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LINKED QUATERNIONIC MAPPINGS AND THEIR ASSOCIATED WITT RINGS

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A quaternionic mapping is a symmetric bilinear mapping $q: G\times G\to B$, where G, B are Abelian groups, G has exponent 2 and contains a distinguished element -1 such that $q(a, a)=q(a, -1)$ $\forall a\in G$. Such a mapping is said to be linked if $q(a, b)=q(c, d)$ implies the existence of $x\in G$ such that $q(a, b)=q(a, x)$ and $q(c, d)=q(c, x)$. The Witt ring $W(q)$ of such a mapping q can be defined to be the integral group ring $\boldsymbol{Z}[G]$ factored by the ideal generated by $1+(-1)$ and the elements $(a+b)-(c+d)$ such that $ab=cd$ and $q(a, b)=q(c, d)$. If q is the quaternionic mapping associated to a field or semi-local ring A with $2\in A^{\cdot}$, then q is linked, and $W(q)$ is the Witt ring of free bilinear spaces over A. This paper gives a ring-theoretic description of the class of rings $W(q)$, q linked. In particular, all such rings are shown to be strongly representational in the terminology of Kleinstein and Rosenberg.

1. Introduction. Throughout this section, F will denote a field or semi-local ring with $2\in F^{\cdot}$ such that all residue class fields contain more than 3 elements. Let B_F denote the Brauer group of F, $G_F = F^{\cdot}/F^{\cdot 2}$, and let $q_F: G_F \times G_F \to B_F$ denote the quaternion algebra mapping. Then q_F satisfies

(A) q_F is symmetric and bilinear, i.e.,

$$\forall a, b, c\in G_F,\ q_F(a, b) = q_F(b, a)$$

and

$$q_F(a, bc) = q_F(a, b)q_F(a, c)\ .$$

(B) $\forall a\in G_F$, $q_F(a, a) = q_F(a, -1)$.

In the case F is a field (A) is [8, 2.11, p. 61] and (B) is [8, 2.6, p. 58]. The corresponding results for semi-local rings may be found in [2, p. 22–29].

It is well known that isometry of (quadratic) forms over F is describable in terms of q_F. For forms of dimension one and two we have $(a)\cong(b)\Leftrightarrow a=b$, and $(a, b)\cong(c, d)\Leftrightarrow ab=cd$ and $q_F(a, b)=q_F(c, d)$. The proof of this statement given for fields in [8, 2.9, p. 60] will work as well in the semi-local ring case. For higher dimensional forms $f\cong g\Leftrightarrow\exists$ a sequence of forms $f=f_0, f_1, \cdots, f_k=g$ such that for each $i=1, \cdots, k$, f_i is obtained from f_{i-1} by replacing two diagonal entries a, b by c, d with $(a, b)\cong(c, d)$. For the proof of this last assertion, see [11, Satz 7] in case F is a field, and [7, Lemma 1.14] in case F is a semi-local ring.

In turn, this gives a description of the Witt ring W_F of quadratic forms over F in terms of q_F: W_F is the integral group ring $Z[G_F]$ factored by the ideal generated by $1+(-1)$ and the elements $(a+b)-(c+d)$ such that $ab=cd$ and $q_F(a, b)=q_F(c, d)$.

More generally, consider an abstract mapping $q: G \times G \rightarrow B$ where G and B are Abelian groups and G has exponent 2 (i.e., $a^2=1\ \forall a \in G$). If such a mapping satisfies properties (A) and (B) above for some distinguished element $-1 \in G$, we will say q is a *quaternionic mapping*. If this is the case, we can certainly define isometry of (abstract) forms by the above formulas (see [4]), and construct an associated (abstract) Witt ring $W(q)$. Certainly some of the classical quadratic form theory will carry over to this abstract situation.

The goal of this paper is to develop a much more refined theory. The key observation is that q_F has an additional important property.

(L) $\quad q_F(a, b)=q_F(c, d) \Rightarrow \exists x \in G_F$ such that $q_F(a, b)=q_F(a, x)$

and $q_F(c, d)=q_F(c, x)$. In case F is a field, this is an exercise in Lam's book [8, p. 69, 12]. Here is a sketch of the proof in the semi-local case: First note that

$$(1) \quad q_F(a, b)=q_F(c, d) \Leftrightarrow (1,-a) \otimes (1,-b) \cong (1,-c) \otimes (1,-d),$$

using [2, 1.19, p. 29]. Expanding and using Witt cancellation, this, in turn, is equivalent to $(-b, ab) \oplus (d,-cd) \cong (a,-c) \oplus (1,-1)$. Thus, by transversality [3, 2.7(c)], $\exists x \in G_F$ such that $(-b, ab) \cong (-x, ax)$ and $(d,-cd) \cong (x,-cx)$. It follows easily from this (for example, use (1) again), that $q_F(a, b)=q_F(a, x)$ and $q_F(c, d)=q_F(c, x)$.

A quaternionic mapping $q: G \times G \rightarrow B$ is said to be *linked* if it satisfies (L). In this paper, we examine the form theory associated to a linked quaternionic mapping and develop properties of the associated Witt ring $W(q)$. In Theorem 2.6 the following *cancellation property* for forms is shown to hold:

$$f \cong f' \text{ and } f \oplus g \cong f' \oplus g' \Rightarrow g \cong g'.$$

It follows from this that each form has a well-defined anisotropic part and Witt index, and that $W(q)$, as a set, can be described as the equivalence classes of forms with respect to Witt equivalence, exactly as in [11]. In Theorem 2.7, the following *representation property* for forms is proved:

$$D(f \oplus g) = \cup \{D(a, b) \mid a \in D(f),\ b \in D(g)\}\ .$$

(Here, $D(f)$ denotes the set of elements of G represented by the form f.) This implies that $W(q)$ is representational in the terminology of [5]. The exact relationship between linked quaternionic mappings and representational Witt rings is presented in Theorem 3.8 following the introduction of the signed discriminant and the Witt invariant. In Theorem 3.11, it is proved that $W(q)$ is reduced (i.e., has nilradical equal to zero) if and only if q satisfies

$$\text{(R)} \qquad \forall a \in G,\ q(a, a) = 1 \Rightarrow a = 1\ .$$

This special case is of interest since, as pointed out in [5], the reduced representational Witt rings are just the Witt rings of spaces of orderings as presented, for example, in [9].

