ taking values in $\{-1, -2, \cdots, -r\}$*

$$\phi\left(\begin{matrix} u_1, u_3, u_2 \\ \alpha_1, \alpha_3, \alpha_2 \end{matrix}\,\middle|\, k, r-k\right) = \sum_{m=0}^{r} \frac{(-1)^{m+r}(-r)_m(\alpha_2+1)_{r-k}(-r-\alpha_3)_m}{(2m+\alpha_1+\alpha_3+2)_{r-k}m!\,(m+\alpha_1+\alpha_2+1)_m} \times {}_4F_3\left(\begin{matrix} k-r, -\alpha_1-\alpha_3-k-r-1, -m, m+\alpha_1+\alpha_2+1 \\ -r, \alpha_2+1, -\alpha_3-r \end{matrix}; 1\right) \times \phi\left(\begin{matrix} u_1, u_2, u_3 \\ \alpha_1, \alpha_2, \alpha_3 \end{matrix}\,\middle|\, m, r-m\right) .$$

We have found the polynomials in V_r which belong to specific $S(\eta_1)\times S(\eta_2\cup\eta_3)$-submodules and are $S(\eta_1)\times S(\eta_2)\times S(\eta_3)$ invariant and which, restricted to appropriate S_N-orbits in $\boldsymbol{R}^N$, yield families of Hahn polynomials in two variables. Further the connection coefficients relating different bases, coming from $S(\eta_1) \times S(\eta_2 \cup \eta_3)$ and $S(\eta_1 \cup \eta_3) \times S(\eta_2)$ (etc.) decompositions, provide a finite group setting for the $6 - j$ symbols, originally calculated for the compact group $SU(2)$ by Racah. The techniques used here should be useful in the more difficult problem of finding orthogonal bases for functions on S_N which are biinvariant for $S(\eta_1) \times S(\eta_2) \times S(\eta_3)$ and the spherical functions.

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UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VA 22903