PACIFIC JOURNAL OF MATHEMATICS
Vol. 96, No. 2, 1981

ON WEAK RESTRICTED ESTIMATES AND ENDPOINT PROBLEMS FOR CONVOLUTIONS WITH OSCILLATING KERNELS (I)

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Throughout we consider $K(t) = e^{i|t|^a}/|t|^b$, $a > 0$, $a \neq 1$, $b < 1$ and $t \in \boldsymbol{R}$. Here we consider for fixed $\lambda, \mu > 0$ the function $B(\lambda, \mu; K) = B(\lambda, \mu) = \sup_{\chi_\lambda \chi_\mu} \int \chi_\lambda(x) K * \chi_\mu(x) dx$ where the sup is taken over all "characteristic" functions χ_λ, χ_μ with complex signs (i.e., χ_μ is a measurable function for which $|\chi_\mu| = 1$ on E, $|\chi_\mu| = 0$ off E and $|E| \leqq \mu$ ($\mu > 0$)). We estimate $B(\lambda, \mu; K)$ within constant factors from above and below. This settles the endpoint problems for these kernels, at least in the weak restricted sense.

0. **Introduction.** This paper is concerned with (L_p, L_q)-mapping properties of the operator

$$g = K * f\ , \qquad g(x) = \int K(x - y) f(y) dy \qquad (x, y \in \boldsymbol{R}^n)\ ,$$

in particular with (weak restricted) estimates

$$\left| \int \chi_\lambda (K * \chi_\mu) \right| \leqq c_{pq} \lambda^{1/q'} \mu^{1/p} (1/q + 1/q' = 1)\ , \tag{1}$$

where, e.g., χ_μ denotes a "characteristic" function with complex signs, i.e., a measurable function with $|\chi_\mu| = 1$ on E, $|\chi_\mu| = 0$ off E, $|E| \leqq \mu (\mu > 0)$. Let us denote by $B(\lambda, \mu) \equiv B(\lambda, \mu; K)$ the quantity

$$\sup_{\chi_\lambda \cdot \chi_\mu} \left| \int \chi_\lambda (K * \chi_\mu) \right| = \sup_{\chi_\lambda \cdot \chi_\mu} \left| \iint K(x + y) \chi_\lambda(x) \chi_\mu(y) dx dy \right|\ ,$$

where the sup varies over all characteristic functions χ_λ, χ_μ with fixed $\lambda > 0$, $\mu > 0$. Our present problem will be to estimate $B(\lambda, \mu)$ as closely as possible from above and below.

In earlier papers [4], [9], [13], [14] we already discussed the mapping properties for oscillating kernels. In [4] we gave, in part, the mapping properties for the kernels

$$K(t) = \frac{e^{i|t|^a}}{|t|^b} (0 \neq t \in \boldsymbol{R}) \text{ with } a > 0,\ a \neq 1,\ b < 1 \tag{2}$$

except for the endpoints. By means of the function $B(\lambda, \mu)$ we settle the endpoint problems in the weak restricted sense. Furthermore, we determine $B(\lambda, \mu)$ within constant factors from above and below. Thus for the kernels K in (2) we can determine the set $S(K) = S$ of all mapping points $(1/p, 1/q)$ in the Riesz triangle $0 \leqq 1/q \leqq 1/p \leqq 1$. This will imply weak as well as strong mapping properties at all boundary points except possibly at the vertices of S (this is obtained through interpolation). Recently, in the paper [10; Theorem 5 and Remark 2] we were able to prove that the kernels in (2) satisfy strong mapping properties at the vertices B, B' and C of S.

In [5] and along with Stein in [7], C. Fefferman solved the (L^p, L^q) mapping problem for these kernels in (2) with $a < 0$ (when $a < 0$ we assume further that K has compact support). Just recently Fefferman had pointed out to us that he also knew how to solve the mapping problem when $0 < a < 1$ (of course all of his results apply as well to n-dimensions). To be more precise the methods used by Fefferman assume that the kernels $K(t) = e^{i|t|^a}/(1 + |t|)$ (here $t \in \boldsymbol{R}$, $0 < a$, $a \neq 1$) satisfy at least a regularity condition at infinity, i.e., for some $0 < \theta < 1$,

$$\int_{|x| \geqq 2|y|^{1/(1-\theta)}} |K(x - y) - K(x)|\,dx \leqq B\,, \quad \text{for} \quad |y| \geqq 1\,,$$

B a positive constant independent of y. One can easily show that the kernels $K(t) = (1 + |t|)^{-1}e^{i|t|^a}$ where $a > 1$, do not satisfy such a regularity condition for any $0 < \theta < 1$.