2. The form theory.

Throughout, assume that $q\colon G \times G \to B$ is a *linked quaternionic mapping*. Recall, from the introduction, this means G, B are Abelian groups, G has exponent 2 and a distinguished element -1, and q satisfies

(A) q is symmetric and bilinear,

(B) $q(a, a) = q(a, -1)$ $\forall a \in G$, and

(L) $q(a, b) = q(c, d) \Rightarrow \exists x \in G$ such that $q(a, b) = q(a, x)$ and $q(c, d) = q(c, x)$.

It is worth pointing out, to begin with, that $\forall a, b \in G$, $q(a, b)^2 = q(a, b^2) = q(a, 1) = 1$. In particular, the subgroup of B generated by the image of q has exponent 2. Also, note that $q(a, -a) = q(a, -1)q(a, a) = q(a, -1)^2 = 1$.

By a *form of dimension* $n \geqq 1$ (over G) is meant an n-tuple $f = (a_1, \cdots, a_n)$ with $a_1, \cdots, a_n \in G$. The *discriminant* and *Hasse invariant* of such a form f are defined by

$$(2) \qquad d(f) = \prod_i a_i, \text{ and } s(f) = \prod_{i>j} q(a_i, a_j)\ .$$

The *sum* of f and g, with f as above and $g = (b_1, \cdots, b_m)$, is defined by $f \oplus g = (a_1, \cdots, a_n, b_1, \cdots, b_m)$. *Isometry* of one and two dimensional forms is defined by

(3) $(a) \cong (b) \Leftrightarrow a = b$, and

(4) $(a, b) \cong (c, d) \Leftrightarrow ab = cd$ and $q(a, b) = q(c, d)$.

For forms of dimension $n \geqq 3$, *isometry* is defined inductively by

$$(5) \qquad (a_1, \cdots, a_n) \cong (b_1, \cdots, b_n) \Leftrightarrow \exists a, b, c_3, \cdots, c_n \in G$$

such that $(a_2, \cdots, a_n) \cong (a, c_3, \cdots, c_n)$, $(b_2, \cdots, b_n) \cong (b, c_3, \cdots, c_n)$ and $(a_1, a) \cong (b_1, b)$. It will follow from 2.4 that this definition coincides with the one given in the introduction.

THEOREM 2.1. *If* $b_1, \cdots, b_n$ *is a permutation of* $a_1, \cdots, a_n$, *then* $(a_1, \cdots, a_n) \cong (b_1, \cdots, b_n)$.

Proof. We may assume $n \geqq 3$. If $b_1 = a_i$, $i \geqq 2$, take $a = a_i$, $b = a_1$, and take $c_3, \cdots, c_n$ to be the elements left after a_1, a_i are deleted from $a_1, \cdots, a_n$. Note that $a, c_3, \cdots, c_n$ is a permutation of $a_2, \cdots, a_n$; $b, c_3, \cdots, c_n$ is a permutation of $b_2, \cdots, b_n$; and b_1, b is a permutation of a_1, a, so the result is true by induction. On the other hand, if $b_1 = a_1$, take $a = b = a_2$, and $c_i = a_i$, $i \geqq 3$. □

THEOREM 2.2. *If* $f \cong g$ *then* $\dim(f) = \dim(g)$, $d(f) = d(g)$, *and* $s(f) = s(g)$. *The converse holds for forms of dimension* $n \leqq 3$.

Proof. It is clear that the theorem and its converse hold for 1 and 2 dimensional forms, by (3) and (4). (Note: if f is 1-dimensional, then $s(f) = 1$, by definition.) Now let $f = (a_1, \cdots, a_n)$, $g = (b_1, \cdots, b_n)$, $n \geqq 3$. First suppose $f \cong g$, and choose $a, b, c_3, \cdots, c_n$ as in (5). Then, by induction, $a_2 \cdots a_n = ac_3 \cdots c_n$, $b_2 \cdots b_n = bc_3 \cdots c_n$, and $a_1 a = b_1 b$, so $a_1 a_2 \cdots a_n = a_1 a c_3 \cdots c_n = b_1 b c_3 \cdots c_n = b_1 b_2 \cdots b_n$. Also, using

$$s(f \oplus h) = s(f) \cdot s(h) \cdot q(d(f), d(h)) \tag{6}$$

(this is easily verified), we have, by induction,

$$\begin{aligned} s(f) &= s(a_2, \cdots, a_n) q(a_1, a_2 \cdots a_n) \\ &= s(a, c_3, \cdots, c_n) q(a_1, ac_3 \cdots c_n) \\ &= s(c_3, \cdots, c_n) q(a, c_3 \cdots c_n) q(a_1, ac_3 \cdots c_n) \\ &= s(c_3, \cdots, c_n) q(aa_1, c_3 \cdots c_n) q(a_1, a) \\ &= s(c_3, \cdots, c_n) q(bb_1, c_3 \cdots c_n) q(b_1, b) = \cdots = s(g) . \end{aligned}$$

Now suppose $n = 3$, $d(f) = d(g)$, and $s(f) = s(g)$. Thus $a_3 = a_1 a_2 x$, $b_3 = b_1 b_2 x$ where x denotes the common discriminant. Thus using properties (A) and (B) of q,

$$\begin{aligned} s(a_1, a_2, a_1 a_2 x) &= q(a_2, a_1 a_2 x) q(a_1, a_2) q(a_1, a_1 a_2 x) \\ &= q(a_2, a_1 a_2 x) q(a_1, a_1 x) = q(a_2, a_1 a_2 x) q(a_2, -a_2) q(-a_1 x, a_1 x) q(a_1, a_1 x) \\ &= q(a_2, -a_1 x) q(-x, a_1 x) = q(a_2, -a_1 x) q(-x, -a_1 x) q(-x, -1) \\ &= q(-a_2 x, -a_1 x) q(-x, -1) . \end{aligned}$$

