PACIFIC JOURNAL OF MATHEMATICS
Vol. 93, No. 1, 1981

# LIE ALGEBRAS AND HOPF ALGEBRAS

JOHN H. REINOEHL

**For a finite-dimensional Lie algebra $L$ over a field of characteristic 0, the structure of the Hopf algebra of representative functions of $L$, $\mathscr{H}(L)$, is known in some detail from Hochschild, Illinois J. Math., 1959 and 1960. Depending on these results, we demonstrate sufficient conditions for a Hopf algebra over an algebraically closed field of characteristic 0 to be isomorphic to $\mathscr{H}(L)$ for some Lie algebra $L$.**

Our Theorem 2.1 gives sufficient conditions for a Hopf algebra $A$ to be isomorphic to the algebra $\mathscr{H}(L)$ of representative functions for some Lie algebra $L$, and describes how $L$ may be obtained. To accomplish this, we rely on Hochschild's results concerning the structure of $\mathscr{H}(L)$ in [2] and [3]. For Theorem 2.1, suppose $A$ is a Hopf algebra over an algebraically closed field of characteristic 0 such that (1) $A$ is an integral domain, (2) there is a group isomorphism $\rho$ from the additive group of the primitive elements $P$ of $A$ onto the multiplicative group $Q$ of the group-like elements of $A$, (3) there is a finitely-generated subalgebra $B$ of $A$ such that $A$ is a free $B$-module with basis $Q$ and $\gamma(B) \subset B \otimes A$, where $\gamma$ is the comultiplication of $A$, and (4) the semisimple part $B_s$ of $B$ is a Hopf subalgebra of $A$ and has no proper affine unramified extension in any Hopf algebra containing $B_s$. Then if $L$ is the Lie subalgebra of the differentiations on $A$ consisting of those differentiations $\lambda$ satisfying $\lambda(\rho(p)) = \lambda(p)$ for all elements $p$ of $P$, $A \approx \mathscr{H}(L)$.

The preliminaries which follow define essential concepts and describe the dualization process by which $\mathscr{H}(L)$ is obtained from the universal enveloping algebra of $L$. Following the statement of Theorem 2.1, the remainder of the paper, the subsequent propositions, contribute to the proof of Theorem 2.1.

Since $A$ is a Hopf algebra, the elements of $A$ may be regarded as representative functions of the Lie algebra of differentiations on $A$, and hence by the restriction map, as representative functions of $L$. In Proposition 2.2, we show that the map $A \to \mathscr{H}(L)$ so defined is injective by an argument which in effect shows that the restriction of $L$ is algebraically dense in any finitely-generated Hopf subalgebra of $A$ containing $B$. The remainder of the paper shows that our map $A \to \mathscr{H}(L)$ is surjective. Proposition 2.3 shows that the image of $B_s$ is that portion of $\mathscr{H}(L)$ annihilated by the radical of $L$ by left translation. Proposition 2.4 shows similarly that the image of $F[P, Q]$ is the portion of $\mathscr{H}(L)$ annihilated by $[L, L]$ by left translation.

In the proofs of the two propositions just named, the arguments are concerned with a finitely-generated Hopf subalgebra of $A$ containing $B$, so the standard results of affine algebraic group theory are available. Lemma 2.5 is a technical result based on decomposition of submodules of $\mathscr{H}(L)$. This contributes to an induction argument that completes the demonstration that our map $A \to \mathscr{H}(L)$ is surjective.

We rely on [4] for notation and basic results in the theory of affine algebraic groups, and on [6] for results concerning pro-affine algebraic groups. Analogous results were obtained for complex analytic groups in [5]. This paper is a revised version of the author's doctoral dissertation. The author wishes to express his gratitude for the many suggestions offered by Dr. G. Hochschild, his disseration advisor. The referee has also made a number of helpful suggestions.

**1. Preliminaries.** Let $L$ be a Lie algebra over a field $F$. An element of the dual space of the universal enveloping algebra $\mathscr{U}(L)$ of $L$ which vanishes on an ideal of finite codimension is a *representative function* of $L$. The space of representative functions of $L$ is labeled $\mathscr{H}(L)$. A $\mathscr{U}(L)$-module structure for $\mathscr{H}(L)$ is defined for elements $u, v \in \mathscr{U}(L)$ and $f \in \mathscr{H}(L)$ by $(u \cdot f)(v) = f(vu)$ and $(f \cdot u)(v) = f(uv)$. The maps $f \to u \cdot f$ and $f \to f \cdot u$ are termed *left* and *right translations respectively*.

An algebra will be assumed to have an identity element and a Hopf algebra an antipode. A Hopf algebra structure may be defined for $\mathscr{U}(L)$ and the algebra and Hopf algebra structures of $\mathscr{H}(L)$ are induced by duality. The comultiplication of $\mathscr{U}(L)$ is the $F$-algebra homomorphism $d: \mathscr{U}(L) \to \mathscr{U}(L) \otimes \mathscr{U}(L)$ defined by $d(\lambda) = 1 \otimes \lambda + \lambda \otimes 1$ for all $\lambda \in L$. Hence for $f, h \in \mathscr{H}(L)$ and $u \in \mathscr{U}(L)$, $(fh)(u) = (f \otimes h)(d(u))$. Since $d$ is cocommutative, the algebra multiplication is commutative. If the elements of $\mathscr{H}(L) \otimes \mathscr{H}(L)$ are regarded as elements of the dual space of $\mathscr{U}(L) \otimes \mathscr{U}(L)$ in the natural way, the comultiplication $\gamma: \mathscr{H}(L) \to \mathscr{H}(L) \otimes \mathscr{H}(L)$ is defined by the equation $\gamma(f)(u \otimes v) = f(uv)$. The antipode $z$ on $\mathscr{U}(L)$ is the $F$-algebra antimorphism $z: \mathscr{U}(L) \to \mathscr{U}(L)$ defined for $\lambda \in L$ by $z(\lambda) = -\lambda$, and for $f \in \mathscr{H}(L)$ the antipode $\zeta$ is given by $\zeta(f) = f \circ z$. Details for the above are found in [7].

