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A CHARACTERIZATION OF THE ADJOINT L-KERNEL OF SZEGÖ TYPE

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Let G be a bounded regular region in the complex plane and $\hat{L}(z, u)$ the adjoint L-kernel of Szegö kernel function $\hat{K}(z, \bar{u})$ on G. Then, for any analytic function $h(z)$ on G with a finite Dirichlet integral, it is shown that the equation

$$\frac{1}{\pi}\iint_G |h'(z)|^2 dxdy = \int_{\partial G}\int_{\partial G} |(h(z_1)-h(z_2))\hat{L}(z_1, z_2)|^2 \, |dz_1| \, |dz_2|$$

holds. Furthermore, for any fixed nonconstant $h(z)$, we show that the function $\hat{L}(z_1, z_2)$ on $G\times G$ is characterized by that equation in some class.

1. Introduction and statement of result. Let S denote an arbitrary compact bordered Riemann surface. Let $W(z, t)$ be a meromorphic function whose real part is the Green's function $g(z, t)$ with pole at $t \in S$. The differential id $W(z, t)$ is positive along ∂S. For simplicity, we do not distinguish between points $z \in S \cup \partial S$ and local parameters z. For an arbitrary integer q and for any positive continuous function $\rho(z)$ on ∂S, let $H^q_{p,\rho}(S)[p \geqq 1]$ be the Banach space of analytic differentials $f(z)(dz)^q$ on S of order q with finite norms

$$\left\{\frac{1}{2\pi}\int_{\partial S} |f(z)(dz)^q|^p \rho(z)[\mathrm{id}\ W(z, t)]^{1-pq}\right\}^{1/p} < \infty \ ,$$

where $f(z)$ means the Fatou boundary value of f at $z \in \partial S$. Let $K_{q,t,\rho}(z, \bar{u})(dz)^q$ be the reproducing kernel for $H^q_{2,\rho}(S)$ which is characterized by the reproducing property

$$f(u) = \frac{1}{2\pi}\int_{\partial S} f(z)(dz)^q \overline{K_{q,t,\rho}(z, \bar{u})(dz)^q}\rho(z)[\mathrm{id}\ W(z, t)]^{1-2q}$$
$$\text{for all } f(z)(dz)^q \in H^q_{2,\rho}(S) \ .$$

See [9]. Let $L_{q,t,\rho}(z, u)(dz)^{1-q}$ denote the adjoint L-kernel of $K_{q,t,\rho}(z, \bar{u})(dz)^q$. The function $L_{q,t,\rho}(z, u)(dz)^{1-q}$ is a meromorphic differential on S of order $1 - q$ with a simple pole at u having residue 1. Moreover,

$$\overline{K_{q,t,\rho}(z, \bar{u})(dz)^q}\rho(z)[\mathrm{id}\ W(z, t)]^{1-2q} = \frac{1}{i}L_{q,t,\rho}(z, u)(dz)^{1-q} \text{ along } \partial S \ . \tag{1.1}$$

We note that $|K_{q,t,\rho}(z, \bar{u})|$ and $|L_{q,t,\rho}(z, u)|$ can be extended continuously on ∂S. In addition, $K_{q,t,\rho}(z, \bar{u}) = \overline{K_{q,t,\rho}(u, \bar{z})}$ and $L_{q,t,\rho}(z, u) = -L_{1-q,t,\rho^{-1}}(u, z)$ on S.

If S is a bounded regular region *in the plane*, then we can define the kernels for arbitrary real values of q. In this case, for $q = 1/2$ and $\rho(z) \equiv 1$, we have the classical Szegö kernels $\hat{K}(z, \bar{u}) = K_{1/2,t,1}(z, \bar{u})/2\pi$ and $\hat{L}(z, u) = L_{1/2,t,1}(z, u)/2\pi$. Cf. [8] and [9].

