

# INCOMPRESSIBILITY OF MAPS AND THE HOMOTOPY INVARIANCE OF ČECH COHOMOLOGY

ALLAN CALDER AND FRANK WILLIAMS

The  $\Omega$ -compressibility dimension of a space  $Y$  is the largest integer  $r$  for which every map  $f: X \rightarrow Y$  from a normal space with dimension less than  $r$ , the loop map  $\Omega f: \Omega X \rightarrow \Omega Y$  is compressible. Bounds are determined for the  $\Omega$ -compressibility dimension of Eilenberg-Maclane spaces of type  $(Z, 2n)$  and  $(Z_p k, 2n)$ . In application this is used to settle the question as to when Čech cohomology based on finite covers is a homotopy invariant functor.

**1. Compressibility: Background and statement results.** A map is called *compressible* if it is homotopic to a map whose image is contained in a compact subset of the target space. Let  $\Omega$  denote the loop functor. A map  $f: X \rightarrow Y$  is called  $\Omega$ -*compressible* if  $\Omega f: \Omega X \rightarrow \Omega Y$  is compressible.

Obviously if  $\Omega Y$  has the homotopy type of a compact space then any map into  $Y$  is  $\Omega$ -compressible. On the other hand if  $\Omega Y$  has nonzero homology (or homotopy) groups in infinitely many dimensions then it is easy to see that the identity map  $1_Y: Y \rightarrow Y$  is  $\Omega$ -incompressible. In particular that will be the case when  $Y$  is a finite complex with nonzero (reduced) homology.

We shall be primarily concerned with the case  $Y = K(G, n)$ , an Eilenberg-Maclane space. Any essential map  $f: S^{2n+1} \rightarrow K(Z, 2n+1)$  is  $\Omega$ -incompressible since  $\Omega f$  induces an isomorphism,

$$(\Omega f)^*: H^*(\Omega K(Z, 2n+1); Q) \longrightarrow H^*(\Omega S^{2n+1}; Q),$$

in rational cohomology. The following fundamental result is due to Weingram [11] (see also [9]).

**WEINGRAM'S THEOREM.** *For any finitely generated abelian group  $G$ , every essential map  $f: S^{2n+1} \rightarrow K(G, 2n+1)$  is  $\Omega$ -incompressible.*

For even dimensional spheres the situation is more complicated: if  $p$  is an odd prime then any map  $S^{2n} \rightarrow K(Z_p k, 2n)$  is  $\Omega$ -compressible, since  $S^{2n-1}$  is a  $H$ -space mod  $p$  (see [9] for details). Similarly, if  $n = 1, 2$  or  $4$  then any map  $S^{2n} \rightarrow K(G, 2n)$  is  $\Omega$ -compressible for any  $G$ .

Any map from an  $n$ -dimensional (covering dimension) normal space into a CW-complex can be homotopically deformed into a map whose image is in the  $n$ -skeleton of the target space [see e.g., 6 p.

356]. The  $n$ -skelton of  $K(Z, n)$  (or  $K(Z_p k, n)$ ) can be considered to be  $S^n$ . So if the inclusion of  $S^n$  is  $\Omega$ -compressible then so is every map from any  $n$ -dimensional space into  $K(Z, n)$  (or  $K(Z_p k, n)$ ). When this is the case, it seems reasonable to ask what the dimension of the domain must be in order that there be an  $\Omega$ -incompressible map into  $K(Z, n)$  (or  $K(Z_p k, n)$ ).

**DEFINITION 1.1.** The  $\Omega$ -compressibility dimension of a space  $Y$ , denoted by  $c(Y)$ , is the greatest integer  $r$  such that if  $X$  is a normal space of dimension less than  $r$  then every map  $X \rightarrow Y$  is  $\Omega$ -compressible. If no such greatest integer exists we set  $c(Y) = \infty$ .

If  $Y = K(G, n)$ , we denote  $c(Y)$  by  $c(G, n)$ .

**EXAMPLE 1.2.** From preceding remarks, we have:

- (1)  $c(Y) = \infty$  if  $\Omega Y$  is homotopic to a compact space. Thus  $c(Z, 2) = \infty$ .
- (2)  $c(Z, 2n + 1) = c(Z_p k, 2n + 1) = 2n + 1$ .
- (3)  $c(Z_p k, 2n) > 2n$  if  $n \geq 1$  and  $p$  is odd.
- (4)  $c(G, 2n) > 2n$  if  $n = 1, 2$  or  $4$ .

