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AN ANALOGUE OF KOLMOGOROV'S INEQUALITY FOR A CLASS OF ADDITIVE ARITHMETIC FUNCTIONS

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Let f be a complex valued additive number theoretic function (i.e., $f(mn)=f(m)+f(n)$ if m and n are relatively prime). This paper shows that $\sum D^2(p^\alpha)p^{-\alpha} = O(D^2(n))$ or $\sum |f(p^\alpha)|p^{-\alpha}=O(D(n))$ (where the summations are over those $p^\alpha \leqq n$, p^α being a prime raised to a power) is sufficient to guarantee that the following analogue of Kolmogorov's inequality holds:

$$\nu_n\left\{\operatorname*{Max}_{k\leqq n} |f_k(m)-A(k)| > tD(n)\right\}=O(t^{-2})$$

where, if $p^\alpha || m$ denotes the fact that p^α divides m but $p^{\alpha+1}$ does not (i.e., p^α exactly divides m), then

$$A(n)=\sum_{p^\alpha\leqq n} f(p^\alpha)p^{-\alpha} ,$$

$$D^2(n)=\sum_{p^\alpha\leqq n} |f(p^\alpha)|^2p^{-\alpha} ,$$

$$f_k(m)=\sum_{\substack{p^\alpha\leqq k\\ p^\alpha || m}} f(p^\alpha) ,$$

and

$$\nu_n(\mathscr{S})=n^{-1}\sum_{\substack{m\leqq n\\ m\in\mathscr{S}}} 1$$

for any set $\mathscr{S}$.

It is known that

$$\sum_{m\leqq n} |f(m) - A(n)|^2 \leqq c_0nD^2(n)$$

holds for all additive functions for some absolute constant c_0. This implies the analogue of Chebyshev's inequality. Hence it is of interest to determine whether the analogue of Kolmogorov's inequality also holds for all such functions. The author could not do this for all additive functions, but did find various sufficient conditions to guarantee the result. The two which were most general and verifiable for specific functions are stated in the opening paragraph.

The author proved his result in two stages. First he determined in Theorem 1 necessary and sufficient conditions for

$$\sum_{m\leqq n} \operatorname*{Max}_{k\leqq n} |f_k(m) - A(k)|^2 \leqq cnD^2(n)$$

to hold (which implies the analogue of Kolmogorov's inequality).

The more manageable problem that resulted provided the basis for proving the result stated in the opening paragraph as well as an approach that might eventually help lead to the full solution of the problem.

1. Preliminaries. Let p^α and q^β represent primes raised to a power. Given an integer m, let $L(m)$ be the largest p^α such that $p^\alpha \| m$ and let $S(m)$ be the smallest such p^α. If $q^\beta \| m$ with $q^\beta > S(m)$, we shall denote by $r^\gamma(m, q^\beta)$ the largest exact prime-power divisor of m which is less than q^β.

The following well known facts are freely used in this article:

$$\sum_{p^\alpha \leqq n} p^{-\alpha} \log p^\alpha = O(\log n)$$

and

$$\sum_{p^\alpha \leqq n} p^{-\alpha} = \log\log n + B + O(\log^{-1} n)$$

where B is an absolute constant. The next lemma represents an extension of a known result of sieve methods.

LEMMA. *Given $1.9 \leqq b \leqq c \leqq n$, let $\mathscr{S} = \mathscr{S}(n, c, b)$ be the set of those m, $m \leqq n$, such that $p^\alpha \| m$ implies either $p^\alpha \leqq b$ or $p^\alpha \geqq c$. Then there exists an absolute constant c_1 such that*

$$\sum_{m \in \mathscr{S}} 1 \leqq c_1 n (\log b) \log^{-1} c \,. \tag{1.1}$$

Proof. If for any $z \geqq 2$ we let

$$\mathscr{U} = \{p\colon p < z \quad \text{and} \quad p \leqq b \quad \text{or} \quad p \geqq c\}$$

and $\mathscr{S}' = \{m\colon m \leqq n \text{ and } p \nmid m \text{ if } b < p < \min(c, z)\}$, then it is known [1, p. 104] that

$$\sum_{m \in \mathscr{S}'} 1 \leqq z^2 + n \log^{-1} z \prod_{p \in \mathscr{U}} (1 - p^{-1})^{-1} \,.$$