Let me add that P. Sjölin in [17] has solved some of these mapping problems in n-dimensions.

1. An interpolation theorem with respect to the kernel. The usual interpolation theorems refer to two function spaces and a corresponding decomposition $f = f_1 + f_2$. Here we consider a decomposition of the kernel:

$$K = K_1 + K_2\,.$$

We make use of the decreasing rearrangement K^* of K (if it exists), so that

$$\sup_{\chi_\mu} \left| \int K(x)\chi_\mu(x)dx \right| = \int_0^\mu K^*(t)dt \qquad (x \in \boldsymbol{R}^n,\ t \in \boldsymbol{R})\,.$$

We also make use of the (distributional) Fourier transform

$$\hat{K}(x) = \int e^{2ix\cdot y}K(y)dy\,.$$

With these notations we have the following

THEOREM 1. *If K_1^* exists and $\hat{K}_2$ is bounded then*

$$B(\lambda, \mu) \leqq \lambda \int_0^\mu K_1^*(t)dt + (\lambda\mu)^{1/2} \| \hat{K}_2 \|_\infty .$$

Since $B(\lambda, \mu) = B(\mu, \lambda)$ there is a second inequality with λ and μ interchanged.

Proof. Corresponding to the decomposition of K we have

$$B(\lambda, \mu; K) \leqq B(\lambda, \mu; K_1) + B(\lambda, \mu; K_2) .$$

Here the first term can be estimated using

$$\| K_1 * \chi_\mu \|_\infty \leqq \int_0^\mu K_1^*(t)dt$$

and the second term can be estimated using

$$\| K_2 * \chi_\mu \|_2 \leqq \| \hat{K}_2 \|_\infty \mu^{1/2} .$$

REMARK. Observe that

$$K_1^{**}(\mu) = \frac{1}{\mu} \int_0^\mu K_1^*(t)dt \leqq \frac{1}{\lambda} \int_0^\lambda K_1^*(t)dt \qquad \text{for} \quad \lambda \leqq \mu ,$$

so that interchanging λ with μ produces a weaker inequality in the case $\lambda \leqq \mu$. Example (2) can be used to show that the "right" decomposition of K gives sharp estimates for all λ, μ which is rather surprising in view of the two simple estimates used. Also note that

$$\| g \|_q^* = \sup_{\lambda > 0, \chi_\lambda} \lambda^{-1/q'} \left| \int \chi_\lambda g \right| , \qquad 1 < q < \infty$$

defines a weak q-norm of g (which is equivalent to the usual weak q-norm). Therefore

$$N_q(\mu) = \sup_{\chi_\mu} \| K * \chi_\mu \|_q^* = \sup_{\lambda > 0} \lambda^{-1/q'} B(\lambda, \mu)$$

can be calculated via $B(\lambda, \mu)$. Note that $N_q(\mu)$ is a logarithmically convex function in $1/q$ (μ fixed) since it is a sup of such functions.

2. Upper estimates for B.

From now on we consider the kernel

$$K(t) = \frac{e^{i|t|^a}}{|t|^b} (0 \neq t \in \boldsymbol{R}) \quad \text{with} \quad a > 0,\ a \neq 1,\ b < 1 .$$

By $c, c_1, c_2, \cdots$ we generically denote suitable positive constants which depend only on a and b. We introduce $K_w (w > 0)$, $K^w (w \geqq 0)$, and $K_{u,v} (0 \leqq u < v \leqq \infty)$ by $K_w^{(t)} = K(t)$ for $|t| < w$, $K^w = K$ for

$|t| \geqq w$, $K_{u,v} = K$ for $u \leqq |t| < v$ and $K_w = 0$, $K^w = 0$, $K_{u,v} = 0$ elsewhere.