In this paper we show that  $c(Z, 2n)$  and  $c(Z_p k, 2n)$  are finite for most values of  $n$ . Specifically in § 3 we prove:

**THEOREM 1.3.** If  $n \geq 1$ ,  $c(Z_p k, 2n) \leq 2n + p - 1$ .

**THEOREM 1.4.** If  $n \geq 2$ ,  $c(Z, 2n) \leq 2n + 9$ .

We also wish to consider the concept analogous to  $c(G, n)$  when  $G = Q$ . Since  $K(Q, n)$  is not of finite type, we define  $c^*(Y)$  in the same way as  $c(Y)$  but restricting the domain spaces  $X$  to be compact. Thus if  $G$  is finitely generated,  $c^*(G, n) = c(G, n)$ . The following table summarizes the known results for  $c(G, n)$  and  $c^*(G, n)$  for  $G = Z_p k, Z$  and  $Q$ .

TABLE 1

$n \setminus G$	$c(G, n) = c^*(G, n)$			$c^*(G, n)$
$n$	$Z_2 k$	$Z_p k$ ( $p \neq 2$ )	$Z$	$Q$
1	$\infty$	$\infty$	1	1
odd $> 1$	$n$	$n$	$n$	$n$
2			$\infty$	
4, 8, 16	$n < c \leq n + 3$	$n < c \leq n + 2p - 1$	$n \leq c \leq n + 9$	$\infty$
other even	$n \leq c \leq n + 3$			

The entries in Table 1 come from 1.2, 1.3, 1.4, and Table 2 in the next section.

It is still an open question as to what the exact values of  $c(G, n)$  are in the last line of the table.

**2. Čech cohomology: Background and statement of results.** In 1932 Čech defined cohomology groups for an arbitrary space  $X$  using the system of nerves of finite covers of  $X$ , [5]. These groups denoted by  $H_f^*(X)$ , have been ignored in favor of those based on arbitrary or locally finite covers, since Dowker showed that  $H_f^1(X)$  is not a homotopy invariant of the space  $X$ , [7]. However it was subsequently shown in [1] that if  $k > 1$  and  $G$  is finitely generated then  $H_f^k(-; G)$  is a homotopy invariant functor on the category of finite dimensional normal spaces.

The homotopy invariance of  $H_f^k(-; G)$  for larger categories was investigated in [4] using category-theoretic methods. Of primary importance there was the following theorem, in which  $\beta$  denotes the Stone-Čech functor.

**THEOREM 2.1.** *On the category of completely regular Hausdorff spaces the functors  $H_f^k(-; G)$  and  $[\beta -, K(G, k)]$  are naturally isomorphic.*

*Proof.* By [8] p. 282 and [6] p. 366.

This theorem was used to apply results from [2, 3] concerning the homotopy invariance of functors of the form  $[\beta -, Y]$ , to  $H_f^k(-; G)$ . We summarize the known results in the following table, which holds when  $C$  is any full subcategory of the category of completely regular Hausdorff spaces that contains all countable CW-complexes.

TABLE 2. Is  $H_f^n(-; G)$  a homotopy functor on  $C$ ?

$n \backslash G$	$Z_p k$	$Z$	$Q$
1	YES	NO	NO
2		YES	YES
$2l + 1$ ( $l \geq 1$ )	NO	NO	NO
$2l$ ( $l > 1$ )			YES

In § 4 we will complete the table by proving the following theorem.

**THEOREM 2.2.** *The functors  $H^{2n}(-; Z_p k)$  for  $n \geq 1$  and  $H^{2n}(-; Z)$  for  $n > 1$  are not homotopy invariant.*

Thus the blanks in the table can be filled in with “NO’s”.

**3. Proofs of Theorems 1.3 and 1.4.** The proofs of the theorems will be applications of a basic technical result, 3.1, the method of proof of which is essentially due to Goldfeather [9]. We attempt to follow Goldfeather’s proof as closely as possible so as to make it clear where our arguments extend his and we include complete details, as the subtle nature of incompressibility seems to require fastidious exposition.