Since

$$\Big| \prod_{p \leqq x} (1 - p^{-1}) - e^{-\gamma} \log^{-1} x \Big| < e^{-\gamma} \log^{-3} x$$

for $x > 1$ where γ is Euler's constant [3, p. 70], it follows that if we choose $z = n^{1/2} \log^{-1/2} n$ and assume $3 \leqq b \leqq c \leqq z$, then

$$\sum_{m \in \mathscr{S}'} 1 \leqq \frac{n}{\log n} + \frac{e^\gamma (1 + \log^{-2} 3)}{(2/3)(1 - \log^{-2} 3)^2} \frac{n \log b}{\log c} \leqq 168 n (\log b) \log^{-1} c \,.$$

For the cases where $1.9 \leqq b < 3$ or $z < c \leqq n$ or $z \leqq b \leqq c \leqq n$, note that it follows from the last result that

$$\sum_{m \in \mathscr{S}'} 1 \leqq \frac{168n(\log b)(\log 3)(\log n)}{(\log 1.9)(\log z)(\log c)} \leqq 910 \frac{n \log b}{\log c}.$$

Now if we let $\mathscr{S}''$ be the set of m, $m \leqq n$, such that there exists a $p^\alpha \| m$ for which $b < p < c$ and $p^\alpha \geqq c$, then

$$\sum_{m \in \mathscr{S}} 1 \leqq \sum_{m \in \mathscr{S}'} 1 + \sum_{m \in \mathscr{S}''} 1 .$$

Since

$$\begin{aligned}\sum_{m \in \mathscr{S}''} 1 &\leqq \sum_{b<p<c} \sum_{\log c/\log p \leqq \alpha \leqq \log n/\log p} \sum_{\substack{m \leqq n \\ p^\alpha \| m}} 1 \\ &\leqq n \sum_{b<p<c} \sum_{\alpha \geqq \log c/\log p} p^{-\alpha} \\ &\leqq 2nc^{-1} \sum_{p<c} 1 \leqq \frac{2n}{c}\left(\frac{1.26c}{\log c}\right)\frac{\log b}{\log 1.9} \\ &\leqq 4n(\log b)\log^{-1} c\end{aligned}$$

we see that choosing $c_1 = 914$ yields (1.1). This completes the proof.

2. General necessary and sufficient conditions.

The next theorem is of theoretical significance. However, since it is not easy to apply the results to specific functions it is not very practical.

THEOREM 1. *Given an additive complex valued arithmetic function f, necessary and sufficient conditions for*

$$\sum_{m \leqq n} \operatorname*{Max}_{k \leqq n} |f_k(m) - A(k)|^2 \leqq c_2 n D^2(n) \tag{2.1}$$

to hold for some constant c_2 are:

$$\sum_{m \leqq n} \operatorname*{Max}_{q^\beta \| m} |f_{q^\beta}(m) - A(q^\beta)|^2 \leqq c_3 n D^2(n) \tag{2.2}$$

and

$$\sum_{m \leqq n} \operatorname*{Max}_{\substack{q^\beta \| m \\ S(m) < q^\beta < L(m)}} \operatorname*{Max}_{r^\gamma(m, q^\beta) \leqq k < q^\beta} |A(k) - A(r^\gamma)|^2 \leqq c_3 n D^2(n) \tag{2.3}$$