By standard estimates (van der Corput, eg.,) we obtain for all real x and all (admissible) w

$$|\hat{K}^w(x)| \leqq c(w+1)^{1-(a/2)-b} \quad \text{if} \quad b \geqq 1 - \frac{a}{2},$$

$$|\hat{K}_w(x)| \leqq cw^{1-(a/2)-b} \quad \text{if} \quad b < 1 - \frac{a}{2}.$$

Furthermore

$$\int_0^\mu (K_w)^* \leqq \begin{cases} c \min(\mu^{1-b}, w^{1-b}) & \text{if} \quad b \geqq 0 \\ c \min(\mu w^{-b}, w^{1-b}) & \text{if} \quad b < 0, \end{cases}$$

$$\int_0^\mu (K^w)^* \leqq c \min(\mu^{1-b}, \mu w^{-b}) \quad \text{if} \quad b \geqq 0.$$

Accordingly we distinguish between the four cases

(I) $\qquad b \geqq 1 - \dfrac{a}{2}, \qquad b \geqq 0,$

(II) $\qquad b \geqq 1 - \dfrac{a}{2}, \qquad b < 0$ (implies $a > 2$),

(III) $\qquad b < 1 - \dfrac{a}{2}, \qquad b \geqq 0$ (implies $a < 2$),

(IV) $\qquad b < 1 - \dfrac{a}{2}, \qquad b < 0.$

In case (IV) there will be no nontrivial estimates. In the other cases we employ Theorem 1. Each of the cases (I), (II), (III) will be subdivided according to $\lambda \leqq \mu^{1-a}$ (subscript 1) or $\lambda \geqq \mu^{1-a}$ (subscript 2). Our estimates for $B(\lambda, \mu) = B(\lambda, \mu; a, b)$ are as follows:

Case I. Using $K_1 = K$, $K_2 = 0$, we obtain

(i) $\qquad B(\lambda, \mu) \leqq c\lambda\mu^{1-b}, \qquad [B(\lambda, \mu) \leqq c\mu\lambda^{1-b}].$

The first inequality is better than the second one if $\lambda \leqq \mu$; they agree if $b = 0$.

Using $K_1 = K_w$, $K_2 = K^w$ with $w = (\mu/\lambda)^{1/a} \leqq \mu$, we obtain

$$\begin{aligned} B(\lambda, \mu) &\leqq c\lambda w^{1-b} + c(\lambda\mu)^{1/2} w^{1-(a/2)-b}, \\ &\leqq c\lambda^{(a+b-1)/a}\mu^{(1-b)/a} \quad \text{in case } (\mathrm{I}_2). \end{aligned}$$

Since

$$\lambda\mu^{1-b} \leqq \lambda^{(a+b-1)/a}\mu^{(1-b)/a} \quad \text{iff} \quad \lambda \leqq \mu^{1-a}$$

the last inequality above holds also in case (I_1) since we have the better inequality (i) there. So we have without restrictions [by symmetry]

$$\text{(ii)} \qquad B(\lambda, \mu) \leqq c\lambda^{(a+b-1)/a}\mu^{(1-b)/a}\,, \qquad [B(\lambda, \mu) \leqq c\mu^{(a+b-1)/a}\lambda^{(1-b)/a}]\,.$$

Here the left inequality is better for $\lambda \leqq \mu$ since $a + b - 1/a \geqq 1 - b/a$; the inequalities agree if $b = 1 - (a/2)$.

In total we have four inequalities in case (I). They come in pairs, and we may always pick the first one, since $\lambda \leqq \mu$ can be assumed by symmetry. Then (i) is relevant (better) for (I_1) and (ii) for (I_2).

Case (II). Letting $K_1 = K_w$, $K_2 = K^w$ with $w = (\mu/\lambda)^{1/a} \leqq \mu$, we obtain

$$\begin{aligned} B(\lambda, \mu) &\leqq c\lambda w^{1-b} + c(\lambda\mu)^{1/2}w^{1-(a/2)-b}\,, \\ &\leqq c\lambda^{(a+b-1)/a}\mu^{(1-b)/a} \quad \text{in case } (\mathrm{II}_2)\,. \end{aligned}$$

With $w \geqq \mu$ we obtain

$$B(\lambda, \mu) \leqq c\lambda\mu w^{-b} + c(\lambda\mu)^{1/2}w^{1-(a/2)-b}\,, \qquad w = (\lambda\mu)^{-1/(a-2)} \geqq \mu\,,$$

$$B(\lambda, \mu) \leqq c(\lambda\mu)^{(a+b-2)/(a-2)} \quad \text{in case } (\mathrm{II}_1)\,.$$

Observe that

$$(\lambda\mu)^{(a+b-2)/(a-2)} \leqq \lambda^{(a+b-1)/a}\mu^{(1-b)/a} \quad \text{iff} \quad \lambda \leqq \mu^{1-a}$$

so that the first inequality extends to (II_1) and the second inequality to (II_2). Thus we get [by symmetry] the three unrestricted inequalities

$$\text{(ii)} \qquad B(\lambda, \mu) \leqq c\lambda^{(a+b-1)/a}\mu^{(1-b)/a}\,, \qquad [B(\lambda, \mu) \leqq c\mu^{(a+b-1)/a}\lambda^{(1-b)/a}]\,,$$

$$\text{(iii)} \qquad B(\lambda, \mu) \leqq c(\lambda\mu)^{(a+b-2)/(a-2)}\,.$$

Of the first two the first one is better for $\lambda \leqq \mu$ which can always be assumed. Then (ii) is relevant for (II_2) and (iii) for (II_1).