By a divided polynomial algebra over  $Z$  we shall mean a free graded abelian group with generators  $x_1, x_2, \dots$  such that the algebra structure is given by  $(x_i)^i = (i!)x_i$ .

Let  $\rho_p: H^*(X; Z) \rightarrow H^*(X; Z_p)$  be reduction mod  $p$ .

**THEOREM 3.1.** *Let  $f: X \rightarrow Y$ . Suppose that there exists  $x \in H^*(\Omega X; Z)$  such that*

- (a)  *$x$  generates a divided polynomial algebra that is a summand of  $H^*(\Omega X; Z)$ ;*
- (b)  *$\rho_p(x) \in \text{Im } \{(\Omega f)^*: H^*(\Omega Y; Z) \rightarrow H^*(\Omega X; Z_p)\}$ ; and*
- (c)  *$\tilde{H}^k(X; Q) = 0$  for  $k \neq \deg(x) + 1$ .*

*Then  $f$  is  $\Omega$ -incompressible.*

*Proof.* Let  $N = \text{degree of } x$ . By hypothesis there exist:

- (1)  $\chi \in H^*(\Omega Y; Z_p)$  such that  $(\Omega f)^*(\chi) = \rho_p(x)$ ;
- (2)  $\eta: \Omega Y \rightarrow K(Z_p, N)$  given by  $\eta^*(\varepsilon_N) = \chi$ ; and
- (3) a fibration  $g: \Omega X \rightarrow K(Z_p, N)$  given by  $g^*(\varepsilon_N) = \rho_p(x)$  where  $\varepsilon_N$  denotes a generator of  $H^N(K(Z_p, N))$ .

Let  $F$  denote the fiber of  $g$  and consider the commutative diagram;

$$\begin{array}{ccc} F & \longrightarrow & F \\ i_K \downarrow & & \downarrow i \\ \Omega X[K] & \xrightarrow{j_K} & \Omega X \\ g_K \downarrow & & \downarrow g \\ K(Z_p, N)^{(K)} & \xrightarrow{\text{incl.}} & K(Z_p, N) \end{array}$$

where  $\Omega X[K] = g^{-1}(K(Z_p, N)^{(K)})$ , the portion of  $\Omega X$  lying over the  $K$ -skeleton of  $K(Z_p, n)$ .

Suppose now that the divided polynomial algebra generated by  $x$  is given by elements  $x_k \in H^{kN}(\Omega X; Z)$ ,  $k = 1, 2, \dots$ , such that  $x^k = k! x_k$ .

Since  $\rho_p i^*(x) = i^*(\rho_p(x)) = i^* g^*(\varepsilon_N) = 0$ , and since  $i^*: H^N(\Omega X; Z) \rightarrow H^N(F; Z)$  is a monomorphism (by the Serre exact sequence) we see that  $i^*(x)$  is a nonzero element of  $H^N(F; Z)$  that is divisible by

p. Let  $y$  be an element such that  $i^*(x) = py$ . Now

$$i^*(x^k) = p^k y^k,$$

so that

$$i^*(x_k) = \frac{1}{k!} p^k y^k.$$

Let  $\sigma_k$  be the highest power of  $p$  that divides  $i^*(x_k)$  and let  $y_k$  be such that  $i^*(x_k) = p^{\sigma_k} y_k$ . Note that  $\lim_k \sup(\sigma_k) = \infty$ , since  $(1/k!)p^k$  is divisible by arbitrarily high powers of  $p$ .

**LEMMA 3.2.** *For each integer  $K \geq 0$  there exists an integer  $\gamma(K)$  such that*

$$p^{\gamma(K)} H^*(F; Z) \subset \text{Im}(i_K^*).$$

*Proof.* Let  $\mu(l)$  be the exponent of  $p$  in  $H^l(K(Z_p, N); Z)$ . By the Universal Coefficient Theorem, the exponent of  $p$  in  $E_2^{l,*}$  in the Serre spectral sequence for  $g$  is less than or equal to