Here, as always, $-a$ denotes the element $(-1)(a) \in G$. We record this result:

$$s(a_1, a_2, a_1 a_2 x) = q(-a_1 x, -a_2 x) q(-x, -1) . \tag{7}$$

If we do the same computation for g, we see that the equality of the Hasse invariants implies $q(-a_1x, -a_2x) = q(-b_1x, -b_2x)$. Thus, by (L), $\exists y \in G$ such that $q(-a_1x, -a_2x) = q(-a_1x, y)$ and $q(-b_1x, -b_2x) = q(-b_1x, y)$. Take $c_3 = -xy$, $a = -a_1y$, and $b = -b_1y$. Now it is just a matter of checking $(a_2, a_1a_2x) \cong (a, c_3)$, $(b_2, b_1b_2x) \cong (b, c_3)$ and $(a_1, a) \cong (b_1, b)$. Clearly, the discriminants are the same and

$$\begin{aligned} q(a_2, a_1a_2x) &= q(a_2, a_1a_2x)q(a_2, -a_2) = q(a_2, -a_1x) \\ &= q(-a_2x, -a_1x)q(-x, -a_1x) = q(-a_1x, y)q(-x, -a_1x) \\ &= q(-a_1x, -xy) = q(-a_1x, -xy)q(xy, -xy) = q(-a_1y, -xy) \\ &= q(a, c_3)\ . \end{aligned}$$

Similarly $q(b_2, b_1b_2x) = q(b, c_3)$. Finally, using $q(-a_1x, y) = q(-b_1x, y)$, we have

$$\begin{aligned} q(a_1, a) &= q(a_1, -a_1y) = q(a_1, y) = q(-a_1x, y)q(-x, y) \\ &= q(-b_1x, y)q(-x, y) = q(b_1, y) = q(b_1, -b_1y) = q(b_1, b)\ . \qquad \square \end{aligned}$$

THEOREM 2.3. *Isometry is a transitive relation.*

(*Note.* Since isometry is clearly reflexive and symmetric, this implies it is an equivalence relation.)

Proof. Suppose f, g, h are n dimensional forms with $f \cong g \cong h$. We show $f \cong h$ by induction on n. By 2.2, we may assume $n \geqq 4$. Let the elements $a, b, c \in G$ and the $n-1$ dimensional forms f', g', h' be defined by $f = (a) \oplus f'$, $g = (b) \oplus g'$, $h = (c) \oplus h'$. Thus, by assumption, $\exists a', b', b'', c' \in G$ and $n-2$ dimensional forms i, j such that $f' \cong (a') \oplus i$, $g' \cong (b') \oplus i$, $g' \cong (b'') \oplus j$, $h' \cong (c') \oplus j$, $(a, a') \cong (b, b')$, and $(b, b'') \cong (c, c')$. Thus, by induction, $(b') \oplus i \cong (b'') \oplus j$, so $\exists b_1, b_2 \in G$ and an $n-3$ dimensional form k satisfying $i \cong (b_1) \oplus k$, $j \cong (b_2) \oplus k$, and $(b', b_1) \cong (b'', b_2)$. It follows that $(a, a', b_1) \cong (b, b', b_1) \cong (b, b'', b_2) \cong (c, c', b_2)$, so, using transitivity in the case $n=3$, $\exists a_1, c_1, x \in G$ such that $(a', b_1) \cong (a_1, x)$, $(c', b_2) \cong (c_1, x)$, and $(a, a_1) \cong (c, c_1)$. Take $l = (x) \oplus k$. Then $f' \cong (a') \oplus i \cong (a', b_1) \oplus k \cong (a_1, x) \oplus k = (a_1) \oplus l$, and $h' \cong (c') \oplus j \cong (c', b_2) \oplus k \cong (c_1, x) \oplus k = (c_1) \oplus l$. Thus, by induction, $f' \cong (a_1) \oplus l$ and $h' \cong (c_1) \oplus l$. Since $(a, a_1) \cong (c, c_1)$, this completes the proof. $\square$

COROLLARY 2.4. $f \cong g \Leftrightarrow$ *there exists a sequence of forms* $f = f_0, f_1, \cdots, f_k = g$, $k \geqq 0$, *such that for each* $i = 1, \cdots, k$, f_i *is obtained from* f_{i-1} *by replacing two entries* a, a' *by* b, b' *respectively, where* $(a, a') \cong (b, b')$.

Proof. The implication $(\Rightarrow)$ is immediate, by induction on

$n = \dim(f)$. To prove ($\Leftarrow$), we may assume $n \geqq 3$, and, by 2.3, that $k=1$. Thus, by 2.1, $f \cong (a, a', c_3, \cdots, c_n)$ and $g \cong (b, b', c_3, \cdots, c_n)$. Now it is clear $(a, a', c_3, \cdots, c_n) \cong (b, b', c_3, \cdots, c_n)$. Thus, by 2.3, $f \cong g$. □

LEMMA 2.5. *For arbitrary forms f, g, g', $g \cong g' \Leftrightarrow f \oplus g \cong f \oplus g'$.*

Proof. We may assume f is 1-dimensional, say $f = (a_1)$.

($\Rightarrow$): Define $a, c_3, \cdots, c_n$ by $g = (a, c_3, \cdots, c_n)$ and let $b = a$. Then $f \oplus g \cong f \oplus g'$ by (5).