Case (III). Letting $K_1 = K$, $K_2 = 0$, we obtain

$$\text{(i)} \qquad B(\lambda, \mu) \leqq c\lambda\mu^{1-b}\,, \qquad [B(\lambda, \mu) \leqq c\mu\lambda^{1-b}]\,.$$

The first inequality is better than the second one if $\lambda \leqq \mu$.

Letting $K_1 = K^w$, $K_2 = K_w$ with $w = (\lambda\mu)^{1/(2-a)} \geqq \mu$ we obtain

$$\begin{aligned} B(\lambda, \mu) &\leqq c\lambda\mu w^{-b} + c(\lambda\mu)^{1/2}w^{1-(a/2)-b}\,, \\ &\leqq c(\lambda\mu)^{(2-a-b)/(2-a)} \quad \text{in case } (\mathrm{III}_2)\,. \end{aligned}$$

Since

$$\lambda\mu^{1-b} \leqq (\lambda\mu)^{(2-a-b)/(2-a)} \quad \text{if} \quad \lambda \leqq \mu^{1-a} ,$$

the last inequality above extends to (III_1), i.e., we have without restrictions

$$\text{(iii)} \qquad B(\lambda, \mu) \leqq c(\lambda\mu)^{(2-a-b)/(2-a)} .$$

Again, (i) is relevant for (III_1) and (iii) for (III_2).

Thus we see that if $\lambda \leqq \mu$ then in each of the cases (I), (II), (III) there is just one of the inequalities (i), (ii), (iii) relevant for $\lambda \leqq \mu^{1-a}$ and one for $\lambda \geqq \mu^{1-a}$. This suggests defining the following explicit function $\tilde{B}(\lambda, \mu; a, b)$ for $\lambda \leqq \mu$:

$$\tilde{B} = \lambda\mu^{1-b} \quad \text{for } (\mathrm{I}_1) , \qquad \tilde{B} = \lambda^{(a+b-1)/a}\mu^{(1-b)/a} \quad \text{for } (\mathrm{I}_2) ;$$

$$\tilde{B} = (\lambda\mu)^{(a+b-2)/(a-2)} \quad \text{for } (\mathrm{II}_1) , \qquad \tilde{B} = \lambda^{(a+b-1)/a}\mu^{(1-b)/a} \quad \text{for } (\mathrm{II}_2) ;$$

$$\tilde{B} = \lambda\mu^{1-b} \quad \text{for } (\mathrm{III}_1) , \qquad \tilde{B} = (\lambda\mu)^{(2-a-b)/(2-a)} \quad \text{for } (\mathrm{III}_2) ;$$

along the dividing line $\lambda = \mu^{1-a}$ both definitions in (I), (II), (III) give the same value. For $\lambda \geqq \mu$ we define $\tilde{B}(\lambda, \mu)$ by symmetry. Finally, we set $\tilde{B} = \infty$ in case (IV). Then we can summarize the upper estimates as

THEOREM 2. *Always* $B(\lambda, \mu) \leqq c\tilde{B}(\lambda, \mu)$.

REMARK. We note that K can be replaced by $K_{u,v}$ in all upper estimates, and that these hold true uniformly in u and v.

There is a more compact, but less explicit way to define $\tilde{B}(\lambda, \mu)$: In case (I) we had the four general upper estimates

$$\lambda\mu^{1-b} , \qquad \mu\lambda^{1-b} , \qquad \lambda^{(a+b-1)/a}\mu^{(1-b)/a} , \qquad \mu^{(a+b-1)/a}\lambda^{(1-b)/a} ,$$

except for the constant c. Now $\tilde{B}$ is simply the minimum of these four functions and similarly in the cases (II), (III).

3. Lower estimates for B. Here we prove the opposite inequality for $B(\lambda, \mu; a, b)$.

THEOREM 3. *Always* $B(\lambda, \mu) \geqq c\tilde{B}(\lambda, \mu)$.

