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A FIXED POINT THEOREM IN c_0

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It is proved that if K is the closed convex hull of a weakly convergent sequence in c_0 then each nonexpansive mapping $T: K \to K$ has a fixed point.

1. Introduction. The general problem with which we are concerned is: classify the weakly compact convex subsets K of a Banach space such that every nonexpansive mapping T of K into itself must necessarily have a fixed point. (T is said to be nonexpansive if $\|Tx - Ty\| \leqq \|x - y\|$ for all x and y in K.) We study this problem for the Banach space c_0.[1]

Section II is devoted to the proof of the theorem stated in the abstract, and § III to some extensions of it. For the present we wish to recall some known results in this area, and to explain why the space c_0 may be of special interest.

The problem posed above is of the following type: Let K be a subset of a locally convex topological vector space and $T: K \to K$ a mapping. Give conditions on K and T which insure T will have a fixed point.

The Tychonoff fixed point theorem [14] says if K is compact, convex and T is continuous then T has a fixed point. Banach's fixed point theorem [1] says if K is closed and a subset of a Banach space (more generally a complete metric space) and T is a strict contraction ($\|Tx - Ty\| \leqq \alpha\|x - y\|$ for all x, y in K and some $\alpha < 1$) then T has a unique fixed point.

Our problem may be viewed as combination of these two theorems. Note however that there is a strange feature in this combination: the condition on K concerns the weak topology while that on T concerns the norm topology. The seeming lack of connection between these conditions is what makes the problem so interesting and challenging.

From now on let us assume that K is a given convex weakly compact subset of a Banach space X and $T: K \to K$ is nonexpansive. Of course by translation one may assume $0 \in K$. Then for all $0 < r < 1$, $rT: K \to K$ and rT is a strict contraction. By the Banach theorem rT has a unique fixed point x_r and it is easily seen that $\|Tx_r - x_r\| \to 0$ as $r \to 1$. Thus there always exists a sequence of "approximate fixed points" for T. The points $\{x_r\}_{0 \leqq r < 1}(x_0 = 0)$ form a continuous curve in K. In fact it can be seen that if $0 < r <$

[1] D. Alspach [0] has recently given the first example of a weakly compact convex set K and a nonexpansive mapping on it without a fixed point.

$s < 1$ and $d =$ diameter K then $\|x_r - x_s\| \leqq (s - r)d(1 - r)^{-1}$. Of course if $\{x_r\}$ were norm convergent as $r \to 1$ then its limit would be a fixed point for T.

The most general positive result appears to be a theorem of Kirk [10] which says if K has normal structure then T has a fixed point. (See also [3], [4] and [8].)

A point $x \in K$ is said to be *diametrizing* for K if diameter $K = \sup_{y \in K} \|x - y\|$. K has normal structure if each convex closed subset H of K with a positive diameter contains a point which does not diametrize H. It is known that if X is uniformly convex then K has normal structure [2]. An interesting proof of Kirk's theorem was given by Karlovitz [9] where he proved the following proposition: if K is minimal with respect to T (i.e., no smaller closed convex subset is T invariant) and $\{y_n\}$ is a sequence in K so that $\lim_{n\to\infty} \|Ty_n - y_n\| = 0$ then for all $x \in K$, $\lim_{n\to\infty} \|x - y_n\| =$ diameter K. In the same paper Karlovitz also showed that normal structure is not necessary. He was able to renorm l_2 so that the closed convex hull of the unit vectors failed normal structure, yet still every weakly compact convex set had the fixed point property for nonexpansive mappings.

However all known positive results depend in some way or another on convexity properties of the norm. Our approach to the problem has been to study the case $X = c_0$, a space whose norm fails any nontrivial convexity property. It is also easy to find $K \subset c_0$ that fail normal structure; for example, let K be the closed convex hull of the unit vectors of c_0. (Note that our main result shows that this set has the fixed point property for nonexpansive maps.)

But the space c_0 possesses another property which in a sense compensates for the lack of convexity of the norm, and might indicate that each weakly compact convex subset of c_0 has the fixed point property. Namioka [12] proved that in every weakly compact convex subset K of a Banach space X the set $D = \{x \in K; \{x_n\} \subset K$ and $x_n \xrightarrow{w} x$ implies $\|x_n - x\| \to 0\}$ is a weakly dense G_δ subset of K. We have noticed that if $X = c_0$ then D is in fact *norm* dense in K. Of course T is weakly continuous at each point of D. Thus, if one could find $\{y_n\} \subset K$ with $\|Ty_n - y_n\| \to 0$ and $y_n \xrightarrow{w} y_0 \in D$ then y_0 would be a fixed point of T. Unfortunately we were unable to do this.

R. Haydon and the authors [5] have recently shown that another class of weakly compact subsets of c_0 have the fixed point property for nonexpansive maps. Namely the "coordinatewise star shaped" sets K. Say K is coordinatewise star shaped if there exists a point $x \in K$ so that if $y \in K$ and z lies coordinatewise between

x and y then $z \in K$. Of course such sets need not be convex.

For additional information on the fixed point problem we refer the reader to [13] and the references listed therein.

We wish to thank the referee for his useful suggestions.

We use standard Banach space terminology as may be found in [11]. Let us just mention some of the most frequently used notation. c_0 is the Banach space of all sequences of reals converging to 0. For $x \in c_0$ we denote by $x(n)$ the nth coordinate of x i.e., $x = (x(1), x(2), x(3), \cdots)$. $\|\ \|_\infty$ is the supremum norm on c_0, i.e., $\|x\|_\infty = \sup_n |x(n)|$. If $x \in \ell_1$, $\|x\|_1 = \sum_{i=1}^\infty |x(i)|$. If E is a subset of the positive integers N then $x|E$ is the vector defined by $x|E(n) = x(n)$ if $n \in E$ and $x|E(n) = 0$ if $n \notin E$. $\sim E$ is the complement N/E of E. For $p, q \in N$, $[p, q)$ denotes the set $\{i \in N: p \leqq i < q\}$. For $r \geqq 0 [x-r]^+$ is the vector so that $[x - r]^+(n) = x(n) - r$ if $x(n) \geqq r$ and $[x - r]^+(n) = 0$ otherwise. We write $x_n \overset{w}{\to} x (x_n \overset{w*}{\to} x)$ if $(x_n)_{n=1}^\infty$ converges weakly (weak*) to x.

By $\operatorname{conv}(x_i)_{i \in F}$ we mean the convex hull of $\{x_i: i \in F\}$ and $\overline{\operatorname{conv}}(x_i)$ is the closed convex hull.

II. The main result.

THEOREM 1. *Let K be the closed convex hull of a weakly convergent sequence in c_0, and let $T: K \to K$ be nonexpansive. Then T has a fixed point.*

The general plan of proof is as follows. First we may assume that $K = \overline{\operatorname{con}\{x_i\}_{i=1}^\infty}$ where $x_i \overset{w}{\to} 0$ and $\|x_i\| \leqq 1$ for all i. Let $\{y_n\}_{n=1}^\infty$ be a sequence of approximate fixed points for T ($\|Ty_n - y_n\| \to 0$). By passing to a subsequence we may assume $y_n \overset{w}{\to} y_0$ and $\|y_n - y_0\| \to r$. If $r = 0$ we are done, so we assume $r > 0$. We shall construct a new set $\{w^\varepsilon\}_{\varepsilon > 0}$ of approximate fixed points for T ($\lim_{\varepsilon \to 0} \|Tw^\varepsilon - w^\varepsilon\| = 0$) so that w^ε is norm convergent to some $z \in K$.

A special case. Before proceeding to the general case whose argument is quite technical we briefly sketch the proof in the special case where $K = K_1$ is the closed convex hull of the unit vector basis $\{e_n\}_{n=1}^\infty$ of c_0 (i.e., $K_1 = \{x = (x_1, x_2, \cdots) \in c_0: x_i \geqq 0, \sum_{i=1}^\infty x_i \leqq 1\}$). An understanding of this easier case will make the sequel much more comprehensible. Of course we have no intention of giving all the details twice, and so we shall now take certain liberties.