($\Leftarrow$): By (5) $\exists a, b, c_3, \cdots, c_n$ such that $g \cong (a, c_3, \cdots, c_n)$, $g' \cong (b, c_3, \cdots, c_n)$ and $(a_1, a) \cong (a_1, b)$. Comparing discriminants, this yields $a = b$, so $g \cong (a, c_3, \cdots, c_n) \cong g'$. Thus $g \cong g'$ by 2.3. □

THEOREM 2.6. *Suppose f, f', g, g' are forms satisfying $f \cong f'$. Then $g \cong g' \Leftrightarrow f \oplus g \cong f' \oplus g'$.*

Proof. Since $f \cong f'$, it follows from 2.1 and 2.5 that $f \oplus g \cong f' \oplus g$. Thus, $f \oplus g \cong f' \oplus g' \Leftrightarrow f' \oplus g \cong f' \oplus g' \Leftrightarrow g \cong g'$ by 2.3 and 2.5. □

For $f = (a_1, \cdots, a_n)$, $g = (b_1, \cdots, b_m)$ and $a \in G$ let us define $af := (aa_1, \cdots, aa_n)$, and $f \otimes g := (a_1b_1, \cdots, a_1b_m, \cdots, a_nb_1, \cdots, a_nb_m)$. (Thus $af = (a) \otimes f$.)

THEOREM 2.7. (i) *If $f \cong f'$, then $af \cong af'$.*
(ii) *If $f \cong f'$ and $g \cong g'$, then $f \otimes g \cong f' \otimes g'$.*

Proof. Let $f = (a_1, \cdots, a_n)$. (i) is clear if $n = 1$. Suppose $n = 2$, and that $f' = (a_1', a_2')$. Then $a_1a_2 = a_1'a_2'$ and $q(a_1, a_2) = q(a_1', a_2')$. It follows that af and af' have the same discriminant and $q(aa_1, aa_2) = q(a, a)q(a, a_1a_2)q(a_1, a_2) = q(a, a)q(a, a_1'a_2')q(a_1', a_2') = q(aa_1', aa_2')$. Thus $af \cong af'$. The result for $n \geqq 3$ follows by a simple inductive argument.

To prove (ii), note $f \otimes g \cong a_1g \oplus \cdots \oplus a_ng \cong a_1g' \oplus \cdots \oplus a_ng' \cong f \otimes g'$, using part (i) and 2.6. Similarly $f \otimes g' \cong f' \otimes g'$, so $f \otimes g \cong f' \otimes g'$. □

We say a form f of dimension n *represents* $x \in G$ if $\exists x_2, \cdots, x_n \in G$ such that $f \cong (x, x_2, \cdots, x_n)$. Let us denote by $D(f)$ the set of elements $x \in G$ represented by f in this sense.

THEOREM 2.8. *If f and g are arbitrary forms, then*

$$D(f \oplus g) = \cup \{D(x, y) \mid x \in D(f),\ y \in D(g)\}\ .$$

Proof. To prove the nontrivial inclusion let $f = (a_1, \cdots, a_k)$,

$g=(a_{k+1}, \cdots, a_n)$, and suppose $f\oplus g\cong(b_1, \cdots, b_n)$. Choose $a, b, c_3, \cdots, c_n$ as in (5). Thus $b_1 \in D(a_1, a)$. This completes the proof if $k=1$ (take $x=a_1$, $y=a$). If $k\geqq 2$, then, by induction on k, $\exists x' \in D(a_2, \cdots, a_k)$, $y\in D(g)$ such that $a\in D(x', y)$. Thus, $b_1\in D(a_1, a)\subseteqq D(a_1, x', y)=D(y, a_1, x')$, so by the case $k=1$, $\exists x\in D(a_1, x')$ such that $b_1\in D(y, x)=D(x, y)$. Since $D(a_1, x')\subseteqq D(f)$, this completes the proof. □

Note that $(a, -a)\cong(1, -1)$ $\forall a\in G$ by (4), since $q(a, -a)=1=q(1, -1)$. Any form $(a, -a)$, $a\in G$ will be called a *hyperbolic form.* A form f will be called *isotropic* if $\exists$ a form g such that $f\cong(1, -1)\oplus g$. Otherwise f will be called *anisotropic.* The following version of 2.8 is useful.

COROLLARY 2.9. *Let f, g be forms, and suppose $f\oplus g$ is isotropic. Then $\exists x\in D(f)$ such that $-x\in D(g)$.*

Proof. (Compare to [5, 2.4] and [9, 2.2].) Let a, f', and h be such that $f=(a)\oplus f'$ and $f\oplus g\cong(1, -1)\oplus h\cong(a, -a)\oplus h$. Thus $f'\oplus g\cong(-a)\oplus h$ by 2.6. Suppose $\dim(f')\geqq 1$. Then, by 2.8, $\exists b\in D(g)$, $c\in D(f')$, $d\in G$ such that $(b, c)\cong(-a, d)$. Adding $(a, -b)$ to both sides and cancelling using 2.6, this yields $(a, c)\cong(-b, d)$. Thus, $-b\in D(a, c)\subseteqq D(f)$, i.e., $x=-b$ satisfies the required conditions. If, on the other hand, $\dim(f')=0$, then $x=a$ works. □

3. The Witt ring. We can now define the Witt ring associated to the linked quaternionic mapping q exactly as in [11]. First note that every form f over G decomposes as

$$f\cong f_a\oplus k\times(1, -1) \tag{8}$$

with f_a an anisotropic (possibly zero dimensional) form, and $k\geqq 0$. Here, $k\times g$ denotes $g\oplus\cdots\oplus g$ (k times) or the zero dimensional form if $k=0$. Using the cancellation property 2.6, k is uniquely determined by f, and f_a is determined, up to isometry, by f. Let us refer to f_a as the *anisotropic part of f*, to k as the *Witt index of f*, and to (8) as the *Witt decomposition of f.*

Two forms f, g (not necessarily of the same dimension) are said to be *Witt equivalent*, denoted $f\sim g$, if their anisotropic parts are isometric. It is clear that

$$f\sim g\Rightarrow\dim(f)\equiv\dim(g)\pmod 2 \tag{9}$$

and

$$f\cong g\Leftrightarrow f\sim g \text{ and } \dim(f)=\dim(g)\,. \tag{10}$$

Let us denote by W the set of equivalence classes of forms with respect to Witt equivalence. It is easily verified, using 2.6 and 2.7, that $\oplus$ and $\otimes$ induce binary operations on W, and by the same elementary arguments as in [11], W becomes a commutative ring with unity. We will refer to the ring W so constructed as the *Witt ring associated to* q, and will denote this by writing $W = W(q)$.