for some constant c_3. Similarly, necessary and sufficient conditions for

$$\nu_n\{\operatorname*{Max}_{k \leqq n} |f_k(m) - A(k)| > tD(n)\} \leqq c_2 t^{-2}, \tag{2.4}$$

the analogue of Kolmogorov's inequality, to hold for all real $t > 0$ and for some constant c_2 are:

$$\nu_n\{\operatorname*{Max}_{q^\beta \| m} |f_{q^\beta}(m) - A(q^\beta)| > tD(n)\} \leqq c_3 t^{-2} \tag{2.5}$$

and

$$(2.6) \qquad \nu_n\{ \operatorname*{Max}_{\substack{q^\beta \| m \\ S(m) < q^\beta < L(m)}} \ \operatorname*{Max}_{r^\gamma(m, q^\beta) \leqq k < q^\beta} |A(k) - A(r^\gamma)| > tD(n)\} \leqq c_3 t^{-2}$$

for some constant c_3. *Note that* (2.2) *and* (2.3) *imply* (2.5) *and* (2.6). *Also,* c_2 *depends only on* c_3 (*which may depend on* f).

Proof. Let $T = T(n, m) = \mathrm{Max}_{k \leqq n} |f_k(m) - A(k)|$. Then

$$T \leqq \mathrm{Max}(T_1, T_2, T_3) + T_4 + T_5$$

where

$$T_1 = \operatorname*{Max}_{k < S(m)} |A(k)|$$

$$T_2 = \operatorname*{Max}_{L(m) < k \leqq n} |A(k) - A(L(m))|$$

$$T_3 = \operatorname*{Max}_{r^\gamma(m, L(m)) < k \leqq L(m)} |A(k) - A(r^\gamma)|$$

$$T_4 = \operatorname*{Max}_{q^\beta \| m} |f_{q^\beta}(m) - A(q^\beta)|$$

and

$$T_5 = \operatorname*{Max}_{\substack{q^\beta \| m \\ S(m) < q^\beta < L(m)}} \ \operatorname*{Max}_{r^\gamma(m, q^\beta) \leqq k < q^\beta} |A(k) - A(r^\gamma)| .$$

Using Schwarz's inequality and the lemma we see that

$$\begin{aligned} \sum_{m \leqq n} T_1^2 &\leqq \sum_{m \leqq n} D^2(S(m)) \sum_{p^\alpha < S(m)} p^{-\alpha} \\ &\leqq D^2(n) \sum_{p^\alpha \leqq n} p^{-\alpha} \sum_{m \in \mathscr{S}(n, p^\alpha, 1, 9)} 1 \\ &\leqq c_1 n D^2(n)(\log 1.9) \sum_{p^\alpha \leqq n} p^{-\alpha} \log^{-1} p^\alpha \\ &= O(nD^2(n)) \end{aligned}$$

and

$$\begin{aligned} \sum_{m \leqq n} T_2^2 &\leqq D^2(n) \sum_{m \leqq n} \sum_{L(m) < p^\alpha \leqq n} p^{-\alpha} \\ &\leqq D^2(n) \sum_{p^\alpha \leqq n} p^{-\alpha} \sum_{m \in \mathscr{S}(n, n, p^\alpha)} 1 \\ &\leqq c_1 n D^2(n)(\log n)^{-1} \sum_{p^\alpha \leqq n} p^{-\alpha} \log p^\alpha \\ &= O(nD^2(n)) . \end{aligned}$$