We shall show that $[y_0 - r]^+$ is a fixed point for T. First (since

$y_n \xrightarrow{w} y_0$) we shall assume (by passing to a subsequence) that y_0, $y_1 - y_0$, $y_2 - y_0$, $\cdots$ are disjointly supported elements in K. (Of course in the general case we shall need an argument to show they are in K, and we shall only be able to assume they are "almost" disjointly supported.) Also let us assume $\|y_n - y_0\| = r$ for all n.

Fix $0 < \varepsilon < r/2$. Define $z_i = [(y_i - y_0) - (r - \varepsilon)]^+$ for $i \geqq 1$ and $z_0 = [y_0 - (r - \varepsilon)]^+$. Thus $\{z_i\}_{i=0}^{\infty} \subseteqq K$ and are disjointly supported in

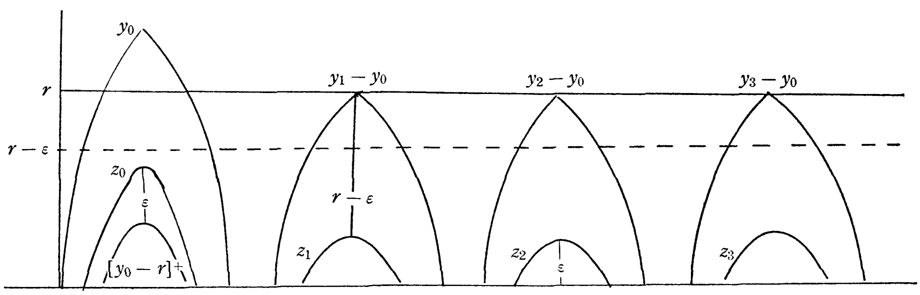

FIGURE 1

c_0 (see Figure 1). Of course z_0 might be 0. Let $w^\varepsilon = \sum_{i=0}^{m} z_i$ where m is the largest integer such that $w^\varepsilon \in K$. If we use $\|x\|_1$ to denote the ℓ_1 norm of a vector $x \in K$, this means that m is the largest integer so that $\|w^\varepsilon\|_1 = \sum_{i=0}^{m} \|z_i\|_1 \leqq 1$. It is easily seen that for $i \geqq 1$

$$\varepsilon \leqq \|z_i\|_1 \leqq \frac{\varepsilon}{r},$$

and so

$$\|w^\varepsilon\|_1 \geqq 1 - \frac{\varepsilon}{r}.$$

Now for $1 \leqq i \leqq m$

$$\|y_i - w^\varepsilon\| = r - \varepsilon.$$

This is because we have divided y_i into y_0 and $y_i - y_0$ and we have chosen w^ε to be of distance $r - \varepsilon$ from each piece. Thus, by the nonexpansiveness of T, $\|Ty_i - Tw^\varepsilon\| \leqq r - \varepsilon$, $1 \leqq i \leqq m$. $\|Ty_i - y_i\|$ could have been made as small as we pleased, thus up to an error which we can control we also have $\|y_i - Tw^\varepsilon\| \leqq r - \varepsilon$. Since w^ε is the (coordinate-wise) small vector in K among all vectors x with $\|y_i - x\| \leqq r - \varepsilon$, $(i \leqq m)$, we conclude that $Tw^\varepsilon \geqq w^\varepsilon$ and hence $Tw^\varepsilon \geqq z_i$ (coordinate-wise) for $0 \leqq i \leqq m$. But the z_i's were disjointly supported and their ℓ_1 norms summed to almost 1 (up to ε/r). Thus Tw^ε is essentially equal to w^ε (again up to ε/r). I.e.,

$\|Tw^\varepsilon - w^\varepsilon\| \to 0$ as $\varepsilon \to 0$. On the other hand it is easily seen that w^ε converges in norm to $[y_0 - r]^+$. This concludes the special case.

The general case. The proof will be divided into two parts. In the first part we shall study the structure of those weakly compact sets in general Banach spaces which are the closed convex hull of a weakly null sequence. In the second and more difficult part we shall apply the results of the first part to prove the theorem in c_0. Our first lemma will enable us to assume $y_n - y_0 \in K$.

LEMMA 1. *Let* $\{\alpha^n\}_{n=1}^\infty \subset B(\ell_1)^+ = \{x \in \ell_1 : x \geqq 0, \|x\|_1 \leqq 1\}$. *If no subsequence of* $\{\alpha^n\}$ *converges in the* ℓ_1 *norm, then there exists a subsequence* $\{\alpha^{n_i}\}$ *of* $\{\alpha^n\}$, *a vector* $\alpha^0 \in B(\ell_1)^+$ *and a sequence* $\{\beta^i\} \subset B(\ell_1)^+$ *so that*

(i) $\alpha^{n_i} \xrightarrow{w^*} \alpha^0$,

(ii) $\|\alpha^{n_i} - \beta^i\|_1 \to 0$,

(iii) $\beta_i - \alpha_0 \in B(\ell_1)^+$ *for* $i = 1, 2, \cdots$.

Proof. By passing to a subsequence we may assume $\alpha^n \xrightarrow{w^*} \alpha^0 \in B(\ell_1)^+$, $\|\alpha^n - \alpha^0\|_1 \to r > 0$ and $\|\alpha^n\|_1 \to \tau > 0$. Clearly $\|\alpha^0\|_1 < \tau$ (or else $\|\alpha^n - \alpha^0\|_1 \to 0$). Let $\varepsilon < \min\{r/10, (\tau - \|\alpha^0\|_1)/10\}$, and choose i_ε such that $\sum_{i \geqq i_\varepsilon} \alpha^0(i) < \varepsilon$. Choose n_ε large enough such that $n \geqq n_\varepsilon$ implies $\|(\alpha^n - \alpha^0)|_{\{i \leqq i_\varepsilon\}}\|_1 < \varepsilon$, and $|\|\alpha^n\|_1 - \tau| < \varepsilon$. Let $\lambda^\varepsilon = \max\{\alpha^{n_\varepsilon}, \alpha^0\} \in \ell_1$. (The max is taken coordinate-wise.) We claim that $\|\alpha^{n_\varepsilon} - \lambda^\varepsilon\|_1 < 2\varepsilon$. Indeed if $I = \{i : i \leqq i_\varepsilon\}$ and $J = N/I$, then $\|(\alpha^{n_\varepsilon} - \lambda^\varepsilon)|_J\|_1 = \sum_{J'} (\alpha^0(j) - \alpha^{n_\varepsilon}(j)) < \varepsilon$ where $J' = \{j \in J : \alpha^0(j) > \alpha^{n_\varepsilon}(j)\}$, while

$$\|(\alpha^{n_\varepsilon} - \lambda^\varepsilon)|_I\|_1 \leqq \sum_{i \in I} |\alpha^{n_\varepsilon}(i) - \alpha^0(i)| < \varepsilon .$$

