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ON THE ORDER OF $\zeta(1/2+it)$ AND $\Delta(R)$

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In this paper we obtain new estimates of $\zeta(1/2+it)$, the value of the Riemann zeta-function on the critical line, and $\Delta(R)$, the remainder in the Dirichlet divisor problem.

The estimates are

$$\zeta\left(\frac{1}{2}+it\right) \ll t^{35/216+\varepsilon} \quad \textbf{for any} \quad \varepsilon > 0$$

and

$$\Delta(R) \ll R^{35/108+\varepsilon} \quad \textbf{for any} \quad \varepsilon > 0 \, ,$$

which improve previously known best results.

Our results are obtained by van der Corput's method to the estimation of trigonometric sums of several variables. The method can be applied to other problems which involve such trigonometric sums. The article consists of three sections. Section 1 contains lemmas, in § 2 we prove the theorem, in § 3 we obtain the consequences of the theorem—the new estimates of $\zeta(1/2+it)$ and $\Delta(R)$.

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1. Notations.

$\Delta(R)$—the remainder in the Dirichlet divisor problem.

C is any absolute constant, ε is any sufficiently small positive constant.

$\Delta_l(f) = \Delta_l(f(x_1, \cdots, x_n))$ is the principal minor of order l of the Hessian of $f(x_1, \cdots, x_n)$.

$R(f_1(\theta), f_2(\theta))$—the resultant of two polynomials $f_1(\theta)$ and $f_2(\theta)$.

$\sigma_{jn} = \sigma_{jn}(\alpha_1, \cdots, \alpha_n)$—the jth elemetary symmetric function of $\alpha_1, \cdots, \alpha_n$.

$f(x) \ll g(x)$ (or $f(x) \gg g(x)$) means that $|f(x)| \leqq cg(x)$ (or $f(x) \geqq cg(x)$) for some c.

$$e(f(x)) = \exp(2\pi i f(x)) \, .$$

For a domain $\mathscr{D} \in R^2$ and real numbers A_1, A_2, A_3, A_4,

$$\mathscr{D}(A_1, A_2, A_3, A_4) = \mathscr{D} \cap \{x_1, x_2) \,|\, A_1 \leqq x_1 \leqq A_2; A_3 \leqq x_2 \leqq A_4\} \, .$$

If $f(x_1, x_2)$ is a continuously differentiable function, then

$$\mathscr{D}(f) = \{(y_1, y_2) \,|\, y_1 = f_{x_1}(x_1, x_2),\ y_2 = f_{x_2}(x_1, x_2),\ (x_1, x_2) \in \mathscr{D}\} \, .$$

All functions are assumed to be sufficiently many times differentiable.

All domains $\mathscr{D}, \mathscr{D}_1, \cdots$ considered will be domains bounded by $O(1)$ algebraic curves of order $O(1)$ so that every straight line intersects $\mathscr{D}$ in $O(1)$ line segments.

Wherever a variable occurs as a summation variable the reference is to integer values of the variable.

LEMMA 1. *Let $f(x)$ be a real function such that*

$$|f^{(l)}(x)| \geqq F_l^{-1} > 0 \quad on \quad [X, X_1], \quad l \geqq 1 .$$

Then

$$\int_X^{X_1} e(f(x))dx \ll (F_l)^{1/l} .$$

This lemma can be proved by induction similarly to Lemma 4.5, p. 62, [2].

LEMMA 2. *Let $1 < X \leqq X_1 \leqq 2X$ and $K \geqq 3$. Let $f(x)$ be a real-valued function, analytic in $\mathscr{D} = \{z \mid |z-x| \leqq \sqrt{cF_2 \log x}, X \leqq x \leqq X_1\}$ where c is some sufficiently large constant and $F_2^{-1} \leqq f''(x) \ll F_2^{-1}$ for $x \in [X, X_1]$. Let $|f^{(l)}(z)| \ll F_l^{-1}$ $(l = 2, \cdots, K)$ for $z \in \mathscr{D}$, and let x_n be the solution of the equation $f'(x_n) = n$. Then*

$$S = \sum_{X \leqq x \leqq X_1} e(f(x))$$

$$= \sum_{f'(X) \leqq n \leqq f'(X_1)} (f''(x_n))^{-1/2} e\Big(\frac{1}{8} + f(x_n) - nx_n\Big) + O(\sqrt{F_2 \log^3 X})$$

and

$$S \ll \frac{X}{\sqrt{F_2}} + \min_{2 \leqq l \leqq K} (F_l)^{1/l} .$$

The proof is similar to the proof of the Theorem 4.9, p. 65, [2] with the use of Lemma 1 (see also Lemma 1 of [1]).