Also,

$$\begin{aligned} \sum_{m \leqq n} T_3^2 &\leqq D^2(n) \sum_{m \leqq n} \sum_{r^\gamma(m, L(m)) < p^\alpha < L(m)} p^{-\alpha} \\ &\leqq D^2(n)(S_1 + S_2 + S_3) \end{aligned}$$

where

$$S_1 = \sum_{n^{1/2} \leqq p^\alpha < n} p^{-\alpha} \sum_{m \leqq n} 1 = O(n)$$

and where (noting that there are no exact divisors of m larger than p^α once m is divided by $L(m)$ in the sum)

$$\begin{aligned}
S_2 &= \sum_{p^\alpha < n^{1/2}} p^{-\alpha} \sum_{p^\alpha < q^\beta \leqq np^{-\alpha}} \sum_{m \in \mathscr{S}(nq^{-\beta}, nq^{-\beta}, p^\alpha)} 1 \\
&\leqq c_1 n \sum_{q^\beta \leqq n} (q^\beta \log nq^{-\beta})^{-1} \sum_{p^\alpha \leqq \operatorname{Min}(q^\beta, nq^{-\beta})} p^{-\alpha} \log p^\alpha \\
&= O(n) + O(n \sum_{q^\beta < n^{1/2}} (\log q^\beta)(q^\beta \log nq^{-\beta})^{-1}) \\
&= O(n) + O[n \log^{-1} n \sum_{q^\beta < n^{1/2}} q^{-\beta} \log q^\beta] \\
&= O(n)
\end{aligned}$$

and

$$\begin{aligned}
S_3 &= \sum_{p^\alpha < n^{1/2}} p^{-\alpha} \sum_{np^{-\alpha} < q^\beta \leqq n} nq^{-\beta} \\
&= O(n) + O\left[n \sum_{p^\alpha < n^{1/2}} p^{-\alpha} \log \frac{\log n}{\log np^{-\alpha}}\right] \\
&= O(n) + O\left[n \sum_{p^\alpha < n^{1/2}} p^{-\alpha} \log\left[1 - \frac{\log p^\alpha}{\log n}\right]^{-1}\right] \\
&= O(n) + O[n \log^{-1} n \sum_{p^\alpha < n^{1/2}} p^{-\alpha} \log p^\alpha] \\
&= O(n) .
\end{aligned}$$

Hence

$$\sum_{m \leqq n} \operatorname{Max}(T_1^2, T_2^2, T_3^2) \leqq c_4 n D^2(n)$$

where c_4 is absolute and does not depend on f. From this it also follows that

$$\nu_n\{\operatorname{Max}(T_1, T_2, T_3) > tD(n)\} \leqq c_4 t^{-2}$$

for all real $t > 0$. This establishes the sufficiency of the conditions.

The fact that $T \geqq T_4$ establishes the necessity of (2.2) and (2.5). The necessity of (2.2) and (2.5) together with $T \geqq T_5 - T_4$ establishes the necessity of (2.3) and (2.6). This completes the proof.

3. A practical sufficient condition.

The next theorem provides an easily verifiable sufficient condition for (2.1) and (2.4) to hold.

THEOREM 2. *Given an additive complex valued arithmetic function f, a sufficient condition for* (2.1) *and* (2.4) *to hold is*

$$\sum_{q^\beta \leqq n} D^2(q^\beta) q^{-\beta} = O(D^2(n)) \tag{3.1}$$

or

$$\sum_{p^\alpha \leqq n} |f(p^\alpha)| p^{-\alpha} = O(D(n)) . \tag{3.2}$$

Proof. It is known [2, p. 31] that there exists an absolute constant c_0 such that for any complex valued additive function g

$$\sum_{m\leqq n} |g(m) - A(n)|^2 \leqq c_0 D^2(n) .$$

Hence if (3.1) holds then

$$\begin{aligned}
&\sum_{m\leqq n} \operatorname*{Max}_{q^\beta || m} |f_{q^\beta}(m) - A(q^\beta)|^2 \\
&\quad \leqq \sum_{q^\beta\leqq n} \sum_{m\leqq nq^{-\beta}} (2|f_{q^\beta}(m) - A(q^\beta)|^2 + 2|f(q^\beta)|^2) \\
&\quad \leqq 2c_0 n \sum_{q^\beta\leqq n} D^2(q^\beta)q^{-\beta} + 2nD^2(n) \\
&\quad = O(nD^2(n))
\end{aligned}$$