Thus the order of magnitude of $B(\lambda, \mu)$ is determined for all λ, μ. Note that the special case $a = 2$, $b = 0$ is essentially the case of the Fourier transform. The proof of Theorem 3 is based on the following result, where

$$\psi_\mu(x) = \int_{-\infty}^{\infty} K(x+t)\chi_\mu(t)dt \qquad (x, t \in \boldsymbol{R})$$

and χ_μ is our characteristic function.

PROPOSITION. *There are constants c_1, c_2 such that to each pair of parameters δ, T with $0 < \delta \leqq \mu$, $T \geqq \delta$ there exists a characteristic function χ_μ satisfying*

$$|\{x: |\psi_\mu(x)| > c_1\delta T^{-b}\}| \geqq c_2 \min(T, \delta^{-1}T^{2-a}) .$$

In case that $a+b>1$ this can be improved to

$$|\{x: |\psi_\mu(x)| > c_1\delta T^{-b}\}| \geqq c_2\mu\delta^{-1} \min(T, \delta^{-1}T^{2-a}) .$$

Proof. Let

$$\chi_\delta(t) = \begin{cases} e^{-it^a} & \text{for } t \in (T, T+\delta) \\ 0 & \text{elsewhere} \end{cases}$$

and observe that for $0 < x \leqq T$

$$|\psi_\delta(x)| = \left|\int_T^{T+\delta} K(x+t)\chi_\delta(t)dt\right| = \left|\int_T^{T+\delta} \frac{e^{i\Delta(x,t)}}{(x+t)^b}dt\right| ,$$

where the second difference

$$\Delta(x, t) = [(x+t)^a - t^a] - [(x+T)^a - T^a] = \int_0^x d\xi \int_T^t d\tau a(a-1)(\xi+\tau)^{a-2}$$

can be estimated as

$$|\Delta| \leqq cx\delta T^{a-2} \leqq 1 \quad \text{if} \quad x \leqq \frac{1}{c}\delta^{-1}T^{2-a} .$$

Hence we obtain

$$|\psi_\delta(x)| > c_1\delta T^{-b} \quad \text{for} \quad 0 < x < X = c_2 \min(T, \delta^{-1}T^{2-a}) .$$

Taking $\chi_\mu = \chi_\delta$ the first part of the proposition is clear.

Now assume $a+b>1$ and observe that

$$\psi_\delta(x) = \pm \int_T^{T+\delta} \frac{e^{-it^a}}{ia|x+t|^{a+b-1}} d(e^{i|x+t|^a}) \longrightarrow 0$$

as $|x| \to +\infty$ using partial integration. We set

$$\chi_\mu(t) = \sum_{j=1}^k \chi_\delta(t+x_j) , \qquad \psi_\mu(x) = \sum_{j=1}^k \psi_\delta(x-x_j) ,$$

where the integer k is defined by

$$k = \left[\frac{\mu}{\delta}\right] \geqq 1 ,$$

the real numbers x_j are selected so that

$$|x_i - x_j| \geqq \max(\delta, X + d) \quad \text{for} \quad i \neq j ,$$

and $d > 0$ is selected so that

$$|\psi_\delta(x)| \leqq \frac{c_1}{2k}\delta T^{-b} \quad \text{for} \quad |x| \geqq d .$$

Observe that the supports of $\chi_\delta(t + x_j)$ are disjoint and that the support of the characteristic function χ_μ has measure $k\delta \leqq \mu$; furthermore

$$\psi_\mu(x) = \int K(x + t)\chi_\mu(t)dt .$$

Finally, if $x - x_i$ is in the interval $I = (0, X)$ then $|x - x_j| \geqq d$ for $j \neq i$, hence

$$|\psi_\mu(x)| > c_1\delta T^{-b} - (k - 1)\frac{c_1}{2k}\delta T^{-b} \geqq \frac{c_1}{2}\delta T^{-b} \quad \text{for} \quad x \in x_i + I .$$

Since the intervals $x_i + I$ are disjoint we have

$$\left|\left\{x: |\psi_\mu(x)| > \frac{c_1}{2}\delta T^{-b}\right\}\right| \geqq kX ,$$

which is the second claim with changed (smaller) constants.

Integrating the function $|\psi_\mu|$ over a suitable subset of size λ gives the following

COROLLARY. *Assume* $0 < \delta \leqq \mu$, $T \geqq \delta$. *Then*

$$B(\lambda, \mu) \geqq c_1\lambda\delta T^{-b} \tag{3}$$

if $\lambda \leqq c_2 \min(T, \delta^{-1}T^{2-a})$; *in case that* $a + b > 1$ *the estimate* (3) *holds even in the larger range* $\lambda \leqq c_2\mu\delta^{-1} \min(T, \delta^{-1}T^{2-a})$.