The only problem with λ^ε is that it may happen $\|\lambda^\varepsilon\|_1 > 1$. Of course $\|\lambda^\varepsilon\|_1 < 1 + 2\varepsilon$. We wish to perturb λ^ε to get an element in $B(\ell_1)^+$ which is still larger than α^0 (and close to α^{n_ε}). To see that this is possible we must show that the mass of λ^ε which lies above α^0 is larger than 2ε. Now this mass is precisely $\|\lambda^\varepsilon - \alpha^0\|_1 = \|\lambda^\varepsilon\|_1 - \|\alpha^0\|_1$, and $\|\lambda^\varepsilon\|_1 - \|\alpha^0\|_1 > \|\alpha^{n_\varepsilon}\|_1 - \|\alpha^0\|_1 > \tau - \varepsilon - \|\alpha^0\|_1 > 10\varepsilon - \varepsilon = 9\varepsilon$. Define β^ε as follows. Let $\delta = \|\lambda^\varepsilon\|_1 - 1$. We know $\delta < 2\varepsilon$. If $\delta \leqq 0$ set $\beta^\varepsilon = \lambda^\varepsilon$. If $\delta > 0$ then we define β^ε to differ from λ^ε only in some coordinates j at which $\lambda^\varepsilon(j) > \alpha^0(j)$. Since $\sum_{\{j : \lambda^\varepsilon(j) > \alpha^0(j)\}} (\lambda^\varepsilon(j) - \alpha^0(j)) > 9\varepsilon$ it is possible to reduce λ^ε at some of these coordinates to get β^ε which satisfies $\beta^\varepsilon \in B(\ell_1)^+$, $\beta^\varepsilon \geqq \alpha^0$, and $\|\beta^\varepsilon - \lambda^\varepsilon\| < 2\varepsilon$ (and hence $\|\beta^\varepsilon - \alpha^{n_\varepsilon}\| < 4\varepsilon$). The lemma follows by repeating this process for $\varepsilon_i \to 0$. □

Let X be a Banach space and let $\{x_i\}_{i=1}^\infty$ be a weakly null sequence in X with $\|x_i\| \leqq 1$ for all i. Define $f: \ell_1 \to X$ by $f(\alpha) = \sum_{i=1}^{\infty} \alpha(i)x_i$ for $\alpha \in \ell_1$. Clearly f is linear and $\|f\| \leqq 1$. Also it is easily seen that $f|B(\ell_1)^+$ is continuous with respect to the w^* topology in ℓ_1 and the weak topology in X.

LEMMA 2. *Let K be the closed convex hull of the weakly null sequence $\{x_i\}$ in X. Let $\{w_n\}$ be a sequence in K. Then if no subsequence of $\{w_n\}$ converges in norm in X, there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$, $y_0 \in K$ and a sequence $\{y_n\}$ in K so that*

(i) $w_{n_i} \xrightarrow{w} y_0$,

(ii) $\|w_{n_i} - y_i\| \to 0$,

(iii) $y_i - y_0 \in K$, *for all* i.

Proof. This follows directly from Lemma 1, the continuity properties of the function f defined above, and the fact that $f(B(\ell_1)^+) = K$. □

For K as in Lemma 2 and $y \in K$ we define

$$|y|_1 = \inf\{\|\alpha\|_1 : \alpha \in B(\ell_1)^+, f(\alpha) = y\} .$$

LEMMA 3. *$|y|_1$ has the following properties:*

(i) *for each $y \in K$ there exists some $\alpha \in B(\ell_1)^+$ so that $f(\alpha) = y$ and $\|\alpha\|_1 = |y|_1$,*

(ii) *$|y|_1 = 0$ if and only if $y = 0$,*

(iii) *if $y \in K$, $t > 0$ and $ty \in K$ then $|ty|_1 = t|y|_1$,*

(iv) *if y, z and $y + z$ are in K then $|y + z|_1 \leqq |y|_1 + |z|_1$,*

(v) $|y|_1 \geqq \|y\|$.

Proof. If $y \in K$, $B(\ell_1)^+ \cap f^{-1}(y)$ is a w^* compact subset of ℓ_1 and the ℓ_1 norm attains its minimum on such sets. This proves (i) and the other properties are equally easy to check. □

LEMMA 4. *Let K be as in Lemma 2, and let $H \neq \emptyset$ be a weakly closed subset of K. Let $\tau = \inf\{|w|_1 : w \in H\}$ and $H' = \{z \in H: |z|_1 = \tau\}$. Then*

(i) $H' \neq \emptyset$,

(ii) *H' is norm compact in X,*

(iii) *if H is convex then H' is convex too.*

Proof. (i) Let $\{z_n\} \subset H$ such that $|z_n|_1 \downarrow \tau$ and choose $\alpha^n \in B(\ell_1)^+$ with $\|\alpha^n\|_1 = |z_n|_1$ and $f(\alpha^n) = z_n$. By passing to a subsequence we may assume $\alpha^n \xrightarrow{w^*} \alpha^0$ and thus since H is weakly closed $z_n = f(\alpha^n) \xrightarrow{w} f(\alpha^0) = z_0 \in H$. Clearly $\|\alpha^0\|_1 \leqq \lim \|\alpha^n\|_1 = \tau$ and so $|z_0|_1 \leqq \tau$.

By definition of τ, $|z_0|_1 = \tau$ and so $z_0 \in H'$.

(ii) Let $\{z_n\} \subset H'$, and let $\{\alpha^n\} \subset B(\ell_1)^+$ be so that $f(\alpha^n) = z_n$, $\|\alpha^n\|_1 = |z_n|_1 = \tau$. Let $\alpha^{n_i} \xrightarrow{w^*} \alpha^0$. The argument in (i) shows $z_0 = f(\alpha^0) \in H'$, and $z_{n_i} \xrightarrow{w} z_0$. We claim $\|z_{n_i} - z_0\| \to 0$. But this follows from the observation that $\|\alpha^{n_i} - \alpha^0\|_1 \to 0$ which is easily checked.

(iii) If $y, z \in H'$ and $0 < t < 1$, then $|ty + (1-t)z|_1 \leqq t|y|_1 + (1-t)|z|_1 = \tau$ and so if H is convex then by the definition of τ, $|ty + (1-t)z|_1 = \tau$. □

Let us review the situation at present. T is a nonexpansive mapping on $K = \overline{\operatorname{con}\{x_i\}}$, $x_i \xrightarrow{w} 0$ and $\|x_i\| \leqq 1$. Let w_n be a sequence of approximate fixed points for $T(\|Tw_n - w_n\| \to 0)$. If some subsequence of $\{w_n\}$ converges in norm its limit is a fixed point for T. If not, then by Lemma 2 there exists a sequence $\{y_n\} \subset K$ so that $y_n \xrightarrow{w} y_0$, $\|Ty_n - y_n\| \to 0$, $y_n - y_0 \in K$ for all n, and $\|y_n - y_0\| \to r > 0$.

For $y \in K$ and $s > 0$ define

$$H(y, s) = \{z \in K : \|y - z\| \leqq s\} ,$$

and

$$H'(y, s) = \{z \in H(y, s) : |z|_1 = \inf\{|w|_1 : w \in H(y, s)\}\} .$$

Clearly $H(y, s)$ is weakly closed and convex and so by Lemma 4, $H'(y, s)$ is nonempty, convex and norm compact in X. The thrust of the proof will be to show $H'(y_0, r)$ is invariant under T, and hence T has a fixed point (in $H'(y_0, r)$). Unfortunately we can prove the invariance of $H'(y_0, r)$ under T only if we assume $X = c_0$. This begins the second stage of the proof. We shall henceforth write $\|x\|$ for $\|x\|_\infty$.