which guarantees that (2.2) holds. Noting that for $m \leqq n$, $q^\beta || m$ and $q^\beta < L(m)$, we must have $q^\beta \leqq n^{1/2}$, it follows from Schwarz's inequality and the lemma that

$$\begin{aligned}
&\sum_{m\leqq n} \operatorname*{Max}_{\substack{q^\beta || m \\ S(m)<q^\beta<L(m)}} \operatorname*{Max}_{r^\gamma(m,q^\beta)\leqq k<q^\beta} |A(k) - A(r^\gamma)|^2 \\
&\quad \leqq \sum_{m\geqq n} \sum_{\substack{q^\beta || m \\ \mathscr{S}(m)<q^\beta<L(m)}} D^2(q^\beta) \sum_{r^\gamma(m,q^\beta)<p^\alpha<q^\beta} p^{-\alpha} \\
&\quad \leqq \sum_{q^\beta\leqq n^{1/2}} D^2(q^\beta) \sum_{p^\alpha<q^\beta} p^{-\alpha} \sum_{m\in\mathscr{S}(nq^{-\beta},q^\beta,p^\alpha)} 1 \\
&\quad \leqq c_1 n \sum_{q^\beta\leqq n} D^2(q^\beta)(q^\beta \log q^\beta)^{-1} \sum_{p^\alpha<q^\beta} p^{-\alpha}\log p^\alpha \\
&\quad = O(n \sum_{q^\beta\leqq n} D^2(q^\beta)q^{-\beta}) \\
&\quad = O(nD^2(n))
\end{aligned}$$

which guarantees that (2.3) holds. Hence (2.1) and (2.4) hold according to Theorem 1.

Now suppose that (3.2) is true and define the additive function g by $g(p^\alpha) = |f(p^\alpha)|$. To avoid confusion let $\hat{A}(n) = \sum g(p^\alpha)p^{-\alpha}$ where the sum is over those $p^\alpha \leqq n$; $A(n)$ is thus reserved for f. Note that $D^2(n)$ is the same for both f and g. We see that

$$\begin{aligned}
&\sum_{m\leqq n} \operatorname*{Max}_{k\leqq n} |f_k(m) - A(k)|^2 \\
&\quad \leqq 2\sum_{m\leqq n} (\hat{A}^2(n) + g^2(m)) \\
&\quad = 2\sum_{m\leqq n} (g(m) - \hat{A}(n))^2 + 4\hat{A}(n)\sum_{m\leqq n} g(m) \\
&\quad \leqq 2c_0 nD^2(n) + 4\hat{A}(n)\sum_{q^\beta\leqq n} g(q^\beta) \sum_{\substack{m\leqq n \\ q^\beta || m}} 1 \\
&\quad \leqq 2c_0 nD^2(n) + 4n[\sum_{p^\alpha\leqq n} |f(p^\alpha)|\, p^{-\alpha}]^2 \\
&\quad = O(nD^2(n)) .
\end{aligned}$$

This completes the proof.

Examples of functions which satisfy (3.1) and (3.2) are the additive functions determined by $f(p^\alpha) = \log p^\alpha$ and $f(p^\alpha) = p^\alpha$. An example of a function that satisfies neither (3.1) nor (3.2) is the one determined by $f(p^\alpha) = 1$. Any nontrivial function f such as $f(p^\alpha) = \log^{-1} p^\alpha$ for which $\sum |f(p^\alpha)| p^{-\alpha}$ and $D^2(n)$ are bounded satisfies (3.2) but not (3.1).

REFERENCES

1. H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, London, 1975.
2. J. Kubilius, *Probabilistic Methods in the Theory of Numbers*, Translations of Mathematical Monographs, Amer. Math. Soc. Vol. 11, Providence, R. I., 1964.
3. J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math., **6** (1962), 64–94.

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