LEMMA 3. *Let $a(x_1, x_2)$ be real twice continuously differentiable function and let $b(x_1, x_2)$ be a function on $\mathscr{D} \in R^2$ such that $a_{x_1}, a_{x_2}, a_{x_1 x_2}$ do not change sign and $|a(x_1, x_2)| \leqq A$ in $\mathscr{D}$. Then for some A_1, A_2, B_1, B_2 and $\mathscr{D}_1 = \mathscr{D}(A_1, A_2, B_1, B_2)$ we have*

$$\Big|\sum_{(x_1,x_2) \in \mathscr{D}} a(x_1, x_2) b(x_1, x_2)\Big| \ll A \Big|\sum_{(x_1,x_2) \in \mathscr{D}_1} b(x_1, x_2)\Big| .$$

This lemma is a simple corollary of Lemma 1 in [3].

LEMMA 4. *Let $\mathscr{D}$ be a domain, contained in the rectangle $\{(x_1, x_2) \mid X_1 \leqq x_1 \leqq 2X_1, X_2 \leqq x_2 \leqq 2X_2\}$, and let $f(x_1, x_2)$ be three times continuously differentiable functions such that:*

(i) *for all integers* $a_{i_l,j_l} \in [-c, c]$ *the functions* $\sum_{i,j} \prod_{l=1}^{c} a_{i_l,j_l} f_{x_1^{i_l} x_2^{j_l}}$ *do not change sign in* $\mathscr{D}$, *where* $i = (i_1, \cdots, i_c)$, $j = (j_1, \cdots, j_c)$, *and the sum is over all possible distinct* (i, j) *such that* $i_l + j_l \leqq 4$ $(l = 1, \cdots, 6)$;

(ii) $|f_{x_1^i x_2^j}| \leqq cF/x_1^i x_2^j$, $1/M_1 \leqq f_{x_1^2} \leqq c/M_1$, $1/M_2 \leqq f_{x_1^2} f_{x_2^2} - (f_{x_1 x_2})^2 \leqq c/M_2$;

(iii) $f(x_1, x_2)$ *satisfies the conditions of Lemma* 2 *as a function of* x_1 *with* $M = M_1$;

(iv) $g(y_1, x_2) = f(x_1(y_1, x_2), x_2) - y_1 x_1(y_1, x_2)$ *satisfies the conditions of Lemma* 1 *as a function of* x_2 *with* $M = M_2/M_1$, *where* $x_1(y_1, x_2)$ *is the function satisfying* $f_{x_1}(x_1(y_1, x_2), x_2) = y_1$. *Let* $\varphi_1 = \varphi_1(y_1, y_2)$, $\varphi_2 = \varphi_2(y_1, y_2)$ *be the solution of the system*

$$f_{x_1}(\varphi_1, \varphi_2) = y_1\,, \qquad f_{x_2}(\varphi_1, \varphi_2) = y_2\,.$$

Then there exist some numbers $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$ *such that for* $\mathscr{D}_1 = \mathscr{D}'(A_1, A_2, A_3, A_4)$, $\mathscr{D}' = \mathscr{D}''(f)$, $\mathscr{D}'' = \mathscr{D}(B_1, B_2, B_3, B_4)$ *we have*

$$\begin{aligned} S &= \Big|\sum_{(x_1,x_2)\in\mathscr{D}} e(f(x_1, x_2))\Big| \\ &\ll \sqrt{M_2}\Big|\sum_{(y_1,y_2)\in\mathscr{D}_1} e(f(\varphi_1, \varphi_2) - y_1\varphi_1 - y_2\varphi_2)\Big| \\ &\quad + X_2\sqrt{M_1 \log^3 X_1} + \sqrt{M_2 \log^3 X_2} + \frac{F}{X_1}\sqrt{M_2 \log^3 X_2} \\ &\ll M_2^{-1/2} \sum_{(x_1,x_2)\in\mathscr{D}} 1 + X_2\sqrt{M_1 \log^3 X_1} + \sqrt{M_2 \log^3 X_2} \\ &\quad + F\sqrt{M_2 \log^3 X_2}/X_1\,. \end{aligned}$$