$$\mu'(l) = \max(\mu(l), \mu(l+1)).$$

Let  $\gamma(K) = \sum_{l=1}^K \mu'(l)$ . Then the group  $p^{\gamma(K)} H^*(F; Z) = p^{\gamma(K)} E_2^{0,*}$  is in the kernel of all the differentials  $d_r$  for  $2 \leq r \leq K$ . Let  $(E_r(K), d_r(K))$  denote the spectral sequence of the fibration  $g_K$ . Since  $E_2^{l,*}(K) = 0$  for  $l > K$ ,  $d_l(K) = 0$  for  $l > K$ . By naturality of the Serre spectral sequence,

$$d_l(K)(p^{\gamma(K)} E_2^{0,*}(K)) = 0 \quad \text{for } 2 \leq r \leq k.$$

Thus

$$p^{\gamma(K)} H^*(F; Z) = p^{\gamma(K)} E_2^{0,*}(K) \subset E_\infty^{0,*}(K) = \text{Im}(i_K^*) \subset H^*(F; Z). \quad \square$$

For each  $K$ , let  $w_k = j_K^*(x_k)$ , so that  $i_K^*(w_k) = p^{\sigma_k} y_k$  and let  $z_k \in H^{kN}(\Omega X[K]; Z)$  be such that  $i_K^*(z_k) = p^{\gamma(K)} y_k$ , as guaranteed by Lemma 3.2. If  $\sigma_k > \gamma(K)$  we have that

$$w_k - p^{\sigma_k - \gamma(K)} z_k \in \text{Ker}(i_K^*).$$

**LEMMA 3.3.** *If  $N$  does not divide  $K$ , there exists an integer  $M(K)$  such that  $M(K) \ker(i_K^*) = 0$ .*

*Proof.* In the Serre spectral sequence for  $g_K$ ,  $H^l(\Omega X[K]; Z)$  is filtered by groups  $D_{i,l-i}$ ,  $0 \leq i \leq K+1$ , such that

- (1)  $D_{0,l} = H^l(\Omega X[K]; Z)$ ,
- (2)  $D_{K+1,l-K-1} = \{0\}$ ,

and

$$(3) \quad D_{i,l-i}/D_{i+1,l-i-1} \approx E_\infty^{i,l-i}[K].$$

Hence  $D_{1,l-1} = \ker(i_K^*: H^l(\Omega X[K]; Z) \rightarrow H^l(F; Z))$ .

We can ensure that  $D_{1,l-1}$  is a finite group by knowing that each  $E_2^{i,l-i} \approx H^i(K(Z_p, N)^{(K)}; H^{l-i}(F; Z))$  is finite for  $i \geq 0$ . We have that  $H^i(K(Z_p, N)^{(K)}; Z)$  is finite except for  $i = 0$  or  $K$ . By condition (c) of the theorem  $H^j(\Omega X; Q) = 0$  if  $N$  does not divide  $j$ . Since  $i^*: H^*(\Omega X; Q) \rightarrow H^*(F; Q)$  is an isomorphism  $E_2^{i,l-1}$  is finite for all  $i \leq 1$ .  $\square$

From now on we assume that  $N$  does not divide  $K$ . Since  $w_k - p^{\sigma_k - \gamma(K)} z_k \in \ker(i_K^*)$ ,  $M(K)[w_k - p^{\sigma_k - \gamma(K)} z_k] = 0$ .

**LEMMA 3.4.** *If  $g$  compresses into  $K(Z_p, N)^{(K)}$ , there exists a map  $r_K: \Omega X \rightarrow \Omega X[K]$  such that  $j_K r_K \sim 1_{\Omega X}$ .*

*Proof.* Immediate from the covering homotopy property (see [9] p. 689).  $\square$

If  $g$  were compressible into the  $K$ -skeleton of  $K(Z_p, N)$  we could apply the map  $r_K$  given by 3.4 to get

$$M(K)x_k = M(K)p^{\sigma_k - \gamma(K)}r_K^*(z_k),$$

which contradicts the hypothesis that each  $x_k$  is a free generator of  $H^{kN}(\Omega X; Z)$ .

Consequently the map  $g: \Omega X \rightarrow K(Z_p, N)$  is not deformable into the  $K$ -skeleton of  $K(Z_p, N)$  for any  $K$  not divisible by  $N$ , hence not deformable into any finite skeleton. Since  $g$  factors as  $\Omega X \xrightarrow{\Omega f} \Omega Y \rightarrow K(Z_p, N)$ , the map  $\Omega f$  is incompressible.