Let $A$ be a Hopf algebra with comultiplication $\gamma$, counit $\varepsilon$ and antipode $z$ over a field $F$. The $F$-algebra homomorphisms $A \to F$ constitute a *pro-affine algebraic group* labeled $\mathscr{G}(A)$, with group operations given by $g_1 g_2 = (g_1 \otimes g_2) \circ \gamma$, $g^{-1} = g \circ z$ and $1_G = \varepsilon$. If $A$ is commutative and finitely-generated as an algebra, i.e., affine, $\mathscr{G}(A)$ is an *affine algebraic group*. The $F$-multiples of the identity element of an algebra or Hopf algebra will be termed *constants*. The *dif-*

*ferentiations* on $A$ are the linear maps $A \to F$ which annihilate the constants and satisfy for any elements $a_1, a_2 \in A$, $\delta(a_1a_2) = \delta(a_1)\varepsilon(a_2) + \varepsilon(a_1)\delta(a_2)$. The differentiations on $A$ determine a Lie algebra labeled $\mathscr{L}(A)$ or $\mathscr{L}(\mathscr{G}(A))$ with Lie algebra bracket given by $[\delta_1, \delta_2] = (\delta_1 \otimes \delta_2 - \delta_2 \otimes \delta_1) \circ \gamma$.

**2. Lie algebras and Hopf algebras.** Prior to the statement of our main result, we state some essential definitions. The *primitive elements* of a Hopf algebra $A$ over a field $F$ with comultiplication $\gamma$ are those elements $p$ such that $\gamma(p) = 1 \otimes p + p \otimes 1$. These elements form an $F$-submodule of $A$ and also an additive subgroup, hence an $F$-group. The *group-like elements* are those nonzero elements $q$ which satisfy $\gamma(q) = q \otimes q$. The group-like elements of $A$ form a subgroup of the group of units of $A$: if $\zeta$ is the antipode of $A$ and $q$ is group-like, then $q^{-1} = \zeta(q)$.

The primitive elements of $\mathscr{H}(L)$ are those elements which annihilate the constants and $L^2\mathscr{U}(L)$. A primitive element $p$ of $\mathscr{H}(L)$ may hence be regarded as a Lie algebra homomorphism $L \to F$. We then define $\exp(p)$ as the unique homomorphism of associative algebras $\mathscr{U}(L) \to F$ such that $\exp(p)_L = p_L$, where the subscript denotes restriction to $L$. A homomorphism of associative algebras $\mathscr{U}(L) \to F$ is a group-like element of $\mathscr{H}(L)$.

If $B$ is a subspace of a Hopf algebra $A$ with comultiplication $\gamma$, $B$ will be termed *left stable* or *right stable* if it satisfies, respectively, $\gamma(B) \subset B \otimes A$ or $\gamma(B) \subset A \otimes B$, and *bistable* if it is both left and right stable. If $B$ is bistable and closed under the antipode of $A$, $B$ is termed *fully stable*. If $A$ is a Hopf algebra consisting of representative functions of a Lie algebra, these notions of right and left stability are equivalent to stability under right and left translation, respectively, by elements of $\mathscr{U}(L)$. This equivalence also holds for translation by elements of $\mathscr{G}(A)$, where for $g_1, g_2 \in \mathscr{G}(A)$ and $a \in A$, $g_1 \cdot a$ is defined by $(g_1 \cdot a)(g_2) = a(g_2g_1)$, etc.

Let $A$ be a Hopf algebra over a field $F$ such that $\mathscr{G}(A)$ separates the elements of $A$; i.e., for any two distinct elements $a_1, a_2 \in A$, an element $g$ of $\mathscr{G}(A)$ exists such that $g(a_1) \neq g(a_2)$. Label the $F$-space generated by left translation of an element $a$ of $A$ by elements of $\mathscr{G}(A)$ as $[a]$. If $[a]$ is semisimple as a representation space of $\mathscr{G}(A)$, $a$ is called a *semisimple element* of $A$. If $B$ is a left stable subalgebra of $A$ and $F$ is of characteristic 0, then $B_s$, the space of semisimple elements of $B$, is also a left stable subalgebra of $B$. $B_s$ is termed the *semisimple part* of $B$.

By *derivation* we shall mean a linear map $D$ from an algebra $A$ into an $A$-module $M$ (usually also $A$) which annihilates the constants and satisfies $D(a_1a_2) = D(a_1)a_2 + a_1D(a_2)$, where $a_1, a_2 \in A$. Let $A$ and

$B$ be affine $F$-algebras such that $A$ contains $B$ and is finitely-generated as a $B$-module. If for every $A$-module $M$, every $B$-linear derivation $A \to M$ is the zero map, then $A$ is termed an *affine unramified extension* of $B$.