LEMMA 5. *$H(y_0, r)$ is invariant under T.*

Proof. Let $z \in H(y_0, r)$. Since $y_n \xrightarrow{w} y_0$ and $\|y_n - y_0\| \to r$ it follows that $\limsup_n \|z - y_n\| \leqq r$. Indeed, if not then we may assume without loss of generality that $\|z - y_n\| \geqq r + \varepsilon$ and $\|y_n - y_0\| < r + \varepsilon/3$ for all n and some $\varepsilon > 0$. Let $0 < \delta < \varepsilon/3$ and choose i_δ so that $i > i_\delta$ implies $|y_0(i)|, |z(i)| < \delta$ (here we use $y_0, z \in c_0$). Choose n_δ so that $n \geqq n_\delta$ implies $|y_0(i) - y_n(i)| < \delta$ for $i \leqq i_\delta$. Fix $n \geqq n_\delta$. If $i \leqq i_\delta$ then $|z(i) - y_n(i)| \leqq |z(i) - y_0(i)| + |y_0(i) - y_n(i)| < r + \delta < r + \varepsilon$ and if $i \geqq i_\delta$ then $|z(i) - y_n(i)| \leqq |z(i)| + |y_0(i)| + \|y_0 - y_n\| \leqq 2\delta + r + \varepsilon/3 < r + \varepsilon$, i.e., $\|z - y_n\| < r + \varepsilon$ which is a contradiction. Thus $\lim_{i\to\infty} \|z - y_{n_i}\| \leqq r$ for some subsequence $\{y_{n_i}\}$ of $\{y_n\}$. Hence $r \geqq \lim \|z - y_{n_i}\| \geqq \lim \|Tz - Ty_{n_i}\| = \lim \|Tz - y_{n_i}\|$. But $Tz - y_{n_i} \xrightarrow{w} Tz - y_0$ and so $\|Tz - y_0\| \leqq r$ or $Tz \in H(y_0, r)$. □

It remains only to show that if $z \in H'(y_0, r)$ (that is $|z|_1 = \tau = \inf\{|w|_1: w \in H(y_0, r)\}$) then $|Tz|_1 = \tau$ too. The word "only" here is misleading since this is the most complicated part of the proof. We shall produce $w^\varepsilon \in K$ for $\varepsilon > 0$, so that $\lim_{\varepsilon\to 0} \|w^\varepsilon - z\| = 0$, and $d_0^\varepsilon \in K$ so that $\lim_{\varepsilon\to 0} \|Tw^\varepsilon - d_0^\varepsilon\| = 0$, and $\lim_{\varepsilon\to 0} |d_0^\varepsilon|_1 = \tau$. It follows that $Tz = \lim_{\varepsilon\to 0} d_0^\varepsilon$ satisfies $|Tz|_1 \leqq \tau$ (apply the function f) and hence $|Tz|_1 = \tau$ or $Tz \in H'(y_0, r)$. Fix $0 < \varepsilon < (1/100)\min(r, r^2)$ and $0 < \delta < \varepsilon^2/100$.

We would like it if $y_0, y_1 - y_0, y_2 - y_0, \cdots$ were disjointly supported in c_0 (as in the special case). Of course this is not necessarily true. But we may assume they are essentially disjoint. More precisely we have the following lemma.

LEMMA 6. *There exist integers* $1 = p_1 < p_2 < p_3 < \cdots$ *and* $1 = q_0 < q_1 < q_2 < \cdots$ *and a subsequence* $\{y'_n\}$ *of* $\{y_n\}$ *so that for all* $n \geqq 1$,

(1) *If* $x \in B_n = \operatorname{conv}\{x_i\}_{i\in[p_n, p_{n+1})}$ *then* $\|x|_{\sim[q_{n-1}, q_{n+1})}\| < \dfrac{\delta}{2^n}$,

$$\|y_0|_{[q_n,\infty)}\| < \frac{\delta}{2^n}, \tag{2}$$

$$\|(y'_n - y_0)|_{\sim[q_n, q_{n+1})}\| < \frac{\delta}{2^n}, \tag{3}$$

$$|\,\|y'_n - y_0\| - r\,| < \frac{\delta}{2^n}, \tag{4}$$

$$\|Ty'_n - y'_n\| < \frac{\delta}{2^n}. \tag{5}$$

Proof. We indicate briefly how to do this. By passing to a subsequence of $\{y_n\}$ we may clearly assume (4) and (5) hold. Set $q_0 = p_1 = 1$. Choose q_1 large enough so that (2) holds for $n = 1$. Choose $y'_1 \in \{y_n\}$ so that for all $j \in [q_0, q_1)$, $|(y'_1 - y_0)(j)| < \delta/2$. This may be done since $(y_n - y_0) \xrightarrow{w} 0$. Then let p_2 be large enough so that for $i \geqq p_2$, $|x_i(j)| < \delta/2^2$ for all $j \in [q_0, q_1)$. Let q_2 be so large that (i) if $i \in [p_1, p_2)$ then $|x_i(j)| < \delta/2$ for $j > q_2$; (ii) (2) holds for $n = 2$ and (iii) if $j \geqq q_2$ then $|(y'_1 - y_0)(j)| < \delta/2$. We have constructed $q_0, q_1, q_2; p_1, p_2$ and y'_1 so that (1) holds for $n = 1$, (2) is satisfied for $n = 1, 2$ and (3) is true for $n = 1$. Moreover we have one half of (1) for $n = 2$ (i.e., $i \geqq p_2$ implies $|x_i(j)| < \delta/2^2$ for $j \in [q_0, q_1)$). Clearly this process can be continued inductively—we omit the simple details. $\square$

To simplify notation, we shall assume henceforth that $\{y_n\}$, not $\{y'_n\}$, satisfies the conclusion of Lemma 6. Note that Lemma 6 has actually blocked the x_i's into sets B_n so that vectors in different B_{2n}'s are essentially disjointly supported in c_0. This blocking trick and the ideas for the more refined versions below come from recent work in L_p theory. (See e.g., [6] and [7].)

Let $0 = m_0 < m_1 < m_2 < \cdots$ be integers so that

$$(6) \qquad (m_n - m_{n-1} - 2)^{-1} < \delta/2^n \quad \text{and} \quad 2^{-m_1} < r\ .$$

We shall use the following

Observation. Set $I_n = [p_n, p_{n+1})$, and let $v = \sum_{i=1}^{\infty} \beta_i x_i \in K$ with $\sum_{i=1}^{\infty} \beta_i \leqq 1$, $\beta_i \geqq 0$. Then for each $n \geqq 1$ there exists an integer u_n, $m_{n-1} < u_n < m_n$ so that $\sum_{i \in I_{u_n}} \beta_i < \delta/2^n$ and thus if $d_n = \sum_{i=p_{u_n}+1}^{p_{u_{n+1}}-1} \beta_i x_i$ for $n \geqq 1$ and $d_0 = \sum_{i=1}^{p_{u_1}-1} \beta_i x_i$, then $\|v - \sum_{n=0}^{\infty} d_n\| < \delta$.

To see this fix n and note

$$1 \geqq \sum_{i=1}^{\infty} \beta_i \geqq \sum_{j=m_{n-1}+1}^{m_n-1} \sum_{i \in I_j} \beta_i \geqq (m_n - m_{n-1} - 2) \min\{\sum_{i \in I_j} \beta_i \colon m_{n-1} + 1 \leqq j \leqq m_n - 1\}\ .$$

Thus $1 \geqq (m_n - m_{n-1} - 2)\sum_{i \in I_{j_0}} \beta_i$ for some j_0, $m_{n-1} < j_0 < m_n$. Let $u_n = j_0$. Then by (6) $\sum_{i \in I_{u_n}} \beta_i < \delta/2^n$. The last statement of the observation follows easily. $\square$

Our next goal is to define w^ε. This will require some preliminary work. For each $k \geqq 1$ let $z_k \in H'(y_{m_k} - y_0, r - \varepsilon)$, and let $\beta^k \in B(\ell_1)^+$ be such that $z_k = \sum_{i=1}^{\infty} \beta^k(i) x_i$ with $\|\beta^k\|_1 = |z_k|_1$. Fix k. By the observation we can find $m_{k-1} < u_k < m_k$ and $m_{k+1} < v_k < m_{k+2}$ so that

$$(7) \qquad \sum_{i \in I_{u_k}} \beta^k(i) < \delta/2^k \quad \text{and} \quad \sum_{i \in I_{v_k}} \beta^k(i) < \delta/2^k\ .$$

Set $z'_k = \sum_{i=p_{u_k}+1}^{p_{v_k}-1} \beta^k(i) x_i$. It is easily checked that $|z'_k|_1 = \sum_{i=p_{u_k}+1}^{p_{v_k}-1} \beta^k(i)$.