We proceed now to the proofs of Theorems 1.1 and 1.2. To show that  $c(G, n) \leq N$ , for some  $N$ , we need to produce an  $N$ -dimensional space  $X$  and a map  $f: X \rightarrow K(G, n)$  such that  $f$  is  $\Omega$ -incompressible. We shall take our spaces  $X$  to be appropriate skeleta of our  $K(G, n)$ 's.

*Proof of Theorem 1.3.* We apply 3.1 by letting  $Y = K(Z_p k, 2n)$ ,  $N = 2n + 2p - 2$ ,  $X = K(Z_p k, 2n)^{(N+1)}$ , and  $f: X \rightarrow Y$  be the inclusion

$$f: K(Z_p k, 2n)^{(N+1)} \subset K(Z_p k, 2n).$$

If  $l \leq N + 1$ ,  $f^*: H^l(Y; G) \rightarrow H^l(X; G)$  is monic for any group  $G$ . Consequently  $f^*(P^1 \beta(\iota_{2n}))$  is a nonzero element of  $H^{N+1}(X; Z_p)$  that is in the kernel of  $\beta$ . (Here  $P^1$  and  $\beta$  denote the Steenrod operations.) Thus there exists a free generator  $\bar{x}$  of  $H^{N+1}(X; Z)$  such that

$$\rho_p(\bar{x}) = f^*(P^1 \beta(\iota_{2n})).$$

Set  $x = \sigma^*(\bar{x})$ . (Here  $\sigma^*$  denotes the cohomology suspension.) Noting that  $N$  is even and that in the Serre spectral sequence of the path space fibration of  $X$ ,  $d_{N+1}: E_{N+1}^{0,*} \rightarrow E_{N+1}^{N+1,*}$  is an isomorphism, we see

that  $x$  generates a divided polynomial subalgebra of  $H^*(\Omega X; \mathbb{Z})$ . Furthermore, if  $\mu$  denotes that multiplication in  $\Omega X$ , then the  $k$ -fold interaction of  $\mu^*$  contains a term

$$(\mu^*)^k(x) = \cdots + k! x \otimes \cdots \otimes x(k \text{ factors}) + \cdots$$

and hence  $x_k = (1/k!)x^k$  is a free generator of  $H^{*k}(\Omega X; \mathbb{Z})$ . Thus condition (a) of Theorem 3.1 is satisfied.

By the naturality of the Steenrod operations, the cohomology suspension, and reduction mod  $p$ , condition (b) is satisfied. Condition (c) is obvious, since  $X$  is the  $(N+1)$ -skeleton of a space  $Y$  whose reduced rational cohomology is trivial.

Hence the hypotheses of Theorem 3.1 are satisfied and Theorem 1.3 is proven.

*Proof of Theorem 1.4.* Suppose  $n > 1$ . We would like to apply 3.1 to  $Y = K(Z, 2n)$ ,  $N = 2n + 8$ , and  $X = K(Z, 2n)^{(N+1)}$ , as in the proof of Theorem 1.3. However we would fail to satisfy condition (c). So instead we consider the composition

$$K(Z_2, 2n - 1)^{(N+1)} \xrightarrow{h} K(Z, 2n)^{(N+1)} \xhookrightarrow{i} K(Z, 2n),$$

where  $h$  is a cellular map induced by a nontrivial map  $K(Z_2, 2n - 1) \rightarrow K(Z, 2n)$ . Then  $h^*(Sq^6 Sq^3 \iota_{2n}) = Sq^6 Sq^3 Sq^1(\iota_{2n-1})$ , and we let  $f: X \rightarrow Y$  be

$$ih: K(Z_2, 2n - 1)^{(N+1)} \rightarrow K(Z, 2n)$$

and take  $\bar{x}$  to be a free generator that reduces to  $Sq^6 Sq^3 Sq^1(\iota_{2n-1})$ . The proof then proceeds as that of Theorem 1.3.