A classical characterization of $L_{q,t,\rho}(z, u)(dz)^{1-q}$ can be now stated as follows:

PROPOSITION (P. R. Garabedian [3, 4], Z. Nehari [6, 7] and S. Saitoh [8, 9]). *The adjoint L-kernel $L_{q,t,\rho}(z, u)(dz)^{1-q}$ is characterized by the following extremal property*

$$K_{q,t,\rho}(u, \bar{u}) = \frac{1}{2\pi}\int_{\partial S} |L_{q,t,\rho}(z, u)(dz)^{1-q}|^2(\rho(z))^{-1}[\mathrm{id}\ W(z, t)]^{2q-1}$$
$$= \min\left\{\frac{1}{2\pi}\int_{\partial S} |F(z, u)(dz)^{1-q}|^2(\rho(z))^{-1}[\mathrm{id}\ W(z, t)]^{2q-1}\right\} .$$

The minimum is taken here over all meromorphic differentials $F(z, u)(dz)^{1-q}$ on S of order $1-q$ with a simple pole at u having residue 1 *and with finite integral*

$$\int_{\partial S} |F(z, u)(dz)^{1-q}|^2[\mathrm{id}\ W(z, t)]^{2q-1} < \infty .$$

In this paper, we establish the following theorem:

THEOREM 1.1. *For any analytic function $h(z)$ on S with a finite Dirichlet integral, we have the equation*

$$\begin{aligned}(1.2)\qquad &\frac{1}{\pi}\iint_S |h'(z)|^2 dxdy \\ &= \frac{1}{4\pi^2}\int_{\partial S}\int_{\partial S} |(h(v) - h(u))L_{q,t,\rho}(v, u)(dv)^{1-q}(du)^q|^2 \\ &\quad \times (\rho(v))^{-1}[\mathrm{id}\ W(v, t)]^{2q-1}\rho(u)[\mathrm{id}\ W(u, t)]^{1-2q},\ z=x+iy .\end{aligned}$$

Furthermore, for any fixed nonconstant $h(z)$, the adjoint L-kernel $L_{q,t,\rho}(v, u)$ $(dv)^{1-q}(du)^q$ is characterized by the following extremal property:

$$\begin{aligned}(1.3)\qquad &\int_{\partial S}\int_{\partial S} |(h(v) - h(u))L_{q,t,\rho}(v, u)(dv)^{1-q}(du)^q|^2 \\ &\quad \times (\rho(v))^{-1}[\mathrm{id}\ W(v, t)]^{2q-1}\rho(u)[\mathrm{id}\ W(u, t)]^{1-2q}\end{aligned}$$

$$= \min \left\{ \int_{\partial S} \int_{\partial S} |(h(v) - h(u))F(v, u)(dv)^{1-q}(du)^q|^2 \right.$$
$$\left. \times (\rho(v))^{-1}[\mathrm{id}\, W(v, t)]^{2q-1}\rho(u)[\mathrm{id}\, W(u, t)]^{1-2q} \right\} .$$

The minimum is taken here over all meromorphic differentials $F(v, u)(dv)^{1-q}(du)^q$ on $S \times S$ such that

$$F(v, u) = \frac{f(u, v)}{h(v) - h(u)} \tag{1.4}$$

for an analytic differential $f(u, v)(du)^q(dv)^{1-q}$ on $S \times S$ satisfying

$$f(z, z) = h'(z) \text{ on } S \tag{1.5}$$

and

$$\int_{\partial S} \int_{\partial S} |f(u, v)(du^q)(dv)^{1-q}|^2[\mathrm{id}\, W(u, t)]^{1-2q}[\mathrm{id}\, W(v, t)]^{2q-1} < \infty . \tag{1.6}$$

In particular, we note that when $q = 1/2$ and $\rho(z) \equiv 1$, we can define the adjoint L-kernels of the Szegö kernels of S with characteristics. Cf. D. A. Hejhal [5] and J. D. Fay [2]. Then, the adjoint L-kernels are, in general, multiplicative functions, but our proof of Theorem 1.1 will show that Theorem 1.1 is still valid for these adjoint L-kernels in a modified form.

2. Preliminaries. Let $\{\Phi_j(z)(dz)^q\}_{j=1}^{\infty}$ and $\{\Psi_j(z)(dz)^{1-q}\}_{j=1}^{\infty}$ be complete orthonormal systems for $H^q_{2,\rho}(S)$ and $H^{1-q}_{2,\rho^{-1}}(S)$, respectively. Let $H = H^q_{2,\rho}(S) \otimes H^{1-q}_{2,\rho^{-1}}(S)$ denote the direct product of $H^q_{2,q}(S)$ and $H^{1-q}_{2,\rho^{-1}}(S)$. The space H is composed of all differentials $f(z_1, z_2)(dz_1)^q(dz_2)^{1-q}$ on $S \times S$ such that

$$f(z_1, z_2) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{j,k}\Phi_j(z_1)\Psi_k(z_2) , \quad \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |A_{j,k}|^2 < \infty . \tag{2.1}$$

The scalar product $(\, , \,)_H$ is given as follows:

$$(f(z_1, z_2), h(z_1, z_2))_H = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{j,k}\overline{B_{j,k}} \tag{2.2}$$

where $h(z_1, z_2) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} B_{j,k}\Phi_j(z_1)\Psi_k(z_2)$ and $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |B_{j,k}|^2 < \infty$. Cf. [1, § 8].