We claim that

$$(8) \qquad \|(y_{m_k} - y_0) - z'_k\| \leqq r - \varepsilon + 2\delta\ .$$

Indeed, let $L_k = [q_{u_k}, q_{v_k})$. Then

$$\|(y_{m_k} - y_0 - z'_k)|_{\sim L_k}\| \leqq \|(y_{m_k} - y_0)|_{\sim L_k}\| + \|z'_k|_{\sim L_k}\| \leqq \frac{\delta}{2^{m_k}} + \frac{\delta}{2^{u_k}} < \delta\ .$$

(Here we have used (3) and (1).)

Also, since $z_k \in H'(y_{m_k} - y_0, r - \varepsilon)$, $\|y_{m_k} - y_0 - z_k\| \leqq r - \varepsilon$ and thus

$$\begin{aligned}\|(y_{m_k} - y_0 - z_k')|_{L_k}\| &\leqq \|(y_{m_k} - y_0 - z_k)|_{L_k}\| + \|(z_k - z_k')|_{L_k}\| \leqq r - \varepsilon \\ &+ \|\sum_{i \in \sim[p_{u_k}, p_{v_k+1})} \beta^k(i)x_i|_{L_k}\| + \|\sum_{i \in I_{u_k}} \beta^k(i)x_i\| + \|\sum_{i \in I_{v_k}} \beta^k(i)x_i\| \leqq r - \varepsilon \\ &+ \sum_{n=1}^{u_k-1} \delta/2^n + \sum_{n=v_k+1}^{\infty} \delta/2^n + \delta/2^{k-1} < r - \varepsilon + 2\delta\ .\end{aligned}$$

(We have used (1) and (7)) (8) follows.

We shall also need

$$\varepsilon/2 \leqq |z_k'|_1 \leqq \varepsilon/r + \delta\ . \tag{9}$$

To see this, note first that

$$\begin{aligned}|z_k'|_1 \geqq \|z_k'\| \geqq \|y_{m_k} - y_0\| - \|y_{m_k} - y_0 - z_k'\| \geqq r - \delta - (r - \varepsilon + 2\delta) \\ = \varepsilon - 3\delta > \varepsilon/2\end{aligned}$$

by the choice of δ, (8) and (4). To prove the right hand inequality in (9) we need only show $|z_k| \leqq \varepsilon/r + \delta$. But $\|(\varepsilon/r + \delta)(y_{m_k} - y_0) - (y_{m_k} - y_0)\| = (1 - \varepsilon/r - \delta)\|y_{m_k} - y_0\| \leqq (1 - \varepsilon/r - \delta)(r + \delta/2^{m_k}) \leqq r - \varepsilon + \delta(1/2^{m_k} - r) < r - \varepsilon$ by (4) and (6).

This implies $(\varepsilon/r + \delta)(y_{m_k} - y_0) \in H(y_{m_k} - y_0, r - \varepsilon)$ and so $|z_k|_1 = \inf\{|w|_1\colon w \in H(y_{m_k} - y_0, r - \varepsilon)\} \leqq |(\varepsilon/r + \delta)(y_{m_k} - y_0)|_1 \leqq \varepsilon/r + \delta$, which proves (9).

Now let z be an element of $H'(y_0, r)$. We will have to distinguish between two cases: If $\|y_0\| < r$ then $0 \in H(y_0, r)$ and clearly 0 is the only element in $H'(y_0, r)$. In this case we will have to show 0 is a fixed point for T. The second case $\|y_0\| \geqq r$ turns out to be slightly more involved. We shall give the detailed proof for the case where $\|y_0\| \geqq r$, and leave the case $\|y_0\| < r$ to the reader.

So let us assume $\|y_0\| \geqq r$. Note that $\|y_0 - z\| = r$, for if $\|z - y_0\| < r$ then $\|tz - y_0\| \leqq r$ for some $0 < t < 1$, thus $tz \in H(y_0, r)$ and $|tz|_1 = t|z|_1 = t\tau < \tau = \inf\{|w|_1\colon w \in H(y_0, r)\}$ which is impossible. Define

$$z_0' = z + (\varepsilon/r)(y_0 - z) = (\varepsilon/r)y_0 + (1 - \varepsilon/r)z\ .$$

Observe that $z_0' \in K$ since it is a convex combination of y_0 and z; moreover $z_0' \in H(y_0, r - \varepsilon)$ since $\|y_0 - z_0'\| = (1 - \varepsilon/r)\|y_0 - z\| = (1 - \varepsilon/r)\cdot r = r - \varepsilon$.

Also

$$|z_0'|_1 \leqq (1 - \varepsilon/r)|z|_1 + \varepsilon/r|y_0|_1 \leqq (1 - \varepsilon/r)\tau + \varepsilon/r\ . \tag{10}$$

Let k_0 be large enough so that

(11) $$\|(y_0 - z_0')|_{[q_{m_{4(k_0-1)}},\infty)}\| < \delta \quad \text{and} \quad \|z_0'|_{[m_{4(k_0-1)},\infty)}\| < \delta ,$$

and let m be the greatest integer so that

(12) $$|z_0'|_1 + \sum_{k=k_0}^{k_0+m} |z_{4k}'|_1 = 1 - \eta \leqq 1 .$$

By (9) we see that

(13) $$0 \leqq \eta \leqq \varepsilon/r + \delta \quad \text{and} \quad (1 + m) < 2/\varepsilon .$$

Define

$$w^\varepsilon = z_0' + \sum_{k=k_0}^{k_0+m} z_{4k}' . \qquad \text{(See Figure 2.)}$$

By (12) $w^\varepsilon \in K$. The remainder of the proof involves some estimates which, we have a strong feeling, will not thrill the reader. We apologize for this. To start we wish to estimate $\|w^\varepsilon - y_0\|$. We have seen above that $\|z_0' - y_0\| = r - \varepsilon$, and $\|z_{4k}'|_{\sim J_{4k}}\| \leqq \delta/2^{m_{4k}}$ if $J_{4k} = [q_{m_{4k-1}}, q_{m_{4k+2}})$ (by (1)). The intervals $\{J_{4k}\}_{k=k_0}^{k_0+m}$ are disjoint and by (11), if $J = \bigcup_{k=k_0}^{k_0+m} J_{4k}$ then $\|(z_0' - y_0)|_J\| < \delta$. Furthermore $\|z_{4k}'\| \leqq |z_{4k}'|_1 \leqq \varepsilon/r + \delta$ and so $\|\sum_{k=k_0}^{k_0+m} z_{4k}'|_J\| \leqq \varepsilon/r + \delta + \sum_{k=k_0}^{k_0+m} \delta/2^{m_{4k}} < \varepsilon/r + 2\delta$. Putting all this together we have

$$\begin{aligned}\|w^\varepsilon - y_0\| = \left\|z_0' - y_0 + \sum_{k=k_0}^{k_0+m} z_{4k}'\right\| = \max\Bigg\{&\left\|\left(z_0' - y_0 + \sum_{k=k_0}^{k_0+m} z_{4k}'\right)\Big|_{\sim J}\right\| , \\ &\left\|\left(z_0' - y_0 + \sum_{k=k_0}^{k_0+m} z_{4k}'\right)\Big|_J\right\|\Bigg\} .\end{aligned}$$

The first term on the right side is bounded by $r - \varepsilon + \sum_{k=k_0}^{k_0+m} \delta/2^{m_{4k}} < r - \varepsilon + \delta$ and the second is bounded by $\varepsilon/r + 2\delta + \delta$. Thus

(14) $$\|w^\varepsilon - y_0\| \leqq r - \varepsilon + \delta .$$

The above estimates also yield that

(15) $$\left\|\sum_{k=k_0}^{k_0+m} z_{4k}'\right\| \leqq \max\{\varepsilon/r + 2\delta, \delta\} = \varepsilon/r + 2\delta .$$

Thus

$$\begin{aligned}\|w^\varepsilon - z\| = \left\| z_0' + \sum_{k=k_0}^{k_0+m} z_{4k}' - z\right\| &= \left\|\varepsilon/r(y_0 - z) + \sum_{k=k_0}^{k_0+m} z_{4k}'\right\| \\ &\leqq \varepsilon + \varepsilon/r + 2\delta ,\end{aligned}$$

and so

(16) $$\lim_{\varepsilon\to 0} \|w^\varepsilon - z\| = 0 ,$$

which looks promising. One further estimate we shall require is

$$\|w^{\varepsilon} - y_{m_{4k}}\| < r - \varepsilon + 6\delta \quad \text{for} \quad k_0 \leqq k \leqq k_0 + m \, . \tag{17}$$

To prove this set $A = [1, q_{m_{4(k_0-1)}})$ and $B = \sim A$. $\|(w^{\varepsilon} - y_{m_{4k}})|_A\| \leqq \|w^{\varepsilon} - y_0\| + \|(y_0 - y_{m_{4k}})|_A\| \leqq r - \varepsilon + \delta + \delta/2^{m_{4k}} < r - \varepsilon + 2\delta$ by (14) and (3). $\|(w^{\varepsilon} - y_{m_{4k}})|_B\| \leqq \|w^{\varepsilon} - (y_{m_{4k}} - y_0)|_B\| + \|y_0|_B\|$.