THEOREM 2.1. *Let $A$ be a commutative Hopf algebra over an algebraically closed field of characteristic* 0 *such that:* (1) *$A$ is an integral domain;* (2) *there is a group isomorphism $\rho$ of the additive $F$-group $P$ of the primitive elements of $A$ onto the multiplicative group $Q$ of the group-like elements of $A$;* (3) *there is a finitely-generated left-stable subalgebra $B$ of $A$ such that $A = B[Q]$ and $A$ is a free $B$-module with basis $Q$; and* (4) *the semisimple part $B_s$ of $B$ is fully stable and has no affine proper unramified extension in any Hopf algebra containing $B_s$.*

*Label as $L$ the Lie subalgebra of $\mathscr{L}(A)$ consisting of those elements $\lambda$ which satisfy $\lambda(\rho(p)) = \lambda(p)$ for all $p \in P$. Then $A$ is isomorphic to $\mathscr{H}(L)$.*

Before starting the proof, we remark that the necessity of (1)-(4), if $A \approx \mathscr{H}(L)$ for some $L$, is known from [1], [2], and [3]. For (1), see [2, pg. 501]; for (2), [3, pg. 617-618]; for (3), [3, Thm. 1]; and for (4), [2, Thm. 5.1] and [1, Thms. 3.1 and 4.1]. Hence conditions (1)-(4) characterize those Hopf algebras $A$ such that $A \approx \mathscr{H}(L)$ for some Lie algebra $L$.

*Proof.* Let $\gamma$ be the comultiplication of $A$. For elements $a \in A$ and $\delta \in \mathscr{L}(A)$, the left translate $\delta \cdot a$ is defined to be $(i \otimes \delta) \circ \gamma(a)$, where $i$ denotes the identity map; similarly, $a \cdot \delta = (\delta \otimes i) \circ \gamma(a)$. If $\varepsilon$ is the counit of $A$, then the canonical homomorphism $\theta: A \to \mathscr{H}(\mathscr{L}(A))$ is given, for $u \in \mathscr{U}(\mathscr{L}(A))$, by $\{\theta(a)\}(u) = \varepsilon(u \cdot a)$. The translations by elements of $\mathscr{L}(A)$ may be shown to be derivations on $A$.

Since $F$ is an algebraically closed field of characteristic 0 and $A$ is an integral domain, $\mathscr{U}(\mathscr{L}(A))$ separates the elements of $A$, hence the canonical map $\theta: A \to \mathscr{H}(\mathscr{L}(A))$ is an injection. We define the Hopf algebra map $\Phi: A \to \mathscr{H}(L)$ by $\Phi(a) = \theta(a)_{\mathscr{U}(L)}$. In what follows, we shall identify an element $a$ of $A$ with its image by $\theta$ and write $a(u)$ or $u(a)$ for $\varepsilon(u \cdot a)$.

We show $P \subset B$. Let $p$ be a primitive element of $A$. Then $p = b_0 + \sum b_q q$ where $b_0$ and each $b_q$ are elements of $B$, and for each $q$, $q \neq 1$. We thus have $\gamma(p) = 1 \otimes p + p \otimes 1 = \gamma(b_0) + \sum \gamma(b_q)(q \otimes q)$. Since $B$ is left stable and the elements of $Q$ are free over $B$, necessarily $\gamma(b_0) = 1 \otimes p + b_0 \otimes 1$. Consequently, if $\mu$ is the algebra multiplication of $A$, $\mu \circ (\varepsilon \otimes i) \circ \gamma(b_0) = b_0 = \mu \circ (\varepsilon \otimes i)(1 \otimes p + b_0 \otimes 1) =$

$p + \varepsilon(b_0)$, hence $p = b_0$.

Let $\delta$ be a differentiation on $A$. To demonstrate the existence of an element $\lambda \in L$ which coincides with $\delta$ on $B$, define first $\lambda(b) = \delta(b)$ for all $b \in B$. Since $P \subset B$ and $\rho: P \to Q$ is a bijective map, $\lambda$ may be defined on $Q$ by $\lambda(\rho(p)) = \delta(p) = \lambda(p)$. This is verified to determine a differentiation on $F[Q]$ since for any group-like element $q$ of $A$, if $\varepsilon$ is the counit of $A$, $\varepsilon(q) = 1$. Since $A = B \otimes F[Q]$, $\lambda$ is a differentiation on $A$, and evidently $\lambda_B = \delta_B$ and $\lambda \in L$.

By the argument of [5, pg. 1145], which we don't repeat, if $C$ is a fully stable subalgebra of $A$ containing $B$, then $C = B[Q \cap C]$.

Observe that for arbitrary elements $p \in P$ and $q \in Q$,

$$[\mathscr{L}(A), \mathscr{L}(A)](p) = [\mathscr{L}(A), \mathscr{L}(A)](q) = 0 .$$

Thus $[\mathscr{L}(A), \mathscr{L}(A)] \subset L$ and $L$ is an ideal of $\mathscr{L}(A)$.

PROPOSITION 2.2. *The kernel of $\Phi$ is* (0).

*Proof.* Label the kernel of $\Phi$ by $I$; $I$ is an ideal of $A$. Assume $I$ is not (0) and let $a = \sum_{i=1}^{n} b_i q_i \in B[Q]$, with each $b_i \neq 0$, be a nonzero element of $I$ chosen with $n$ as small as possible. Since $aq_1^{-1} \in I$, we may assume $q_1 = 1$. Let $C$ be the smallest fully stable subalgebra of $A$ containing $B[q_2, \cdots, q_n]$. Since $B[q_2, \cdots, q_n]$ is finitely-generated, $C$ is also finitely-generated. Since $C$ contains $B$, the definition of the Lie subalgebra $L$ of $\mathscr{L}(A)$ implies that the restriction map $L \to L_C$ is injective. Label $\mathscr{G}(C)$ by $G$ and let $G_L$ be the smallest algebraic subgroup of $G$ whose Lie algebra contains $L_C$. Denote by $Y$ the stabilizer of $C \cap I$ in $G$ with respect to left translation; then $Y$ is an algebraic subgroup of $G$ and $\mathscr{L}(Y)$ is the stabilizer of $C \cap I$ in $\mathscr{L}(G)$ by left translation. Since $L$ clearly stabilizes $C \cap I$, $\mathscr{L}(Y) \supset L_C$, and thus $Y \supset G_L$ so $G_L$ stabilizes $C \cap I$.