Also, if

$$|f_{x_2^3}(f_{x_1^2})^3 - 3f_{x_2^2 x_1} f_{x_1 x_2}(f_{x_1^2})^2 + 3f_{x_1^2 x_2}(f_{x_1 x_2})^2 f_{x_1^2} + f_{x_1^3}(f_{x_1 x_2})^3| \geqq M_3^{-1}\,,$$

then

$$S \ll M_2^{-1/2} \sum_{(x_1,x_2)\in\mathscr{D}} 1 + X_2\sqrt{M_1 \log^3 X_1} + (F/X_1 + 1)\cdot\sqrt{M_3/M_2^3}\,.$$

Proof. Taking the derivatives, we get

$$\frac{\partial x_1(y_1, x_2)}{\partial y_1} = \frac{1}{f_{x_1^2}}\,, \qquad \frac{\partial x_1(y_1, x_2)}{\partial x_2} = -\frac{f_{x_1 x_2}}{f_{x_1^2}}\,,$$

$$g_{x_2} = f_{x_2}\,, \qquad g_{x_2^2} = [f_{x_1^2} f_{x_2^2} - (f_{x_1 x_2})^2]/f_{x_1^2}\,,$$

$$g_{x_2^3} = [f_{x_2^3}(f_{x_1^2})^3 - 3f_{x_2^2 x_1} f_{x_1 x_2}(f_{x_1^2})^2 + 3f_{x_1^2 x_2}(f_{x_1 x_2})^2 f_{x_1^2} + f_{x_1^3}(f_{x_1 x_2})^2]/(f_{x_1^2})^3\,.$$

Now we apply successively Lemmas 2 and 3:

$$\begin{aligned} S &= \Big|\sum_{x_2}\sum_{x_1} e(f(x_1, x_2))\Big| \\ &= \left|\sum_{x_2}\left(\sum_{y_1} (f_{x_1^2})^{-1/2} e(g(y_1, x_2)) + O(\sqrt{M_1 \log^3 X_1})\right)\right| \\ &\ll \sqrt{M_1}\Big|\sum_{(y_1,x_2)\in\mathscr{D}} e(g(y_1, x_2))\Big| + O(X_2\sqrt{M_1 \log^3 X_1}) \end{aligned}$$

where

$$\mathscr{D}_2 = \mathscr{D}_3(a_1, a_2, b_1, b_2) ,$$
$$\mathscr{D}_3 = \{(y_1, x_2) \mid y_1 = f_{x_1}(x_1, x_2), (x_1, x_2) \in \mathscr{D}\} ,$$

because $(f_{x_1^2})^{-1/2} = (f_{x_1^2}(x_1(y_1, x_2), x_2))^{-1/2}$ satisfies the conditions of Lemma 3 on function $a(x_1, x_2)$ with $A = CM_1^{1/2}$.

Using (i) we can verify that the function

$$f_{x_2^2}(\varphi_1, \varphi_2) - (f_{x_1x_2}(\varphi_1, \varphi_2))^2/f_{x_1^2}(\varphi_1, \varphi_2)$$

also satisfies the same conditions with $A = C\sqrt{M_2/M_1}$, and we can apply again Lemma 2 first, then Lemma 3 to get

$$\begin{aligned} S &\ll \sqrt{M_1}\cdot \Big| \sum_{y_1} \Big[\sum_{y_2} (g_{x_2^2}(y_1, x_2(y_1, y_2))^{-1/2} e(g(y_1, x_2(y_1, y_2)) - y_2 x_2(y_1, y_2)) \\ &\quad + O\Big(\sqrt{\frac{M_2}{M_1}} \log^3 X_2 \Big) \Big] \Big| + X_2 \sqrt{M_1 \log^3 X_1} \\ &\ll \sqrt{M_1} \cdot \sqrt{M_2/M_1} \cdot \Big| \sum_{(y_1, y_2) \in \mathscr{D}_1} e(g(y_1, x_2(y_1, y_2)) - y_1 x_2(y_1, y_2)) \Big| \\ &\quad + \sqrt{M_2 \log^3 X_2} \sum_{y_1} 1 + X_2 \sqrt{M_1 \log^3 X_1} , \end{aligned}$$

where

$$\mathscr{D}_1 = \mathscr{D}_4(c_1, c_2, d_1, d_2) ,$$
$$\mathscr{D}_4 = \{(y_1, y_2) \mid y_2 = g_{x_2}(y_1, x_2), (y_1, x_2) \in \mathscr{D}_2\} ,$$