Now $\|y_0|_B\| < \delta$ by (2) and $\|(w^{\varepsilon} - (y_{m_{4k}} - y_0))|_B\| = \max\{\|(w^{\varepsilon} - (y_{m_{4k}} - y_0))|_{J_{4k}}\|, \|(w^{\varepsilon} - (y_{m_{4k}} - y_0))|_{B/J_{4k}}\|\}$ where as above $J_{4k} = [q_{m_{4k-1}}, q_{m_{4k+2}})$. $\|(w^{\varepsilon} - (y_{m_{4k}} - y_0))|_{J_{4k}}\| \leqq \|(w^{\varepsilon} - z'_{4k})|_{J_{4k}}\| + \|(z'_{4k} - (y_{m_{4k}} - y_0))|_{J_{4k}}\| \leqq 3\delta + r - \varepsilon + 2\delta = r - \varepsilon + 5\delta$ (by the definition of w^{ε}, (11), (1) and (8)). Also $\|(w^{\varepsilon} - (y_{m_{4k}} - y_0))|_{B/J_{4k}}\| \leqq \|z'_0|_{B/J_{4k}}\| + \|\sum_{j=k_0}^{k_0+m} z'_{4j}|_B\| + \|(y_{m_{4k}} - y_0)|_{B/J_{4k}}\| \leqq \delta + \varepsilon/r + 2\delta + \delta/2^{m_{4k}} < \varepsilon/r + 4\delta$. Compiling all this we get (17).

By (17) and (5) we have

$$\begin{aligned} \|Tw^{\varepsilon} - y_{m_{4k}}\| &\leqq \|Tw^{\varepsilon} - Ty_{m_{4k}}\| + \|Ty_{m_{4k}} - y_{m_{4k}}\| \\ &\leqq \|w^{\varepsilon} - y_{m_{4k}}\| + \|Ty_{m_{4k}} - y_{m_{4k}}\| \leqq r - \varepsilon + 7\delta \, . \end{aligned} \tag{18}$$

Let $\beta = \{\beta_i\}_{i=1}^{\infty} \in B(\ell_1)^+$ be such that $Tw^{\varepsilon} = \sum_{i=1}^{\infty} \beta_i x_i$ and $|Tw^{\varepsilon}|_1 = \sum_{i=1}^{\infty} \beta_i$.

By our earlier observation (using (6)) there exist integers ℓ_k, $m_{4k-2} < \ell_k < m_{4k-1}$ for $k_0 \leqq k \leqq k_0 + m$ so that $\sum_{i=p\ell_k}^{p\ell_{k+1}-1} \beta_i < \delta/2^{4k-1}$ and choose $\ell_{k_0+m+1} > m_{4(k_0+m)+2}$ such that $\sum_{k=\ell_{k_0+m+1}}^{\infty} \beta_i \leqq \delta/2^{4(k_0+m+1)}$. Define $d_0^{\varepsilon} = \sum_{i=1}^{p\ell_{k_0}} \beta_i x_i$, $d_k^{\varepsilon} = \sum_{i=p\ell_{k+1}}^{p\ell_{k+1}-1} \beta_i x_i$ for $k_0 \leqq k \leqq k_0 + m$ and $\widetilde{Tw^{\varepsilon}} = d_0^{\varepsilon} + \sum_{k=k_0}^{k_0+m} d_k^{\varepsilon}$.

It follows that

$$\|Tw^{\varepsilon} - \widetilde{Tw^{\varepsilon}}\| < \delta \, . \tag{19}$$

Now by (1)

$$\|d_k^{\varepsilon}|_{\sim[q_{\ell_k}, q_{\ell_{k+1}})}\| < \delta/2^{\ell_{k+1}} \quad \text{and} \quad \|d_0^{\varepsilon}|_{\sim[1, q_{\ell_{k_0}})}\| < \delta \, . \tag{19'}$$

Thus d_0^{ε} and the d_k^{ε}'s $k_0 \leqq k \leqq k_0 + m$ are "essentially disjointly supported" each one having essential support in $[q_{\ell_k}, q_{\ell_{k+1}})$. Also by (11) z'_0 is essentially supported in $[1, q_{m_{4(k_0-1)}}) \subseteq [1, q_{\ell_{k_0}})$, and for $k_0 \leqq k \leqq k_0 + m$, z'_k is essentially supported in $[q_{u_k}, q_{v_k}) \subseteq [q_{\ell_k}, q_{\ell_{k+1}})$. (See Figure 2.)

By (19) and (18) we get

$$\|y_{m_{4k}} - \widetilde{Tw^{\varepsilon}}\| < r - \varepsilon + 8\delta \, . \tag{20}$$

We claim

$$\|d_k^{\varepsilon} - (y_{m_{\cdot k}} - y_0)\| < r - \varepsilon + 10\delta \, ; \quad k_0 \leqq k \leqq k_0 + m \, . \tag{21}$$

As usual we shall need several intermediate estimates. Let $A_k = [q_{\ell_k}, q_{\ell_{k+1}})$. Then $\|(d_k^{\varepsilon} - \widetilde{Tw^{\varepsilon}})|_{A_k}\| \leqq 3\delta/2$ by the definition of $\widetilde{Tw^{\varepsilon}}$ and (19'). Also $\|y_0|_{A_k}\| < \delta/2$ by (2), and thus by (20), $\|(d_k^{\varepsilon} -$