We now show $B \cap I = (0)$. $L_C$ is an ideal of $\mathscr{L}(G)$ hence for any $u \in \mathscr{U}(L_C)$ and $s \in \mathscr{L}(G)$, $su = us + v$, where $v \in \mathscr{U}(L_C)$. For any element $b \in B \cap I$, we have $(s \cdot b)(u) = b(us) = b(su - v) = b(su) = s(u \cdot b)$. An element $\lambda \in L_C$ exists such that $\lambda_B = s_B$, whence $s(u \cdot b) = \lambda(u \cdot b) = b(\lambda u) = 0$ so we we conclude that $s \cdot b \in B \cap I$. This implies that $b$ is zero on $\mathscr{U}(\mathscr{L}(G))$, hence on $G$; thus $b = 0$ and $B \cap I = (0)$.

It follows that $b_1$ is nonzero on $\mathscr{U}(L_C)$ and hence on $G_L$. Thus translating our element $a \in C \cap I$ on the left by an element of $G_L$, it is seen that we may assume $b_1(1_G) \neq 0$, and multiplying by a constant, that $b_1(1_G) = 1$. Then for any $g \in G_L$,

$$g \cdot a = g \cdot b_1 + \sum_{i=2}^{n} (g \cdot b_i) q_i(g) q_i .$$

Since $(g \cdot b_1)a - b_1(g \cdot a)$ is an element of $C \subset I$, we obtain from the minimality of $n$ that $(g\cdot b_1)a - b_1(g\cdot a) = 0$. Hence for all $i$ between 2 and $n$, $(g\cdot b_i)b_i(1_G) = b_1(g\cdot b_i)q_i(g)$. Evaluating this expression for $g = 1_G$, we obtain:

$$(1) \qquad b_1(g)b_i(1_G) = b_i(g)q_i(g) \quad \text{for all} \quad g \in G_L \, .$$

For the duration of this proposition, for any element $c \in C$, we shall denote the restriction $c_{G_L}$ by $c'$. Since $b_1(1_G) = 1$, $a \neq b_1$, so $n > 1$. There exists $p \in P$ such that $\rho(p) = q_2 \neq 1$, and thus $p \neq 0$. Then for $i = 2$, we may write (1) as $b_1'b_2(1_G) = b_2'\rho(p)'$. Now consider the ideal $K$ of $B'$ consisting of those elements $b' \in B'$ such that $b'\rho(p)' \in B'$. Considering the left translates of $b_2'\rho(p)'$ by elements of $G_L$, since $B'$ is left stable, it is seen that $K$ has no zeros in $G_L$. By a previous remark, $C = B \otimes F[Q \cap C]$, so for the polynomial algebra $C'$ of $G_L$, we have $C' = B' \otimes F[Q \cap C]'$. Thus every $F$-algebra homomorphism $B' \to F$ is the evaluation at an element of $G_L$. Since $B'$ is finitely-generated, it follows that $K = B'$ and thus $\rho(p)' \in B'$.

We now show the existence of a differentiation $\delta$ on $C'$ such that $\delta(p') \neq \delta(\rho(p)')$. We accomplish this by showing that $p'$ and $\rho(p)'$ are algebraically independent. In that case, $p'$ is algebraically free over $F[\rho(p)', \rho(-p)']$ and thus a differentiation $\delta$ may be defined on $F[p', \rho(p)', \rho(-p)']$ such that $\delta(p') = 1$ and $\delta(\rho(p)') = \delta(\rho(-p)') = 0$. Since $F[p', \rho(p)', \rho(-p)']$ is fully stable, applying first Theorem 6.4 and then Theorem 7.5 of [4], one shows that $\delta$ may be extended to $C'$.

Assume that $p'$ and $\rho(p)'$ are algebraically dependent. $\mathscr{U}(\mathscr{L}(A))$ separates the elements of $A$ and any primitive element of $A$ annihilates $F$ and $\mathscr{L}(A)^2\mathscr{U}(\mathscr{L}(A))$. Since $p \neq 0$, there is necessarily a $\sigma \in \mathscr{L}(A)$ such that $\sigma(p) = 1$. Choose $\nu \in L$ such that $\nu_B = \sigma_B$, then $\nu(p') = \nu(\rho(p)') = 1$. Translating on the left by $\nu$, we obtain a derivation $D$ on $C'$; $D(p') = \nu\cdot p' = 1$ and $D(\rho(p)') = \nu\cdot\rho(p)' = \rho(p)'$. By assumption, we have polynomials $f_i(t) \in F[t]$ such that:

$$(2) \qquad \sum_{i=0}^{m} f_i(p')[\rho(p)']^i = 0 \, .$$

We choose $m$ as small as possible with not all $f_i = 0$. Since $[\rho(p)']^{-1} = \rho(-p)' \in C'$, we may assume that $f_0(t) \neq 0$. Applying $D$ $k$ times to (2), we obtain

$$\sum_{i=0}^{m} f_{ik}(p')[\rho(p)']^i = 0, \text{ where } f_{i0}(t) = f_i(t) \text{ and } f_{ij+1}(t) = if_{ij}(t) + f_{ij}^*(t); \ f^* \text{ denotes the formal derivative of } f \, .$$

For large enough $k$, necessarily $f_{0k}(t) = 0$. The minimality of $m$ then implies that for each $i$, $f_{ik}(t) = 0$. From our recursion formula, it is clear, if $i > 0$ and $j > 0$, that if $f_{ij}(t) = 0$, then $f_{ij-1}(t) = 0$, so $n = 0$.

In this case, if $f_0(t)$ is of degree $r$, applying $D$ $r-1$ times to $f_0(p') = 0$, we obtain $p' = 0$, a contradiction, so $p'$ and $\rho(p')$ are algebraically independent.