REMARK. Note that in the first case of the corollary the supports of χ_μ and χ_λ can be taken to be single intervals, namely $(T, T + \delta)$ and $(0, X)$ respectively. In the second case we take k translates of each of these intervals which are spread apart by a minimum amount (at least the size of these intervals).

In order to apply the corollary we fix λ, μ arbitrarily and select optimal values of δ, T among those which are permitted. In terms of $\log\delta$ and $\log T$ this is a simple linear programming problem.

In case (IV), cf. §2, we get

$$B(\lambda, \mu) = \infty \quad \text{for all } \lambda, \mu ,$$

since (IV) can be broken down into the subcases

$$(\mathrm{IV}_1) \qquad a \leqq 2 , \qquad b < 0[\delta = \min(\mu, c_2\lambda^{-1}), \; T \longrightarrow +\infty] ,$$

$$(\mathrm{IV}_2) \qquad a > 2 , \qquad b < 2 - a[\delta = c_2\lambda^{-1}T^{2-a}, \; T \longrightarrow +\infty] ,$$

$$(\mathrm{IV}_3) \qquad \begin{array}{l} a > 2 , \qquad b < 1 - \dfrac{a}{2} , \\ b \geqq 2 - a(>1-a)\left[\delta = \left(\dfrac{c_2\mu}{\lambda}\right)^{1/2} T^{1-(a/2)}, \; T \longrightarrow +\infty\right] . \end{array}$$

In what follows we assume $\lambda \leqq \mu$ since $B(\lambda, \mu)$ is symmetric. Remember that (i) was relevant for (I_1) and (III_1), i.e., if $b \geqq 0$ and $\lambda \leqq \mu^{1-a}$. We will now show that

$$(\mathrm{i}') \qquad B(\lambda, \mu) \geqq c\lambda\mu^{1-b} \quad \text{if} \quad \lambda \leqq \mu^{1-a} ,$$

so that (i) is optimal where it is relevant: simply choose in the corollary

$$T = \frac{\mu}{c_2} , \qquad \delta = \min(1, c_2^{-1}, c_2^{a-1})\mu .$$

Remember that (ii) was relevant for (I_2) and (II_2), i.e., if $b \geqq 1 - (a/2)$ and $\lambda \geqq \mu^{1-a}$. We will now show that

$$(\mathrm{ii}') \qquad B(\lambda, \mu) \geqq c\lambda^{(a+b-1)/a}\mu^{(1-b)/a} \quad \text{if} \quad b \geqq 1 - \frac{a}{2} , \qquad \lambda \geqq \mu^{1-a} ,$$

so that (ii) is optimal where it is relevant: since $a + b > 1$, $(\mu/\lambda)^{1/a} \leqq \mu$ simply choose in the corollary

$$T = \left(\frac{\mu}{\lambda}\right)^{1/a} , \qquad \delta = \min(1, c_2, c_2^{1/2})\left(\frac{\mu}{\lambda}\right)^{1/a} .$$

Finally, remember that (iii) was relevant in (II_1) and (III_2), i.e., in case (II) with $\lambda \leqq \mu^{1-a}$ and in case (III) with $\lambda \geqq \mu^{1-a}$. We will now show that

$$(\mathrm{iii}') \qquad B(\lambda, \mu) \geqq c(\lambda\mu)^{(a+b-2)/(a-2)} \quad \text{if} \quad \begin{cases} a > 2, \; \lambda \leqq \mu^{1-a} \\ \text{or } a < 2, \; \lambda \geqq \mu^{1-a} , \end{cases}$$

so that (iii) is optimal where it is relevant: since $(\lambda\mu)^{1/(2-a)} \geqq \mu$ simply choose in the corollary

$$T = \frac{1}{c_2}(\lambda\mu)^{1/(2-a)} , \qquad \delta = \min(1, c_2^{-1}, c_2^{a-1})\mu .$$

The lower estimates are summarized in Theorem 3.

4. Applications. In this section we will determine the weak restricted mapping set $S(K)$ as consequence of our estimates for $B(\lambda, \mu; a, b)$, and we will also determine the order of magnitude of $N_q(\mu)$.