We remark in passing that we have the following description of $W(q)$.

THEOREM 3.1. *$W(q)$ is isomorphic to the integral group ring $\boldsymbol{Z}[G]$ factored by the ideal generated by $1 + (-1)$ and the elements $(a + b) - (c + d)$ where $(a, b) \cong (c, d)$.*

Proof. On the basis of 2.4 the proof is the same as in the classical case, cf. [8, Exc. 1, p. 49]. □

Denote by $I(q)$ the ideal of even dimensional forms in $W(q)$. Clearly $W(q)/I(q) \cong \boldsymbol{Z}/2\boldsymbol{Z}$. Since $(a, b) \sim (1, a) - (1, -b)$, $I(q)$ is generated additively by the *1-fold Pfister forms* $(1, -a)$, $a \in G$. Thus $I^k(q)$ is generated additively by the *k-fold Pfister forms* $(1, -a_1) \otimes (1, -a_2) \otimes \cdots \otimes (1, -a_k)$, $a_1, \cdots, a_k \in G$.

We now modify the discriminant and Hasse invariant in a standard way (eg. see [8, p. 123]) to obtain invariants with respect to Witt equivalence. Namely, we define the *signed discriminant* and the *Witt invariant* by

$$d_{\pm}(f) = (-1)^{\alpha} d(f), \text{ where } \alpha = n(n-1)/2,\ n = \dim(f)\,, \tag{11}$$

and

$$w(f) = s(f)q(-1, d(f))^{\varepsilon} q(-1, -1)^{\eta}\,, \tag{12}$$

where $\varepsilon = (n-1)(n-2)/2$, $\eta = (n+1)(n)(n-1)(n-2)/24$, and $n = \dim(f)$.

THEOREM 3.2. (i) *$d_{\pm}$: $W(q) \to G$ is well-defined.*

(ii) *The restriction of $d_{\pm}$ to the additive group $I(q)$ is a group homomorphism.*

(iii) *$I(q)/I^2(q) \cong G$.*

Proof. Suppose $f = g$ in $W(q)$. We may assume $f \cong g \oplus k \times (1, -1)$ for some $k \in \boldsymbol{Z}$. By 2.2, $d(f) = d(g)$ if $k \equiv 0 \pmod 4$ and $d(f) = -d(g)$ if $k \equiv 2 \pmod 4$. Consequently, $d_{\pm}(f) = d_{\pm}(g)$ and (i) is proved. Suppose $f_1, f_2 \in I(q)$ and $\dim f_1 = m_1$, $\dim f_2 = m_2$. Then

$$\begin{aligned} d_{\pm}(f_1 \oplus f_2) &= (-1)^{(m_1+m_2)(m_1+m_2-1)/2} d(f_1 \oplus f_2) \\ &= (-1)^{m_1(m_1-1)/2}(-1)^{m_2(m_2-1)/2}(-1)^{m_1 m_2} d(f_1)d(f_2) = d_{\pm}(f_1)d_{\pm}(f_2) \end{aligned}$$

and (ii) is proved. Since $d_{\pm}((1, -a) \otimes (1, -b)) = d_{\pm}(1, -a, -b, ab) = 1$, the kernel of $d_{\pm}: I(q) \to G$ contains $I^2(q)$. Since $(1, -a) \oplus (1, -b) \sim (1, -ab) \oplus (1, -a) \otimes (1, -b)$, every element $f \in I(q)$ has the form $f = (1, -a)$ modulo $I^2(q)$. Hence $d_{\pm}(f) = 1 \Leftrightarrow d_{\pm}(1, -a) = 1 \Leftrightarrow a = 1 \Rightarrow f \in I^2(q)$. Thus the kernel is exactly $I^2(q)$. This proves (iii). □

THEOREM 3.3. (i) *If f is an arbitrary form and g is a form satisfying $d_{\pm}(g) = 1$, $\dim(g) \equiv 0 \pmod 2$, then $w(f \oplus g) = w(f)w(g)$.*

(ii) *$w: W(q) \to B$ is well-defined.*

(iii) *$w: I^2(q) \to B$ is a group homomorphism with $I^3(q) \subseteq \ker(w)$.*

Proof. (i) Note that $\dim(f \oplus g) = \dim(f) + \dim(g)$, $d(f \oplus g) = d(f)d(g)$ and $s(f \oplus g) = s(f)s(g)q(d(f), d(g))$ by (6). By hypothesis, $\dim(g) = 2k$, and $d_{\pm}(g) = 1$, so $d(g)$ is either 1 or -1 depending on whether k is even or odd. The conclusion of (i) now follows from a lengthy (but elementary) computation.

(ii) Taking $g = (1, -1)$ in (i), we have $w(f \oplus (1, -1)) = w(f)w(1, -1) = w(f)$. It follows from this and 2.2, that $f \sim h \Rightarrow w(f) = w(h)$.