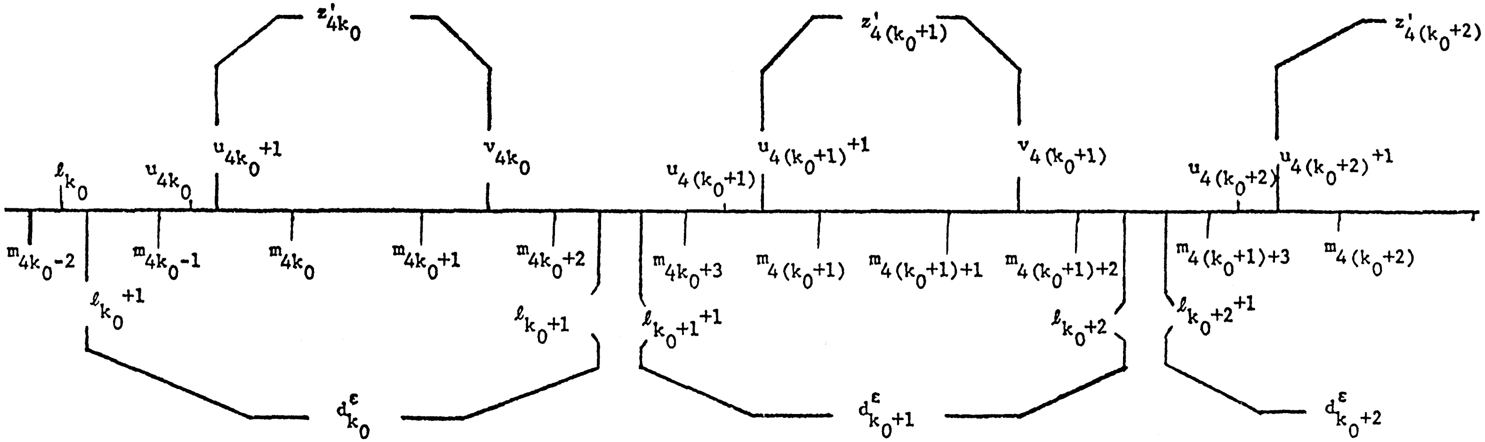

FIGURE 2. Let us illustrate by examples how to use Figure 2: z'_{4k_0} is pictured supported on $[u_{4k_0}+1. v_{4k_0}]$. Thus, z'_{4k_0} is a convex combination of 0 and $\{x_i: i\in[p_{u_{4k_0}}+1, p_{v_{4k_0}})\}$, and by (1) it is essentially supported on the interval $[q_{u_{4k_0}}, q_{v_{4k_0}})$ in c_0. $d^{\varepsilon}_{k_0}$ is pictured supported on $[\ell_{k_0}+1, \ell_{k_0+1})$; thus $d^{\varepsilon}_{k_0}$ is a convex combination of 0 and $\{x_i: i\in[p_{\ell_{k_0}+1}, p_{\ell_{k_0+1}})\}$ and is essentially supported in c_0 on $[q_{\ell_{k_0}}, q_{\ell_{k_0+1}})$.

$(y_{m_4k} - y_0))|_{A_k}\| \leqq \|(d_k^\varepsilon - \widetilde{Tw^\varepsilon})|_{A_k}\| + \|(\widetilde{Tw^\varepsilon} - y_{m_4k})|_{A_k}\| + \|y_0|_{A_k}\| \leqq 3\delta/2 + r - \varepsilon + 8\delta + \delta/2 = r - \varepsilon + 10\delta$. Also

$$\|(d_k^\varepsilon - (y_{m_4k} - y_0))|_{\sim A_k}\| \leqq \|d_k^\varepsilon|_{\sim A_k}\| + \|(y_{m_4k} - y_0)|_{\sim A_k}\| \leqq \delta$$

by (3), and (19′). This proves (21).

Next we wish to show that

$$|d_k^\varepsilon|_1 \geqq |z'_{4k}|_1 - 30\delta/r\ . \tag{22}$$

To see this define

$$A = \{n \in N: |d_k^\varepsilon(n) - (y_{m_4k} - y_0)(n)| > r/2\} \quad \text{and} \quad B = N/A\ .$$

If $n \in A$ and in addition $|d_k^\varepsilon(n)| > |(y_{m_4k} - y_0)(n)|/2$ then $|d_k^\varepsilon(n)| > r/6$. But then $|d_k^\varepsilon|_1 \geqq \|d_k^\varepsilon\| \geqq |d_k^\varepsilon(n)| > r/6 > |z'_{4k}|_1$ by (9) and so (22) holds. Thus we may assume that for $n \in A$

$$|d_k^\varepsilon(n)| \leqq |(y_{m_4k} - y_0)(n)|/2\ . \tag{23}$$

We wish to show that under these circumstances $d_k^\varepsilon + (30\delta/r)(y_{m_4k} - y_0) \in H(y_{m_4k} - y_0, r - \varepsilon)$. (22)then follows since $z_{4k} \in H'(y_{m_4k} - y_0, r - \varepsilon)$ and $|z'_{4k}|_1 \leqq |z_{4k}|_1$.

Suppose $n \in A$ and $(y_{m_4k} - y_0)(n)$ is positive. By (23) and the definition of A

$$(y_{m_4k} - y_0)(n) \geqq r/3 \tag{24}$$

and so

$$\begin{aligned} &|(y_{m_4k} - y_0)(n) - [d_k^\varepsilon(n) + (30\delta/r)(y_{m_4k} - y_0)(n)]| \\ &\quad = (y_{m_4k} - y_0)(n) - d_k^\varepsilon(n) - (30\delta/r)(y_{m_4k} - y_0)(n) \\ &\quad \leqq r - \varepsilon + 10\delta - (30\delta/r)(r/3) = r - \varepsilon\ . \end{aligned}$$

Here the expression in the absolute value sign is positive by (23) and the definition of A, and we applied (24) and (21) to obtain the inequality. A similar argument works if $(y_{m_4k} - y_0)(n) < 0$.

On the other hand, if $n \in B$ then

$$|(y_{m_4k} - y_0)(n) - [d_k^\varepsilon(n) + (30\delta/r)(y_{m_4k} - y_0)(n)]| \leqq r/2 + 30\delta/r < r - \varepsilon\ .$$

This proves (22).

Let up summarize the current situation. Given $z \in H'(y_0, r)$ we have constructed vectors $w^\varepsilon \in K$ so that (16) $\lim_{\varepsilon\to 0}\|w^\varepsilon - z\| = 0$, and vectors $d_0^\varepsilon, d_k^\varepsilon, k_0 \leqq k \leqq k_0 + m$ such that $\widetilde{Tw^\varepsilon} = d_0^\varepsilon + \sum_{k=k_0}^{k_0+m} d_k^\varepsilon$ satisfies $\|Tw^\varepsilon - \widetilde{Tw^\varepsilon}\| < \delta < \varepsilon^\varepsilon/100$ (by (19)).

By the definition of $\widetilde{Tw^\varepsilon}$ and (22) we have

$$1 \geqq |\widetilde{Tw^\varepsilon}|_1 = |d_0^\varepsilon|_1 + \sum_{k=k_0}^{k_0+m} |d_k^\varepsilon|_1 \geqq |d_0^\varepsilon|_1 + \sum_{k=k_0}^{k_0+m} |z'_{4k}|_1 - (30\delta/r)(m+1)\ .$$

Thus by (10), (12), (13) and the fact that $\delta < \varepsilon^2/100$

$$\begin{aligned}|d_0^\varepsilon|_1 &\leqq 1 - \sum_{k=k_0}^{k_0+m} |z'_k|_1 + (30\delta/r)(m+1) \leqq |z'_0|_1 + \eta + (30\delta/r)(2/\varepsilon) \\ &\leqq |z'_0|_1 + \varepsilon/r + \delta + 60\varepsilon/100r \leqq (1-\varepsilon/r)\tau + 2\varepsilon/r + \varepsilon^2/100 \\ &\quad + 60\varepsilon/100r\ .\end{aligned}$$

In particular

$$\limsup_{\varepsilon\to 0} |d_0^\varepsilon|_1 \leqq \tau \equiv |z|_1\ .$$

We want to show that $d_0^\varepsilon \in H(y_0, r)$ and so $|d_0^\varepsilon|_1 \geqq \tau$, from which it follows that $\lim_{\varepsilon\to 0} |d_0^\varepsilon|_1 = \tau$.