Let $\delta$ be a differentiation on $C'$ such that $\delta(p') \neq \delta(\rho(p)')$ and let $\pi$ be the restriction map $C \to C'$. Then $\delta \circ \pi$ is a differentiation on $C$. On $B \subset C$, $\delta \circ \pi$ coincides with some element $\lambda \in L$. Moreover, since $\rho(p)' \in B'$, a $b \in B$ exists such that $b' = \rho(p)'$. Since $\lambda$ annihilates the kernel of $\pi$, $\lambda$ induces a differentiation $\lambda^\circ$ on $C'$ such that $\lambda^\circ \circ \pi = \lambda$; for any $c \in C$, $\lambda^\circ(c') = \lambda(c)$. Therefore:

$$\lambda(\rho(p)) = \lambda^\circ(\rho(p)') = \lambda^\circ(b') = \lambda(b) = (\delta \circ \pi)(b) = \delta(b') = \delta(\rho(p)') .$$

Also: $\lambda(p) = (\delta \circ \pi)(p) = \delta(p')$. Thus $\lambda(\rho(p)) \neq \lambda(p)$. This is contrary to the definition of $L$, so the assumption that $I$ is nonzero is false. This completes the proof of Proposition 2.2.

Label by $J$ the subspace of $\mathscr{L}(A)$ consisting of those differentiations which annihilate $B$. Then $J$ is a Lie subalgebra of $\mathscr{L}(A)$ because $B$ is left stable. Since the elements of $J$ annihilate the primitive elements of $A$, $J \cap L = (0)$, but since $[J, J] \subset L$, it follows that $J$ is abelian. For an element $\delta \in \mathscr{L}(A)$, let $\lambda$ be the element of $L$ such that $\delta_B = \lambda_B$; then $\delta - \lambda \in J$. Thus we obtain the semidirect sum decomposition $\mathscr{L}(A) = L + J$. From now on, we label the radical of $\mathscr{L}(A)$ by $R$ and the radical of $L$ by $N$. $\mathscr{H}(L)^N$ signifies the portion of $\mathscr{H}(L)$ annihilated by left translation by $N$.

Let $G$ and $G'$ be connected affine algebraic groups over an algebraically closed field of characteristic 0. Then a rational surjection $G' \to G$ with a finite kernel is a *group covering* of $G$, and $G$ is *simply connected* if every group covering of $G$ has a trivial kernel.

PROPOSITION 2.3. $\Phi(B_s) = \mathscr{H}(L)^N$.

*Proof*. Let $B^*$ be the smallest fully stable subalgebra of $A$ containing $B$. Then $B^*$ is finitely-generated, hence the fully stable sublagebra $B_s$ of $B^*$ is also finitely-generated. Label the representation of $\mathscr{L}(A)$ on $B_s$ induced by left translation as $\tau$. From the bistability of $B_s$ it follows that $J$ annihilates $B_s$ by left translation so $J \subset \ker(\tau)$.

From condition (4) of Theorem 2.1, $\mathscr{G}(B_s)$ is a simply-connected, reductive, affine algebraic group [1, Thm. 4.1]. Hence $\mathscr{G}(B_s)/[\mathscr{G}(B_s), \mathscr{G}(B_s)]$ is a simply-connected, reductive, abelian and therefore trivial affine algebraic group [1, Thm. 2.3], so $\mathscr{G}(B_s)$ is semisimple. By arguments analogous to [6, Thm. 2.1], every differentiation on $B_s$ may be extended to a differentiation on $A$, because $B_s$ is fully stable. Hence $\mathscr{L}(A)/\ker(\tau) \approx \mathscr{L}(B_s)$, so $R \subset \ker(\tau)$.

Since $L$ is an ideal of $\mathscr{L}(A)$, $N \subset R$, so $N \subset \ker(\tau) \cap L$. We now show $\ker(\tau) \cap L = N$. Let $S$ be any maximal semisimple Lie subalgebra of $L$; then $L = N + S$ and $\ker(\tau) \cap L = N + (\ker(\tau) \cap S)$. Since $[S, S] = S$, the action of $S$ on $Q$ by left translation is trivial. Hence $\ker(\tau) \cap S$ acts trivially by left translation on $B_s[B^* \cap Q]$ which is readily verified to be $B_s^*$, the semisimple part of $B^*$. The kernel of the restriction Lie algebra homomorphism $\mathscr{L}(B^*) \to \mathscr{L}(B_s^*)$ is exactly the Lie algebra of the unipotent radical of $\mathscr{G}(B^*)$, hence the action of $\ker(\tau) \cap S$ on $B^*$ by left translation is locally nilpotent. On the other hand, $\ker(\tau) \cap S$ is a semisimple Lie algebra so the action of $\ker(\tau) \cap S$ on $B^*$ is trivial. Since $S$ acts trivially on $Q$, we have $\ker(\tau) \cap S = (0)$, and hence $\ker(\tau) \cap L = N$. It was observed that $J \subset \ker(\tau)$, consequently $[L(A), J] \subset \ker(\tau) \cap N$, hence $J + N$ is a solvable ideal of $\mathscr{L}(A)$. Therefore $\mathscr{L}(A)/(J + N) = (J + L)/(J + N) \approx L/N$, and so $R = J + N$. Since $\ker(\tau) \cap L = N$, we infer that $\mathscr{L}(B_s) \approx L/N$. Since $N \subset \ker(\tau)$, we have $\Phi(B_s) \subset \mathscr{H}(L)^N$.