$x_2(y_1, y_2)$ is the function satisfying $g_{x_2}(y_1, x_2) = y_2$. Here

$$g_{x_2} = f_{x_2}(x_1(y_1, x_2), x_2)$$

and $x_2(y_1, y_2)$ satisfies the system

$$f_{x_1} = y_1 , \qquad f_{x_2} = y_2 , \qquad \text{i.e.,}$$
$$x_2(y_1, y_2) = \varphi_2(y_1, y_2) ;$$

also, $x_1(y_1, x_2(y_1, y_2))$ satisfies the same system, i.e., $x_1(y_1, y_2) = \varphi_1(y_1, y_2)$;

$$\sum_{y_1} 1 \leqq 1 + (\max f_{x_1}(x_1, x_2) - \min f_{x_1}(x_1, x_2)) \ll 1 + \frac{F}{X_1} .$$

Putting this into the last inequality for S, we prove the lemma.

LEMMA 5. *Let $f(x_1, \cdots, x_n)$ be a real function, and let D be some subdomain of the rectangular parallelepiped $X_i \leqq x_i \leqq 2X_i$; $f_1(x_1, \cdots, x_n, h_1, \cdots, h_n) = \int_0^1 (\partial/\partial t)(f(x_1 + h_1 t, \cdots, x_n + h_n t)dt)$; $q_1, \cdots, q_n$ are positive numbers such that*

$$\frac{q_1}{X_1} = \frac{q_2}{X_2} = \cdots = \frac{q_n}{X_n} = \sqrt[n]{\frac{Q}{N}} \ll 1 ; \quad q_1 \cdots q_n = Q ; \quad X_1 \cdots X_n = N ;$$

$$D_1 = \{(x, h) \mid h \neq 0,\ |h_i| < q_i,\ x \in D,\ x + h \in D\}\ ,$$

$$h = (h_1, \cdots, h_n)\ ; \qquad x = (x_1, \cdots, x_n)\ ; \qquad S = \frac{1}{N} \sum_x e(f(x))$$

$$S_1 = \frac{1}{NQ} \sum_{(x, h) \in \mathscr{D}_1} e(f_1(x, h))\ .$$

Then

$$S \ll \frac{1}{\sqrt{Q}} + \sqrt{S_1 \cdot \frac{1}{N} \sum_x 1}\ .$$

The proof is similar to the proof of Lemma 5.10, p. 91, [2].

2. Theorem.

1. *Notation for the theorem.*

$$\varphi_{k,l}(\theta) = C_{k+3,l} \cdot \theta^3 + C_{k+2,l+1}\sigma_{13}\theta^2 + C_{k+1,l+2}\sigma_{23}\theta + C_{k,l+3}\sigma_{33}$$

$$\varphi_1(\theta) = \varphi_{20}(\theta) \cdot \varphi_{02}(\theta) - (\varphi_{11}(\theta))^2$$

$$\begin{aligned}\varphi_2(\theta) = {} & \varphi_{30}(\theta) \cdot (\varphi_{02}(\theta))^3 - 3\varphi_{21}(\theta)\varphi_{11}(\theta) \cdot (\varphi_{02}(\theta))^2 \\ & + 3\varphi_{12}(\theta) \cdot (\varphi_{11}(\theta))^2 \varphi_{02}(\theta) - \varphi_{03}(\theta) \cdot (\varphi_{11}(\theta))^3\ ;\end{aligned}$$

$$R_1(\sigma_1, \sigma_2, \sigma_3) = R\varphi_{20}(\theta)\ , \quad \varphi_{02}(\theta)\ ;$$

$$\Delta \equiv \Delta(f(x, y)) = f_{x^2}(x, y) \cdot f_{y^2}(x, y) - (f_{xy}(x, y))^2\ ;$$