We begin with $S(K)$. It is convenient to introduce the following four linear functions in $1/q$:

$$\gamma_1 = 1 - b + \frac{1}{q}, \qquad \gamma_2 = 1 - b + (1-a)/q,$$

$$\gamma_3 = \frac{1}{q}, \qquad \gamma_4 = \frac{1 - a - b + 1/q}{1-a}.$$

A point $(1/p, 1/q)$ with $1 \leqq p \leqq \infty$, $1 < q < \infty$ is of weak restricted type if (1) holds, i.e., if both

$$B(\lambda, \mu) \leqq c_{pq} \lambda^{1/q'} \mu^{1/p}, \qquad B(\lambda, \mu) \leqq c_{pq} \mu^{1/q'} \lambda^{1/p}$$

hold (by symmetry). Using the simple cases $\lambda = \mu^{1-a}$ and $\lambda = \mu$ of Theorem 3 we obtain the following necessary conditions

$$\begin{aligned} &\mu^{1-a}\mu^{1-b} \leqq c_{pq}\mu^{(1-a)/q'}\mu^{1/p} \quad \text{for} \quad \mu \geqq 1, \\ &\mu^{1-a}\mu^{1-b} \leqq c_{pq}\mu^{1/q'}\mu^{(1-a)/p} \quad \text{for} \quad \mu \geqq 1, \\ &\mu\mu^{1-b} \leqq c_{pq}\mu^{1/q'}\mu^{1/p} \qquad \text{for} \quad \mu \leqq 1 (b \geqq 0). \end{aligned}$$

By letting $\mu \to +\infty$ resp. $\mu \to +0$ we get the three inequalities

$$\gamma_2 \leqq \frac{1}{p}, \qquad \gamma_4 \leqq \frac{1}{p} (\text{if } a < 1) \quad \text{or} \quad \gamma_4 \geqq \frac{1}{p} (\text{if } a > 1),$$

$$\gamma_1 \geqq \frac{1}{p} (\text{if } b \geqq 0).$$

We can add to this the well known necessary condition $q \geqq p$, i.e., $\gamma_3 \leqq 1/p$. We are going to show that these inequalities are not only necessary but also sufficient (for $1 < q < \infty$), i.e., they describe the set $S(K)$.

To get a clearer geometric picture of the situation we introduce the following five points in the $(1/p, 1/q)$-plane:

$$A = (1-b, 0), \qquad A' = (1, b);$$

$$B = \left(\frac{1-b}{a}, \frac{1-b}{a}\right), \qquad B' = \left(\frac{a+b-1}{a}, \frac{a+b-1}{a}\right)$$

and

$$C = C' = \left(\frac{a+b-2}{a-2}, \frac{b}{2-a}\right) \quad \text{if} \quad a \neq 2.$$

Furthermore we consider the straight lines

$$l_1\colon \gamma_1 = \frac{1}{p} \text{ passing through } A, A' ,$$

$$l_2\colon \gamma_2 = \frac{1}{p} \text{ passing through } A, B, C ,$$

$$l_3\colon \gamma_3 = \frac{1}{p} \text{ passing through } B, B' ,$$

$$l_4\colon \gamma_4 = \frac{1}{p} \text{ passing through } A', B', C' = C .$$

Next we describe the set characterized by the linear inequalities above:

In case (I) we have the closed quadrilateral $[A, A', B', B]$ except the vertex A where $q = \infty$ (see Figure 1). Outside the quadrilateral there are no mapping points as we have seen. The vertices A, A' correspond directly to the inequalities (i); thus A is of strong restricted type and A' is of weak type. Similarly the vertices B, B' correspond directly to the inequalities (ii). Then all the points of the quadrilateral are of weak restricted type by trivial convexity.

In case (II) we have the closed triangle $[B, C, B']$, see Figure 2. Again the vertices correspond to our main upper estimates, in particular C corresponds to (iii).

In case (III) we have the closed triangle $[A, A', C]$ except A, see Figure 3.

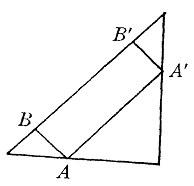

Figure 1
($a = 3,\ b = 1/2$)

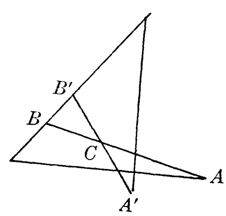

Figure 2
($a = 4,\ b = -1/2$)

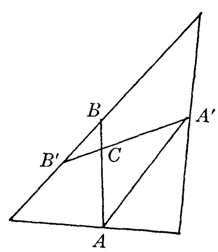

Figure 3
($a = 2/3,\ b = 1/2$)

In case (IV) the set $S(K)$ is empty. The set $S(K)$ can degenerate into a line segment or a single point. We summarize these results as follows:

THEOREM 4. *The weak restricted mapping set* $S(K)$ *is in case* (I) *the closed quadrilateral* $[A, A', B', B]$ *except* A, *in case* (II) *the closed triangle* $[B, C, B']$ *in case* (III) *the closed triangle* $[A, A', C]$

except A, and empty in case (IV).