(iii) By 3.2, $I^2(q)$ consists of those elements of $W(q)$ represented by forms f satisfying $\dim(f) \equiv 0 \pmod 2$ and $d_{\pm}(f) = 1$. Thus the fact that $w: I^2(q) \to B$ is a group homomorphism is a special case of (i). Finally observe that

$$\begin{aligned} s(a(1, -b) \otimes (1, -c)) &= s(a, -ab, -ac, abc) \\ &= q(a, a)q(-ab, -b)q(-ac, abc) = q(a, a)q(-a, -b)q(-ac, b) \\ &= q(a, a)q(-a, -1)q(c, b) = q(-1, -1)q(b, c) \,. \end{aligned}$$

It follows that

$$w(a(1, -b) \otimes (1, -c)) = q(b, c) \quad \forall a, b, c \in G \,. \tag{13}$$

Thus,

$$\begin{aligned} w((1, -a) \otimes (1, -b) \otimes (1, -c)) &= w((1, -b) \otimes (1, -c) \\ &\oplus -a(1, -b) \otimes (1, -c)) = q(b, c)q(b, c) = 1 \quad \forall a, b, c \in G \,, \end{aligned}$$

so $I^3(q) \subseteq \ker(w)$. □

COROLLARY 3.4. *Let $a, b, c, d \in G$. Then the following are equivalent.*

(i) $q(a, b) = q(c, d)$,

(ii) $(1, -a) \otimes (1, -b) \cong (1, -c) \otimes (1, -d)$,
(iii) $(1, -a) \otimes (1, -b) \equiv (1, -c) \otimes (1, -d) (\text{mod } I^3(q))$.

Proof. By (7), $s(-a, -b, ab) = q(a, b)q(-1, -1)$, so (i) $\Rightarrow$ $(-a, -b, ab) \cong (-c, -d, cd)$ by 2.2. This, in turn, clearly implies (ii). The implication (ii) $\Rightarrow$ (iii) is clear. Finally, if one applies w to each member of (iii) and uses (13) and 3.3 (iii), one obtains (i). $\square$

Suppose $q_i: G_i \times G_i \to B_i$ is a linked quaternionic mapping, $i=1, 2$. We will say q_1 and q_2 are *equivalent*, denoted $q_1 \sim q_2$, if $\exists$ a group isomorphism $\alpha: G_1 \cong G_2$ such that $\alpha(-1) = -1$ and $q_1(a, b) = 1 \Leftrightarrow q_2(\alpha(a), \alpha(b)) = 1$ $\forall a, b \in G_1$. Note that $q_1 \sim q_2$ implies

$$(14) \qquad \begin{aligned} q_1(a, b) = q_1(c, d) \Leftrightarrow\ & q_2(\alpha(a), \alpha(b)) \\ & = q_2(\alpha(c), \alpha(d)) \quad \forall a, b, c, d \in G_1 . \end{aligned}$$

This follows since $q(a, b) = q(c, d) \Leftrightarrow \exists x \in G$ such that $q(a, bx) = 1$, $q(c, dx) = 1$ and $q(ac, x) = 1$, by the linkage condition.

COROLLARY 3.5. *Define* $q': G \times G \to I^2(q)/I^3(q)$ *by* $q'(a, b) = (1, -a) \otimes (1, -b) + I^3(q)$. *Then* q' *is a linked quaternionic mapping and* $q \sim q'$.

Proof. This is clear, using 3.2 and 3.4. $\square$

COROLLARY 3.6. *Let* $q_i: G_i \times G_i \to B_i$ *be a linked quaternionic mapping,* $i = 1, 2$. *Then* $q_1 \sim q_2 \Leftrightarrow W(q_1) \cong W(q_2)$.

Proof. ($\Rightarrow$): In view of the definition of $W(q_i)$, it is enough to verify $(a, b) \cong (c, d) \Leftrightarrow (\alpha(a), \alpha(b)) \cong (\alpha(c), \alpha(d))$ $\forall a, b, c, d \in G$. This follows from (14) and the fact that α is a group isomorphism. ($\Leftarrow$): In view of 3.5, it is enough to show $q_1' \sim q_2'$. Now it is clear (since $I(q_i)$ can be characterized as the unique ideal of index 2 in $W(q_i)$) that the given isomorphism $\varphi: W(q_1) \to W(q_2)$ carries $I^k(q_1)$ onto $I^k(q_2)$ $\forall k \geqq 1$, and hence induces isomorphisms $I^2(q_1)/I^3(q_1) \cong I^2(q_2)/I^3(q_2)$ and $G_1 \cong I(q_1)/I^2(q_1) \cong I(q_2)/I^2(q_2) \cong G_2$ (using 3.2). Moreover, we claim that the following diagram

$$\begin{array}{ccc} G_1 \times G_1 & \longrightarrow & I^2(q_1)/I^3(q_1) \\ \updownarrow & & \updownarrow \\ G_2 \times G_2 & \longrightarrow & I^2(q_2)/I^3(q_2) \end{array}$$

commutes. First recall as in 3.2 (iii) that for every $x \in G_1$, $\varphi(1, x)$ can be written in the form $(1, y) \oplus f$ for some $y \in G_2$ and $f \in I^2(q_2)$.

Consequently, for $a_1, b_1 \in G_1$ we have

$$\begin{array}{ccl} (a_1, b_1) & \longrightarrow & (1, -a_1, -b_1, a_1b_1) + I^3(q_1) \\ \Big\downarrow & & \\ (a_2, b_2) & \longrightarrow & (1, -a_2, -b_2, a_2b_2) + I^3(q_2) \end{array}$$

where $\varphi(1, -a_1) = (1, -a_2) \oplus f_1$ and $\varphi(1, -b_1) = (1, -b_2) \oplus f_2$ with $f_1, f_2 \in I^2(q_2)$. Now, by an elementary computation it follows that $\varphi((1, -a_1, -b_1, a_1b_1) + I^3(q_1)) = (1, -a_2, -b_2, a_2b_2) + I^3(q_2)$ and the diagram commutes. Finally, since the isomorphisms $G_i \cong I(q_i)/I^2(q_i)$ carry -1 to 2 and $2 \in I(q_1)$ is mapped to $2 \in I(q_2)$, the isomorphism $G_1 \cong G_2$ carries -1 to -1. This proves $q_1' \sim q_2'$. □

In case $q = q_F$, F a field, the following *Arason-Pfister property* is known to hold $\forall k \geqq 2$.