We let $H_{D(0)}$ denote the subspace in H composed of all differentials which vanish along the diagonal set $D = \{(z, z) \,|\, z \in S\}$ and $(H_{D(0)})^{\perp}$ the orthocomplement of $H_{D(0)}$ in H.

3. Proof of theorem. For $h(z) \in H^0_{1,1}(S)$, we set

$$(3.1) \qquad f_h(u, v) = \int_{\partial S} h(z)\overline{K_{q,t,\rho}(z, \bar{u})K_{1-q,t,\rho^{-1}}(z, \bar{v})dz} .$$

From (1.1) and the residue theorem, we have

$$(3.2) \qquad f_h(u, v) = -2\pi i L_{q,t,\rho}(v, u)(h(v) - h(u))$$

and so

$$(3.3) \qquad f_h(z, z) = -2\pi i h'(z) \text{ on } S .$$

When $h(z)$ has a finite Dirichlet integral, from [12, Theorem 4.1] and [11, Corollary 3.2], we see that $f_h(u, v)(du)^q(dv)^{1-q}$ belongs to $(H_{D(0)})^\perp$. From [12, Corollary 2.1] and [10, Equation (3.2)], we thus obtain (1.2).

Next, suppose that $F^*(v, u)$ attains the minimum in (1.3). Then, in the case such that $h(z)$ is not constant, we set

$$(3.4) \qquad f_h^*(u, v) = F^*(v, u)(h(v) - h(u))$$

and so

$$(3.5) \qquad f_h^*(z, z) = h'(z) \text{ on } S .$$

We note that any $f(u, v)(du)^q(dv)^{1-q} \in H$ satisfying $f(z, z) = h'(z)$ on S is expressible in the form

$$f(u, v) = F(v, u)(h(v) - h(u))$$

for an $F(v, u)$ stated in the theorem. From the extremal property of $f_h^*(u, v)(du)^q(dv)^{1-q}$ in the subspace in H satisfying $f(z, z) = h'(z)$ on S, we see that $f_h^*(u, v)(du)^q(dv)^{1-q} \in (H_{D(0)})^\perp$. Cf. [10, Equation (3.2)]. Therefore, by [12, Theorem 4.2], $f_h^*(u, v)$ is expressible in the form

$$(3.6) \qquad f_h^*(z_1, z_2) = \frac{1}{2\pi}\int_{\partial S} \frac{h^*(\zeta)d\zeta\overline{K_{q,t,\rho}(\zeta, \bar{z}_1)K_{1-q,t,\rho^{-1}}(\zeta, \bar{z}_2)d\zeta}}{\mathrm{id}\, W(\zeta, t)}$$

for a uniquely determined $h^*(z)dz$ in $H^1_{1,1}(S)$. Furthermore, from [12, Equations (4.11) and (4.12)], $h^*(z)$ can be determined as follows:

$$(3.7) \qquad h^*(z) = -W'(z, t)(h(z) - h(t)) .$$

From (3.6) and (1.1), we have

$$(3.8) \qquad f_h^*(u, v) = L_{q,t,\rho}(v, u)(h(v) - h(u)) .$$

We thus have the desired result $F^*(v, u) = L_{q,t,\rho}(v, u)$.

4. Corollary. *In particular, from the proof of Theorem* 1.1, *we obtain*

COROLLARY 4.1. *For any fixed nonconstant analytic function $h(z)$ on S with a finite Dirichlet integral, the unique extremal function which minimizes*

$$\| f(z_1, z_2) \|_{H^{1-q}_{2,\rho^{-1}}(S)\otimes H^q_{2,\rho}(S)}$$

in the subspace in $H^{1-q}_{2,\rho^{-1}}(S) \otimes H^q_{2,\rho}(S)$ *satisfying* $f(z, z) = h'(z)$ *on* S *is given by* $(h(z_1) - h(z_2))L_{q,t,\rho}(z_1, z_2)(dz_1)^{1-q}(dz_2)^q$.

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