**4. Proof of Theorem 2.1.** The connection between compressibility and Čech cohomology is made by the following result.

**LEMMA 4.1.** *If there exists an  $\Omega$ -incompressible map  $f: X \rightarrow K(G, n)$  ( $n > 1$ ), where  $X$  is a finite complex, then  $H_f^*(-; G)$  is not a homotopy functor on the category of countable CW-complexes.*

*Proof.* Specifically, if  $\bar{P}X$  denotes the Milnor universal bundle (path-space) on  $X$ , [10], we shall show that  $H_f^*(\bar{P}X; G) \neq 0$ . Let  $\bar{\Omega}X$  denote the Milnor cellular loopspace of  $X$ . We have a diagram

$$\begin{array}{ccc} \bar{\Omega}X & \xrightarrow{f^+} & \Omega K(G, n) \\ \downarrow & & \downarrow \\ \bar{P}X & \xrightarrow{f^-} & PK(G, n) \\ \downarrow p_1 & & \downarrow p \\ X & \xrightarrow{f} & K(G, n). \end{array}$$

Since  $f^+$  is homotopy-equivalent to  $\Omega f$ , our hypothesis guarantees that  $f^+$  is not deformable into any compact subset of  $\Omega K(G, n)$ .

Now suppose that  $H_f^n(\bar{P}X; G) = 0$ . This means, according to 2.1, that  $[\beta\bar{P}X; K(G, n)] = 0$ . Let  $i: \bar{P}X \rightarrow \beta\bar{P}X$  denote the inclusion. Since  $X$  is compact,  $p_1$  extends to  $\hat{p}_1: \beta\bar{P}X \rightarrow X$  such that  $\hat{p}_1i = p_1$ . By our hypothesis,  $f\hat{p}_1 \sim *$ . Hence there exists a lift  $g: \beta\bar{P}X \rightarrow PK(G, n)$  such that  $pg = f\hat{p}_1$ . Since  $pgi = f\hat{p}_1i = fp_1$ ,  $gi$  is a fiber-preserving map. Since  $\bar{P}X$  is contractible,  $gi \sim f^+$ . Using again the contractibility of  $\bar{P}X$  and the fact that  $n > 1$ ,  $gi$  is vertically homotopic to  $f^+$ . Hence  $f^-$  is homotopic to  $(gi)|\Omega X$ , as maps  $\bar{\Omega}X \rightarrow \Omega K(G, n)$ . Because  $gi$  factors through  $\beta\bar{P}X$  its image is contained in a compact subset. But that is a contradiction to the  $\Omega$ -incompressibility of  $f$ .  $\square$

*Proof of Theorem 2.1.* In the proofs of Theorems 1.3 and 1.4, finite complexes were produced and maps to  $(K(Z_p k, 2n))$  ( $N \geq 1$ ) and  $K(Z, 2n)$  ( $n > 1$ ), respectively that were  $\Omega$ -incompressible. By Lemma 4.1, the functors  $H^{2n}(-; Z_p k)$  ( $n \geq 1$ ) and  $H^{2n}(-; Z)$  ( $n > 1$ ) do not satisfy the homotopy axiom on the category of countable CW-complexes.  $\square$

## REFERENCES

1. A. Calder, *Cohomology of finite covers*, Trans. Amer. Math. Soc., **218** (1976), 349–352.
2. A. Calder and J. Siegel, *Homotopy and extensions*, Categorical Topology Mannheim 1975, Springer Lecture Notes 540 (1976).
3. ———, *Homotopy and uniform homotopy*, Trans. Amer. Math. Soc., **235** (1978), 245–270.
4. ———, *Kan extensions of homotopy functors*, J. Pure Appl. Algebra, **12** (1978), 253–269.
5. E. Čech, *Theorie generale de l'homologie dans un espace quelconque*, Fund. Math., **19** (1932), 149–183.
6. A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, Berlin-Heidelberg, 1972.
7. C. H. Dowker, *Mapping theorems for noncompact spaces*, Amer. J. Math., **69** (1947), 200–242.
8. S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton Univ. Press, Princeton N.J., 1952.
9. J. Goldfeather, *Incompressibility and fibrations*, Illinois J. Math., **21** (1977), 688–702.
10. J. Milnor, *Construction of universal bundles I*, Ann. of Math., **63** (1956), 272–284.
11. S. Weingram, *On the incompressibility of certain maps*, Ann. of Math., **93** (1971), 476–485.

Received May 16, 1980.

BIRKBECK COLLEGE  
UNIVERSITY OF LONDON  
LONDON, ENGLAND  
AND  
NEW MEXICO STATE UNIVERSITY  
LAS CRUCES, NM 88003