REMARKS. Observe that the boundary of $S(K)$ consists of line segments which are neither horizontal nor vertical. Since the endpoints are of weak restricted type or strong restricted type the interior points of these line segments must be strong mapping points by the convexity theorem of Stein-Weiss. The same is true for the interior of $S(K)$ anyway. So the only points which need further clarification with regard to their mapping character are the vertices A, A', B, B', C. It is easy to see that A is strong restricted but not strong and that A' is weak but not strong ($b > 0$). Hence only B, B', C present a problem. We will show in a subsequent paper that C is strong also, which leaves only the character of B, B' partially undecided. This is settled in [10].

Finally we will determine the precise order of magnitude of $N_q(\mu)$. This gives a more detailed picture of the mapping properties and is based on the calculation of

$$\tilde{N}_q(\mu) = \sup_{\lambda>0} \lambda^{-1/q'} \tilde{B}(\lambda, \mu) .$$

Since $\tilde{B}$ is explicit one can work out $\tilde{N}_q$ explicitly also. The calculation is lengthy, but entirely elementary; so we will drop the details. We find that whenever $\tilde{N}_q < \infty$ then it is of the form

$$\tilde{N}_q(\mu) = \mu^{\alpha_q}(\mu \leqq 1) , \qquad \tilde{N}_q(\mu) = \mu^{\beta_q}(\mu \geqq 1) .$$

To describe $\tilde{N}_q$ we distinguish between $a < 1$ and $a > 1$ in our cases (I), (II), (III).

In case (I) with $a < 1$:
If $1/q > b$ then $\tilde{N}_q(\mu) = \infty$ for all μ; if $1/q \leqq b$ then $\tilde{N}_q < \infty$ and $\alpha_q = \gamma_1$, $\beta_q = \max(\gamma_2, \gamma_3, \gamma_4)$.

In case (I) with $a > 1$:
If $1/q > (a + b - 1)/a$ then $\tilde{N}_q(\mu) = \infty$ for all μ; if $1/q \leqq (a + b - 1)/a$ then $\tilde{N}_q < \infty$ and $\alpha_q = \min(\gamma_1, \gamma_4)$, $\beta_q = \max(\gamma_2, \gamma_3)$.

In case (II), where automatically $a > 2$:
If $1/q < -b/(a - 2)$ or $1/q > (a + b - 1)/a$ then $\tilde{N}_q(\mu) = \infty$ for all μ; if $-b/(a - 2) \leqq 1/q \leqq (a + b - 1)/a$ then $\tilde{N}_q < \infty$, $\alpha_q = \gamma_4$, $\beta_q = \max(\gamma_2, \gamma_3)$.

In case (III) with $a < 1$:
If $1/q > b$ then $\tilde{N}_q(\mu) = \infty$ for all μ; if $1/q \leqq b$ then $\tilde{N}_q < \infty$ and $\alpha_q = \gamma_1$, $\beta_q = \max(\gamma_2, \gamma_4)$.

In case (III) with $a > 1$:
If $1/q > b/(2 - a)$ then $\tilde{N}_q(\mu) = \infty$ for all μ; if $1/q \leqq b/(2 - a)$ then $\tilde{N}_q < \infty$ and $\alpha_q = \min(\gamma_1, \gamma_4)$, $\beta_q = \gamma_2$.

In case (IV): $\tilde{N}_q(\mu) = \infty$ for all μ.

Note that also $\tilde{N}_q(\mu)$ is logarithmically convex in $1/q$. In view of Theorems 2 and 3 we have

THEOREM 5. *Always* $c_3\tilde{N}_q(\mu) \leqq N_q(\mu) \leqq c_4\tilde{N}_q(\mu)$.

This determination of N_q will, of course, give the mapping set $S(K)$ again. But it is interesting to see that conversely the linear forms $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ which correspond to the line segments of the boundary of $S(K)$ turn up in the exponents α_q and β_q of $\tilde{N}_q$.

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Received February 16, 1979 and in revised form December 15, 1980. The research of the first author was supported in part by the National Science Foundation.

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