AP (k): If f is a form satisfying $\dim(f) < 2^k$ and $f \in I^k(q)$, then $f \sim 0$.

For the proof see [1]. It is open whether this is true for F a semi-local ring. However we do have the following.

COROLLARY 3.7. *For q an arbitrary linked quaternionic mapping,* AP (k) *holds for $k = 2$, and* 3.

Proof. Let $\dim(f) < 2^k$, $f \in I^k(q)$. Suppose first that $k = 2$, $f = (a, b)$. Applying $d_\pm$ this yields $-ab = 1$, by 3.2, i.e., $b = -a$. Thus $f = (a, -a) \sim 0$. Now suppose $k = 3$. Adding enough hyperbolic forms, we can assume $f = (a_1, a_2, \cdots, a_6)$. Scaling f by $a_1a_2a_3$, if necessary, we can assume $a_3 = a_1a_2$. By 3.2, $d_\pm(f) = 1$, i.e., $a_6 = -a_4a_5$. Thus $f = (a_1, a_2, a_1a_2, a_4, a_5, -a_4a_5) \sim (1, a_1) \otimes (1, a_2) - (1, -a_4) \otimes (1, -a_5) \in I^3(q)$, so $f \sim 0$ by 3.4. □

We now relate the theory just presented with the theoy of representational Witt rings developed in [5]. For the reader's convenience we first record some definitions. Let G be a group of exponent 2. A ring $W = Z[G]/K$ is called an *abstract Witt ring* if the torsion subgroup of W is 2-primary, [7, Def. 3.12]. Throughout this section we will assume without loss of generality that G is a subgroup of the multiplicative group $W^\cdot$, and that $-1 \in G$ (simply replace G by the subgroup of $W^\cdot$ generated by its image and -1). For $r \in W$, $\dim r$ is the smallest number n such that $r = \sum_{i=1}^n g_i$ in W, $g_i \in G$, and $D(r) = \{g \in G \mid r = g + p$ for some $p \in W$ with $\dim p < \dim r\}$, [5, Def. 1.1 and Def. 1.2]. W will be called *representational* if for $r_1 \neq 0$, $r_2 \neq 0$ in W with $\dim(r_1 + r_2) = \dim r_1 + \dim r_2$ and g in $D(r_1 + r_2)$, there exist g_j in $D(r_j)$, $j = 1, 2$

such that $g \in D(g_1 + g_2)$, [5, Def 2.2]. W is *strongly representational* if for $g_1, g_2 \in G$, with $g_1 + g_2 \neq 0$ in W and $g \in D(g_1 + g_2)$ we have $g + gg_1g_2 = g_1 + g_2$, [5, Def 4.1].

It is convenient to associate to W a *theory of forms*. Namely, for $a_i, b_j \in G$, one defines $(a_1, \cdots, a_n) \sim (b_1, \cdots, b_m)$ to mean $a_1 + \cdots + a_n = b_1 + \cdots + b_m$ in W and $(a_1, \cdots, a_n) \cong (b_1, \cdots, b_m)$ to mean $(a_1, \cdots, a_n) \sim (b_1, \cdots, b_m)$ and $n = m$. Isometry so defined clearly satisfies 2.1, 2.3, 2.6, 2.7. Notice that our definitions of dimension determinant, representation, isotropic and anisotropic also make sense for this definition of isometry. Now, W is representational if and only if 2.8 holds for forms over W. This follows quite easily from [5, Prop. 2.29]. Since 2.9 follows from 2.8, 2.9 also holds if W is representational. Now, suppose W is representational and $(a_1, \cdots, a_n) \cong (b_1, \cdots, b_n)$. There exists $a \in D(a_2, \cdots, a_n)$ such that $(a_1, a) \cong (b_1, b)$ for some $b \in G$, by 2.8. Since $a \in D(a_2, \cdots, a_n)$ there exist $c_3, \cdots, c_n \in G$ such that $(a_2, \cdots, a_n) \cong (a, c_3, \cdots, c_n)$. Consequently,

$$\begin{aligned}(a_1, a, b_2, \cdots, b_n) &\cong (b_1, b, b_2, \cdots, b_n) \cong (b, a_1, \cdots, a_n) \\ &\cong (b_1, a_1, a, c_3, \cdots, c_n)_2\ ,\end{aligned}$$

so $(b \cdots, b_n) \cong (b_1, c_3, \cdots, c_n)$ and (5) holds. 2.4 and 3.1 hold also by the same arguments given earlier. Clearly W is strongly representational if and only if $(a, b) \cong (c, d) \Rightarrow ab = cd$ and hence (by an easy application of 2.4) if and only if $f \cong q \Rightarrow d(f) = d(g)$. Consequently, 3.2 holds and hence AP (2) holds (by the proof of 3.7) for strongly representational Witt rings. This proves part of the following.

THEOREM 3.8. *Let W be an abstract Witt ring for G (with G normalized so that $-1 \in G \subseteq W^{\cdot}$). Then*

(i) *W is strongly representational for $G \Leftrightarrow W$ is representational and satisfies* AP (2) *for G.*

(ii) *There exists a linked quaternionic mapping $q: G \times G \rightarrow B$ such that $W = W(q) \Leftrightarrow W$ is representational and satisfies* AP (k), *$k = 2, 3$, for G.*

Proof. (i) We have just proved ($\Rightarrow$). To prove ($\Leftarrow$) suppose $a + b = c + d$ with $a, b, c, d \in G$. Then $ab - cd = a(a+b) - c(c+d) = a(a + b) - c(a + b) = (a - c)(a + b) \in I^2$. By AP (2), this implies $ab - cd = 0$, i.e., $ab = cd$.

(ii) If q is linked, then $W(q)$ is an abstract Witt ring for G by 3.1, it is representational by 2.8, and satisfies AP (k), $k = 2, 3$ by 3.7. This proves ($\Rightarrow$). To prove ($\Leftarrow$) define $q: G \times G \rightarrow I^2/I^3$ by $q(a, b) = (1 - a)(1 - b) + I^3$, where I is the unique ideal of index 2.