To this end set $I = [1, q_{\swarrow k_0})$. $\|d_0^\varepsilon - y_0\| = \max\{\|(d_0^\varepsilon - y_0)|_I\|, \|(d_0^\varepsilon - y_0)|_{\sim I}\|\}$. Now $\|(d_0^\varepsilon - y_0)|_{\sim I}\| \leqq \|d_0^\varepsilon|_{\sim I}\| + \|y_0|_{\sim I}\| < 2\delta$ by (1) and (2) and the definition of d_0^ε. Also

$$\begin{aligned}\|(d_0^\varepsilon - y_0)|_I\| &\leqq \|(d_0^\varepsilon - \widetilde{Tw}^\varepsilon)|_I\| + \|(\widetilde{Tw}^\varepsilon - y_{m_{4k}})|_I\| \\ &\quad + \|(y_{m_{4k}} - y_0)|_I\| \leqq \delta + r - \varepsilon + 8\delta + \delta = r - \varepsilon + 10\delta < r\ .\end{aligned}$$

(Use (1), (3) and (20)). It follows that $\|d_0^\varepsilon - y_0\| \leqq r$, i.e., $d_0^\varepsilon \in H(y_0, r)$ and thus for all ε,

$$|d_0^\varepsilon|_1 \geqq \tau \quad \text{and} \quad \lim_{\varepsilon\to 0} |d_0^\varepsilon|_1 = \tau\ . \tag{25}$$

It remains only to show that

$$\lim_{\varepsilon\to 0} \|\widetilde{Tw}^\varepsilon - d_0^\varepsilon\| = 0\ . \tag{26}$$

By the definition of $\widetilde{Tw}^\varepsilon$, (26) will follow if we show $\lim_{\varepsilon\to 0}\|d_k^\varepsilon\| = 0$ for all $k_0 \leqq k \leqq k_0 + m$, since the d_k^ε's are essentially disjointly supported in c_0. Using the fact that $|d_0^\varepsilon|_1 \geqq \tau$, (22), (10), (12) (13) and $\delta < \varepsilon^2/100$ we get

$$\begin{aligned}1 &\geqq |\widetilde{Tw}^\varepsilon|_1 = |d_0^\varepsilon|_1 + \sum_{k=k_0}^{k_0+m} |d_k^\varepsilon|_1 \geqq \tau + \sum_{k=k_0}^{k_0+m} |d_k^\varepsilon|_1 \geqq \tau + \sum_{k=k_0}^{k_0+m} |z'_k|_1 - 3\varepsilon/5r \\ &\geqq (1-\varepsilon/r)^{-1}(|z'_0|_1 - \varepsilon/r) + \sum_{k=k_0}^{k_0+m} |z'_{4k}|_1 - 3\varepsilon/5r = -\varepsilon/r(1-\varepsilon/r)^{-1} \\ &\quad -3\varepsilon/5r + (1 - r/r - \varepsilon)|z'_0|_1 + |z'_0|_1 + \sum_{k=k_0}^{k_0+m} |z'_{4k}| \\ &= 1 - \eta - \alpha(\varepsilon) = 1 - \beta(\varepsilon)\end{aligned}$$

where $\lim_{\varepsilon\to 0} \alpha(\varepsilon) = \lim_{\varepsilon\to 0} \beta(\varepsilon) = 0$. It follows that (since $\lim_{\varepsilon\to 0} |d_0^\varepsilon| = \lim |z'_0|_1 = \tau$)$\lim_{\varepsilon\to 0} (\sum_{k=k_0}^{k_0+m} |d_k^\varepsilon|_1 - \sum_{k=k_0}^{k_0+m} |z'_{4k}|_1) = 0$. By (9), $|z_{4k}|_1 \leqq \varepsilon/r + \delta$ and by (22), $|d_k^\varepsilon|_1 \geqq |z'_{4k}|_1 - 30\delta/r > |z'_{4k}|_1 - (30/100r)\varepsilon^2$. A simple calculation now shows $\lim_{\varepsilon\to 0} |d_k^\varepsilon|_1 = 0$. Indeed $0 \leqq \sum_{k_0}^{k_0+m} |d_k^\varepsilon|_1 - (|z'_{4k}|_1 - 3/10r^{-1}\varepsilon^2) \leqq \text{small} + (m+1)3\varepsilon^2/(10r) \leqq \text{small} + (2/\varepsilon)(3\varepsilon^2/10r) = \text{small} + (3/5r)\varepsilon = \text{small}$.

Thus each term is small and so $|d_k^\varepsilon|_1$ is small. Since $|d_k^\varepsilon|_1 \geqq \|d_k^\varepsilon\|$, $\lim_{\varepsilon\to 0}\|d_k^\varepsilon\| = 0$ too, and (26) follows.

This completes the proof of the theorem in the case $\|y_0\| \geqq r$. If $\|y_0\| < r$ i.e., $H'(y_0, r) = \{0\}$, we define $z_0' = 0$, and construct w^ε, and $\widetilde{Tw^\varepsilon}$ as before. All the relevant estimates will continue to hold in this case, many of them trivially so. We omit the details.

III. Some extensions of the main result. Clearly not all convex weakly compact subsets of c_0 can be represented as the closed convex hull of a weakly convergent sequence. Moreover some convex weakly compact subsets of c_0 are not even contained in the closed convex hull of any weakly convergent sequence. (Such a set, for example is $K_2 = \{x \in c_0 : x \geqq 0\ \sum_{i=1}^\infty (x(i))^2 \leqq 1\}$.)

However, the proof presented above can be generalized to include a larger class of sets. For example the proof of the special case can be extended to cover the set $\{x = (x(i)) : x(i) \geqq 0$ and $\sum_{i=1}^\infty x(i)^p \leqq 1\}$ where $1 \leqq p < \infty$. More generally we have the following theorem. The set $K(p, w)$ below is the image in c_0 under the formal identity map of the positive cone of a Lorentz sequence space.

THEOREM 2. *Let $1 \leqq p < \infty$, and let $w = (w_1, w_2, \cdots)$ be a decreasing sequence of nonnegatives with $\sum_{i=1}^\infty w_i = \infty$. Then the set $K(p, w) = \{x \in c_0 : x \geqq 0, \sum_{i=1}^\infty (\tilde{x}(i))^p w_i \leqq 1\}$ (where $\tilde{x}$ is the decreasing rearrangement of x) has the fixed point property for nonexpansive mappings.*

Note that for $w = (1, 1, \cdots)$ $K_{(p,w)} = K_p = \{x \in c_0 : x \geqq 0\ \|x\|_p \leqq 1\}$ where $\|\cdot\|_p$ is the ℓ_p norm.

Certainly there are other sets $k \subsetneqq c_0$ with the fixed point property to which the above arguments apply. We did not, however, formally axiomatize the properties required of K to make our proof work. We suspect that every weakly compact convex K in c_0 has the fixed point property.

REFERENCES

0. D. Alspach, *A fixed point free nonexpansive map*, to appear.
1. S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math., **3** (1922), 133–181.
2. L. P. Belluce and W. A. Kirk, *Nonexpansive mappings and fixed points in Banach spaces*, Illinois J. Math., **11** (1967), 474–479.
3. F. E. Browder, *Nonexpansive nonlinear operators in Banach spaces*, Proc. Nat. Acad. Sci., U.S.A., **54** (1965), 1041–1044.
4. D. Gohde, *Zum Prinzip der kontractiven Abbildung*, Math. Nachr., **30** (1965), 251–258.

5. R. Haydon, E. Odell and Y. Sternfeld, *A fixed point theorem for a class of star-shaped sets in G*, to appear.
6. W. B. Johnson, *On quotients of L_p which are quotients of ℓ_p*, Compositio Math., **34** (1977), 69-89.
7. W. B. Johnson and M. Zippin, *On subspaces of quotients of $(\Sigma G_n)_{\ell_p}$ and $(\Sigma G_n)_{c_0}$*, Israel J. Math., **13** (1976), 311-316.
8. L. A. Karlovitz, *On nonexpansive mappings*, Proc. Amer. Math. Soc., **55** (1970), 321-325.
9. ———, *Existence of fixed points of nonexpansive mappings in a space without normal structure*, Pacific J. Math., **66** (1976), 153-159.
10. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, The Amer. Math. Monthly, **72** (1965), 1004-1006.
11. J. Lindenstrauss and L. Tzafriri, *Classical Bananch Spaces I. Sequence Spaces*, Springer-Verlag, 1977.
12. I. Namioka, *Neighborhoods of extreme points*, Israel J. Math., **5** (1967), 145-152.
13. S. Reich, *The fixed point property for nonexpansive mappings*, The Amer. Math. Monthly, **83** (1976), 266-268.
14. A. Tychonoff, *Ein Fixpunktsatz*, Math. Ann., **III** (1935), 767-776.

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