The canonical projection $\mathscr{U}(L) \to \mathscr{U}(L/N)$ dualizes to the natural Hopf algebra injection $\mathscr{H}(L/N) \to \mathscr{H}(L)$, the image of which is $H(L)^N$. Since $\mathscr{L}(\mathscr{H}(L/N)) \approx L/N$ [1, Thm. 6.1], the injection $\Phi\colon B_s \to \mathscr{H}(L)^N$ induces a surjective rational homomorphism $\Gamma\colon \mathscr{G}(\mathscr{H}(L)^N) \to \mathscr{G}(B_s)$ whose differential is a Lie algebra isomorphism. Hence $\Gamma$ is a group covering. But by hypothesis (4) and [1, Thm. 4.1], $\mathscr{G}(B_s)$ is simply connected, hence $\Phi(B_s) = \mathscr{H}(L)^N$. This completes the proof of Proposition 2.3.

Label $\mathscr{G}(B^*)$ as $D$. The Lie subalgebra $L_{B^*}$ of $\mathscr{L}(D)$ is isomorphic to $L$, hence unless otherwise noted, $L$ will be identified with $L_{B^*}$. The Lie subalgebra $J_{B^*}$ of $\mathscr{L}(D)$ will be labeled $J'$; it is seen that $J'$ consists of those elements of $\mathscr{L}(D)$ that are 0 on $B$. Since the restriction map is surjective [cf: 6, Thm. 2.1], we obtain the semidirect sum decomposition $\mathscr{L}(D) = L + J'$. Since $B$ is left stable, the differentiations which are 0 on $B$ coincide with those which annihilate $B$ by right translation. Therefore, if $X$ denotes the right fixer of $B$ in $D$, then $X$ is an algebraic subgroup of $D$ whose Lie algebra is $J'$. From the decomposition $B^* = B \otimes F[B^* \cap Q]$, it may be inferred that $X$ and $\mathscr{G}(F[B^* \cap Q])$ are isomorphic as affine algebraic groups. Since $F[B^* \cap Q]$ is a fully stable subalgebra of $B^*$, it follows that $F[B^* \cap Q]$ is finitely-generated. Since $F[B^* \cap Q]$ is generated by its group-like elements, $X$ is a reductive, abelian, affine algebraic group.

Consider a standard semidirect product decomposition $D = D_u \cdot M$, where $M$ is a maximal reductive subgroup of $D$ chosen to contain $X$ and $D_u$ is the unipotent radical of $D$. Since $\mathscr{L}(M) \supset J'$, we have the semidirect sum decomposition $\mathscr{L}(M) = (\mathscr{L}(M) \cap L) + J'$. If we label $\mathscr{L}(M) \cap L$ as $S$, since $B_s$ is fully stable, we obtain

$\mathscr{L}(B_s) = \mathscr{L}(M)_{B_s} = S_{B_s} + J'_{B_s} = S_{B_s}$. The reductive component $M$ of $D$ is represented faithfully on $B_s^* = B_s \otimes F[B^* \cap Q]$, therefore, in particular, $S$ is also represented faithfully on $B_s^*$. For any primitive element $p \in B^*$, $p(M) = 0$, so $p(S) = 0$. Thus, since $S \subset L$, for any element $q \in B^* \cap Q$, we have $q(S) = 0$. Consequently, the representation of $S$ on $F[B^* \cap Q]$ is trivial, and $S$ is represented faithfully on $B_s$. Hence $S \approx S_{B_s} = \mathscr{L}(B_s)$ and $S$ is semisimple. It is clear that $S$ is a maximal semisimple Lie subalgebra of $\mathscr{L}(D)$ and hence of $L$.

If $D_L$ is the smallest algebraic subgroup of $D$ whose Lie algebra contains $L$, it follows from Proposition 2.2 that the $D_L$-fixed portion of $B^*$ is the constants. Thus $D_L = D$, so by [4, Prop. 13.1], $[\mathscr{L}(D), \mathscr{L}(D)] = [L, L]$. Label the radical $[L, N]$ af $[L, L]$ by $T$. Then $T$ is a nilpotent ideal of both $L$ and $\mathscr{L}(D)$ and $T \subset \mathscr{L}(D_u)$; note also that $[L, L] = [S, S] + [L, N] = S + T$. We label by $D_T$ and $D_S$ the algebraic subgroups of $D$ whose Lie algebras are $T$ and $S$ respectively. It is seen that the algebraic subgroup $D_S D_T$ coincides with $[D, D]$. Since $MD_T \supset D_S D_T$ it follows that $MD_T$ is a normal algebraic subgroup of $D$ and $D/MD_T$ is an abelian, unipotent affine algebraic group, i.e., an algebraic vector group. The polynomial functions of an algebraic vector group are generated by the homomorphisms of that group into $F^+$. This leads to the following result.

PROPOSITION 2.4. *$\Phi(F[P, Q])$ is the natural image of $\mathscr{H}(L/[L, L])$ in $\mathscr{H}(L)$, and the exponential map of $\mathscr{H}(L)$ is given by $\Phi \circ \rho \circ \Phi^{-1}$ on the primitive elements $\Phi(P)$ of $\mathscr{H}(L)$.*

*Proof.* The $F$-space of primitive elements of $B^*$ is the space $P$ of primitive elements of $A$. Considered as functions on $D$, these elements are exactly the elements of $\mathrm{Hom}(D, F^+)$. Since the subgroup $MD_T$ of $D$ is necessarily in the kernel of any such homomorphism, one easily verifies that $P = \mathrm{Hom}(D/MD_T, F^+)$. Thus $F[P]$ is the algebra of polynomial functions of $D/MD_T$ and $\mathscr{L}(D/MD_T)$ is simply $\mathscr{L}(F[P])$. Since $\mathscr{L}(D) = L + J' = N + S + J'$, and $\mathscr{L}(MD_T) = T + S + J'$, we see that $\mathscr{L}(D/MD_T)$ can also be written as $(L/[L, L])_{F[P]}$ or $(N/T)_{F[P]}$. Thus $\Phi(F(P))$ is the algebra of all $\mathscr{U}(L)$-nilpotent representative functions of $L$ that are trivial on $[L, L]$. Further, $\Phi(P)$ is the $F$-space of primitive elements of $\mathscr{H}(L)^{[L,L]}$, which is the image of $\mathscr{H}(L/[L, L])$ by the natural injection.