Since $(1-bc) \equiv (1-b)+(1-c) (\bmod I^2)$ and $(1-a)^2 = 2(1-a)$, q is clearly a quaternionic mapping. Note $(1-a)(1-b) \equiv (1-c)(1-d)$ $(\bmod I^3) \Leftrightarrow -a-b+ab+c+d-cd \in I^3 \Leftrightarrow -a-b+ab+c+d-cd = 0 \Leftrightarrow (1-a)(1-b) = (1-c)(1-d)$ by AP (3). Thus, if $q(a, b) = q(c, d)$, then $(-b+ab)+(d-cd) = a-c$ so by 2.9, AP (2) and part (i) $\exists x \in G$ such that $-b+ab = -x+ax$ and $d-cd = x-cx$. This implies $q(a, b) = q(a, x)$ and $q(c, d) = q(c, x)$ so q is linked. It follows from 3.1 and the corresponding structure result for W that $W = W(q)$. □

It is shown in [7, § 3] that some of the structure results in [10] concerning the nilradical and the reduced Witt ring hold for any abstract Witt ring. For easy reference, we now summarize some of these results. For W an abstract Witt ring, denote by W_t, X, I, and $\mathrm{Nil}(W)$, the torsion subgroup, the set of *signatures* (i.e., ring homomorphisms $\sigma: W \to \boldsymbol{Z}$), the unique ideal of index 2, and the nilradical, respectively, of W.

THEOREM 3.9. *Let W be an abstract Witt ring. Then*
(i) *W_t is 2-primary,*
(ii) *$W_t = \{f \in W \mid \sigma(f) = 0 \ \forall \sigma \in X\}$, and*
(iii) $\mathrm{Nil}(W) = W_t \cap I$.
(More precisely, in (iii), *since* $W_t \subseteqq I$ if $X \neq \Phi$, *whereas* $I \subseteqq W_t = W$, *if* $X = \Phi$, *one has* $\mathrm{Nil}(W) = W_t$, *if* $W_t \neq W$, *and* $\mathrm{Nil}(W) = I$, *if* $W_t = W$.)

The following result is useful in verifying AP (k) in certain cases.

LEMMA 3.10. *Suppose W is an abstract Witt ring for G with $-1 \in G \subseteqq W^{\cdot}$. If I^k is torsion free, then* AP (k) *holds.*

Proof. Suppose f is a form over G, $f \in I^k$, $\dim(f) < 2^k$. Let σ be a signature of W. If $b_1, \cdots, b_k \in G$, then $\sigma(b_i) = \pm 1$ so $\sigma(1, -b_1) \otimes (1, -b_2) \otimes \cdots \otimes (1, -b_k) = 0$ or 2^k. Thus $\sigma(I^k) \subseteqq 2^k \boldsymbol{Z}$. On the other hand, clearly $|\sigma(f)| \leqq \dim(f) < 2^k$. Thus $\sigma(f) = 0$ for all signatures σ of W. It follows, from 3.9 (ii), that f is torsion. Thus, by assumption, $f = 0$. □

Recall [5, 2.24] that if W is an abstract Witt ring which is representational, then so is the reduced ring $W_{\mathrm{red}} = W/\mathrm{Nil}(W)$. Moreover (by [5, 2.30]), the abstract Witt rings which are reduced and representational are just the Witt rings of spaces of orderings in the terminology of [9]. It follows from 3.8 (ii) and 3.10 that all such rings are included in the theory presented here, i.e., are of the form $W(q)$ for some linked quaternionic mapping q. (By 3.9 (iii), W is reduced if and only if I is torsion free, so 3.10 applies.) Here is a characterization of the class of linked quaternionic mappings thus obtained.

THEOREM 3.11. *Let $q: G \times G \rightarrow B$ be a linked quaternionic mapping. Then $W(q)$ is reduced if and only if q satisfies*

$$(R) \qquad q(a, a) = 1 \Rightarrow a = 1 .$$

Proof. By 3.9 (i) and (iii), $W(q)$ is reduced if and only if

$$(\mathrm{R}') \quad 2 \times f \sim 0 \Rightarrow f \sim 0 \quad \forall \text{ even dimensional forms } f \text{ over } G .$$

Thus we must verify $(R) \Leftrightarrow (R')$. Assume (R') and $q(a, a)=1$. Thus $(a, a) \cong (1, 1)$, i.e., $2 \times (1, -a) \sim 0$. Thus, by (R'), $(1, -a) \sim 0$, i.e., $a = 1$. Thus $(R') \Rightarrow (R)$. Now assume (R).

Claim. $D(2 \times f) = D(f)$ $\forall$ forms f over G. For suppose $f = (a_1, \cdots, a_n)$, and that x is represented by $2 \times f \cong (a_1, a_1) \oplus \cdots \oplus (a_n, a_n)$. Thus, by a repeated application of 2.8., $\exists x_i \in D(a_i, a_i)$ such that $x \in D(x_1, \cdots, x_n)$. But $(a_i, a_i) \cong (x_i, x_i)$, i.e., $(a_i x_i, a_i x_i) \cong (1, 1)$, i.e., $q(a_i x_i, a_i x_i) = 1$, so by (R), $x_i = a_i$ $\forall i = 1, \cdots, n$. This proves $x \in D(f)$ and hence proves the claim.

Now suppose (R') fails. Then $\exists$ an anisotropic form $f = (a_1, \cdots, a_n)$ with n even, $n \geqq 2$ and $2 \times f \sim 0$. But then $2 \times (a_1) \oplus 2 \times (a_2, \cdots, a_n) \sim 0$, so by 2.9 and the claim, $-a_1 \in D(a_2, \cdots, a_n)$. This contradicts the fact that f is anisotropic. Thus $(R) \Rightarrow (R')$. $\square$

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