Now consider the elements of $\rho(P) = Q$. For any element $p$ of $P$, if $\lambda \in L \subset \mathscr{L}(A)$, then by definition of $L$, $\lambda(p) = \lambda(\rho(p))$. Thus $\Phi(\rho(p))$ is a group-like element of $\mathscr{H}(L)$ with the property that $\Phi(\rho(p))_L = \Phi(p)_L$. It follows from our remarks concerning the exponentials of $\mathscr{H}(L)$ in the introduction that $\Phi(\rho(p)) = \exp(\Phi(p))$.

From [2, pg. 519], $\mathscr{H}(L/[L, L])$ is generated by its primitive elements and the exponentials of its primitive elements. Thus $\Phi(F[P, Q])$ coincides with the image of $\mathscr{H}(L/[L, L])$ by the natural injection. This completes the proof of Proposition 2.4.

Lemma 2.5 is necessary for the induction argument which follows.

LEMMA 2.5. *Let $K$ be a finite-dimensional Lie algebra over $F$ and suppose there exists a semidirect sum decomposition $K = H + V$ where $V$ is one-dimensional and $H$ is an ideal of $K$. Let $C_1$ and $C_2$ be any left $V$-stable ${}^H\mathscr{H}(K)$-submodules of $\mathscr{H}(K)$ containing ${}^H\mathscr{H}(K)$ such that $(C_1)_{\mathscr{U}(H)} = (C_2)_{\mathscr{U}(H)}$. Then $C_1 = C_2$.*

*Proof.* Since $V$ is one-dimensional, the elements of ${}^H\mathscr{H}(K)$ have the form $\sum_{i=0}^n x_i p^i$, where $p$ is a nonzero primitive element and the $x_i$'s are linear combinations of the elements $\exp(\alpha p)$ where $\alpha \in F$ [2, pg. 519].

Let $\sigma$ be an element of $V$ such that $\sigma(p) = 1$. Let $E$ be the $F$-algebra of $F$-linear endomorphisms of ${}^H\mathscr{H}(K)$ that is generated by left translation by $\sigma$ and multiplication by elements $\exp(\alpha p)$. We shall show in two stages of induction that for any nonzero element $u$ of ${}^H\mathscr{H}(K)$, there is an element $e$ of $E$ such that $e(u) = 1$. First consider an $F$-linear combination of elements $\exp(\alpha p)$, say $x = \sum_{i=1}^k \beta_i \exp(\alpha_i p)$, where each $\alpha_i$, $\beta_i \in F$ and the $\alpha_i$'s are distinct. If $k = 1$, an endomorphism $e$ such that $e(x) = 1$ clearly exists. If $k > 1$, then $s \cdot x - \alpha_k x$ is a linear combination of $k - 1$ elements, so the result follows in this case from the induction assumption.

Now consider $u = \sum_{i=0}^n x_i p^i$ with each $x_i$ a linear combination of elements $\exp(\alpha p)$ and $x_n \neq 0$. Our result is known in the case $n = 0$. If $n > 0$, pick an element $e$ of $E$ such that $e(x_n) = 1$. Then a straightforward calculation shows that $e(x_n p^n) = p^n + \{$terms in smaller powers of $p\}$, so $e(u) = p^n + \sum_{i=0}^{n-1} x_i' p^i$, with the $x_i'$ being new linear combinations of the exponentials. Translating by $\sigma$, $\sigma \cdot e(u) = (n + \sigma \cdot x_{n-1}')p^{n-1} + \{$terms in smaller powers of $p\}$. Since $\sigma \cdot x_{n-1}'$ cannot be a constant other than 0, $\sigma \cdot e(u)$ has degree $n - 1$ in $p$. Hence the desired conclusion follows from the inductive hypothesis.

Let $C$ be any left $V$-stable ${}^H\mathscr{H}(K)$-submodule of $\mathscr{H}(K)$ containing ${}^H\mathscr{H}(K)$. We show $C = {}^H\mathscr{H}(K) \otimes C^V$. Suppose that $C$ contains elements not in ${}^H\mathscr{H}(K) \otimes C^V$. Then, since $\mathscr{H}(K) = {}^H\mathscr{H}(K) \otimes \mathscr{H}(K)^V$ [2, pg. 515], we may pick a nonzero element $c = \sum_{i=1}^r u_i \otimes v_i \in C \backslash {}^H\mathscr{H}(K) \otimes C^V$ with $u_i \in {}^H\mathscr{H}(K)$, $v_i \in \mathscr{H}(K)^V$ and $r$ as small as possible. From the minimality of $r$, it is immediate that no $v_i$ is contained in $C^V$ and the $u_i$'s are linearly independent. Necessarily also, if $r > 1$, no nontrivial linear combination of the $v_i$'s is contained in $C^V$. If this were so, multiplying by a constant and reordering if necessary, we would have $w = v_1 + \sum_{i=2}^{r} \beta_i v_i$ where $w \in C^V$ and $\beta_i \in F$ for all $i$. Then the element $c - w \otimes u_1$ of $C \backslash {}^H\mathscr{H}(K) \otimes C^V$, which is nonzero since the $u_i$'s are linearly independent, could be written as the sum of $r - 1$ terms.

Choose an element $e \in E$ such that $e(u_r) = 1$. Then from the definition of $E$, $e(c) = \sum_{i=1}^{r-1} e(u_i) \otimes v_i + u_r \in C$. If for each $i$, $e(u_i) \in F$, then

$$e(c) = \sum_{i=1}^{r} e(u_i)v_i \in C \cap \mathscr{H}(K)^V = C^V$$

an immediate contradiction, so we assume that some $e(u_i) \notin F$. Then $r > 1$ and

$$\sigma \cdot e(c) = \sum_{i=1}^{r-1} \sigma \cdot e(u_i) \otimes v_i \in C$$

is nonzero. Since $\sigma \cdot e(c)$ can be written with fewer than $r$ terms, $\sigma \cdot e(c) \in {}^H\mathscr{H}(K) \otimes C^V$. Thus if $\sigma \cdot e(c)$ is written in the form $\sum_{j=1}^{t} u'_j \otimes w_j$ where the elements $u'_j$ are $F$-linearly independent elements of ${}^H\mathscr{H}(K)$ and the $w_j$'s are nontrivial $F$-linear combinations of the $v_i$'s, necessarily each $w_j \in C^V$. This is again a contradiction.

Thus in particular, $C_1 = {}^H\mathscr{H}(K) \otimes C_1^V$ and $C_2 = {}^H\mathscr{H}(K) \otimes C_2^V$. Combining these equations with our hypothesis, $(C_1^V)_{\mathscr{U}(H)} = (C_1)_{\mathscr{U}(H)} = (C_2)_{\mathscr{U}(H)} = (C_2^V)_{\mathscr{U}(H)}$. From the semidirect sum decomposition $K = V + H$, the restriction maps $C_i^V \to (C_i^V)_{\mathscr{U}(H)}$ are bijective for $i = 1, 2$. Thus we obtain that $C_1^V = C_2^V$ and $C_1 = C_2$. This completes the proof of Lemma 2.5.

Recall that $N$ is the radical of $L$ and consider a chain of $F$-subspaces of $L$: $H_0 = T \subset \cdots \subset H_i \subset \cdots \subset H_m = N$ for which the dimension of $H_{i+1}$ is one greater than that of $H_i$ for all $i$. Since $[H_i, H_i] \subset [H_i, H_{i+1}] \subset [L, N] \subset T$, $H_i$ is a Lie subalgebra of $L$ and an ideal of $H_{i+1}$ for all $i$. We employ induction on $i$ to show that $\mathscr{H}(L)_{\mathscr{U}(H_i)} = ({}^SA)_{\mathscr{U}(H_i)}$, in particular for $H_i = N$. By [2, pg. 516], $\mathscr{H}(L)_{\mathscr{U}(H_i)}$ is the algebra of all representative functions of $H_i$ whose restrictions to $\mathscr{U}(T)$ are nilpotent representative functions of $T$, i.e., representative functions of $T$ for which the $F$-space of left $\mathscr{U}(T)$-translates is nilpotent as a left $\mathscr{U}(T)$-module.

Corresponding to the semidirect product decomposition $D = M \cdot D_u$, we have the tensor product decomposition $B^* = {}^MB^* \otimes B^{*D_u}$, where ${}^MB^*$ is the right $M$-fixed part of $B^*$. $({}^MB^*)_{D_T}$ is the algebra of polynomial functions of the unipotent affine algebraic group $D_T$, hence, regarded as representative functions on $\mathscr{U}(T)$, $({}^MB^*)_{\mathscr{U}(T)}$ consists of the nilpotent representative functions of $T$. Recall that ${}^SA$ is the right $S$-annihilated part of $A$. Since ${}^{S}A \supset M_{B^*}$, it follows that $({}^{S}A)_{\mathscr{U}(T)} \supset ({}^{M}B^*)_{\mathscr{U}(T)} = \mathscr{H}(L)_{\mathscr{U}(T)} \supset ({}^{S}A)_{\mathscr{U}(T)}$, which establishes the case $i = 0$.

For $k < m$, if $\sigma$ is any element of $H_{k+1} \backslash H_k$, we choose $V = F\sigma$ and apply Lemma 2.5. Clearly $({}^{S}A)_{\mathscr{U}(H_{k+1})}$ and $\mathscr{H}(L)_{\mathscr{U}(H_{k+1})}$ are left $V$-stable. Also, recalling from Proposition 2.4 that $\mathscr{L}(F[P])$ can be written as $(N/T)_{F[P]}$, it is seen from [2, pg. 519] that

$$^{H_k}\mathscr{H}(H_{k+1}) = (^{H_k}(F[P, Q]))_{\mathscr{U}(H_{k+1})} \subset ({}^{S}A)_{\mathscr{U}(H_{k+1})} .$$

Therefore the inductive hypothesis and Lemma 2.5 give

$$({}^{S}A)_{\mathscr{U}(H_{k+1})} = \mathscr{H}(L)_{\mathscr{U}(H_{k+1})} .$$

Thus we obtain $({}^{S}A)_{\mathscr{U}(N)} = \mathscr{H}(L)_{\mathscr{U}(N)}$. In view of the semidirect sum decomposition $L = N + S$, it follows that $\Phi({}^{S}A) = {}^{S}\mathscr{H}(L) \approx \mathscr{H}(L)_{\mathscr{U}(N)}$.

Finally, $\Phi(A) = \Phi({}^{S}A)\Phi(B_s) = {}^{S}\mathscr{H}(L)\mathscr{H}(L)^N = \mathscr{H}(L)$. Hence $\Phi$ is surjective, and therefore $\Phi$ is indeed a Hopf algebra isomorphism $A \rightarrow \mathscr{H}(L)$ and the proof of Theorem 2.1 is complete.

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Received March 20, 1978 and in revised form June 9, 1980.

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NEW ORLEANS, LA