$$\Phi_{2,0}(f(x, y)) = -f_{y^2}(x, y)\ ;$$

$$\Phi_{1,1}(f(x, y)) = f_{xy}(x, y)\ ;$$

$$\Phi_{0,2}(f(x, y)) = -f_{x^2}(x, y)\ ;$$

$$\begin{aligned}\Phi_{i,j+1}(f(x, y)) = {} & \Big(-f_{xy}(x, y)\frac{\partial}{\partial x}\Big(\frac{\Phi_{i,j}(f(x, y))}{\Delta^{2i+2j-3}}\Big) \\ & + f_{x^2}(x, y)\frac{\partial}{\partial y}\Big(\frac{\Phi_{i,j}(f(x, y))}{\Delta^{2i+2j-3}}\Big)\Big)\Delta^{2i+2j-1}\ ;\end{aligned}$$

$$\begin{aligned}\Phi_{i+1,j}(f(x, y)) = {} & f_{y^2}(x, y)\frac{\partial}{\partial x}\Big(\frac{\Phi_{i,j}(f(x, y))}{\Delta^{2i+2j-3}}\Big) \\ & - \frac{\partial}{\partial y}\Big(\frac{\Phi_{i,j}(f(x, y))}{\Delta^{2i+2j-3}}\Big) f_{xy}(x, y)\Delta^{2i+2j-1}\ ;\end{aligned}$$

$\varphi_{i+3}(\theta)$ $(i=0, 1, 2, 3)$ are the polynomials of θ, obtained respectively from $\Phi_{3,2}(f(x, y))$, $\Phi_{2,3}(f(x, y))$, $\Phi_{1,4}(f(x, y))$ and $\Phi_{0,5}(f(x, y))$ by the substitution of $\varphi_{k,l}(\theta)$ for $f_{x^k y^l}(x, y)$; $\varphi_7(\theta) = \varphi_3(\theta)\varphi_5(\theta) - (\varphi_4(\theta))^2$; $\varphi_8(\theta) = \varphi_4(\theta) \cdot \varphi_6(\theta) - (\varphi_5(\theta))^2$

$$\psi_{k,l}(\theta) = C_{k+3,l}\theta^2 + C_{k+2,l+1}\sigma_{12}\theta + C_{k+1,l+2}\sigma_{22}$$

$$\psi_1(\theta) = \psi_{2,0}(\theta)\psi_{0,2}(\theta) - (\psi_{1,1}(\theta))^2\ .$$

$\psi_2(\theta)$ is the polynomial obtained from $\Phi_{0,4}(f(x, y))$ by substitution of $\psi_{k,l}(\theta)$ for $f_{x^k y^l}(x, y)$.

2. THEOREM. *Let $f(x, y)$ be a real function; $|f(x, y)| \leqq F$; $f_{x^k y^l}(x, y) = C_{k,l}(f(x, y))/x^k y^l + 0(\delta F/X^k Y^l)$, $(x, y) \in D$; $\delta \ll N^{-1/3}$; D is the domain of Lemma* 4; *$X \geqq Y$; $C_{k,l}$ are numbers such that each of the following systems of equations as functions of $\sigma_{13}(\alpha_1, \alpha_2, \alpha_3)$, $\sigma_{23}(\alpha_1, \alpha_2, \alpha_3)$, $\sigma_{33}(\alpha_1, \alpha_2, \alpha_3)$ doesn't have any solution*:

$$(1)\qquad \varphi_{20}(1) = \varphi_{02}(1) = \frac{\partial \varphi_{20}(1)}{\partial\theta} = \frac{\partial \varphi_{02}(1)}{\partial\theta} = 0$$

$$(2)\qquad \varphi_{20}(1) = \varphi_{02}(1) = \varphi_{11}(1) = \frac{\partial \varphi_{20}(1)}{\partial\theta} = 0$$

$$(3)\qquad \varphi_{20}(1) = \varphi_{02}(1) = \varphi_{11}(1) = \frac{\partial \varphi_{02}(1)}{\partial\theta} = 0$$

$$(4)\qquad \varphi_{20}(1) = \frac{\partial \varphi_{20}(1)}{\partial\theta} = \frac{\partial^2 \varphi_{20}(1)}{\partial\theta^2} = \varphi_{11}(1) = 0$$

$$(5)\qquad \varphi_{02}(1) = \frac{\partial \varphi_{02}(1)}{\partial\theta} = \frac{\partial^2 \varphi_{02}(1)}{\partial\theta^2} = \varphi_{11}(1) = 0$$

$$(6)\qquad \varphi_{20}(1) = \varphi_{02}(1) = \varphi_{11}(1) = \frac{\partial_{20}\varphi(1)}{\partial\theta}\cdot\frac{\partial \varphi_{02}(1)}{\partial\theta} - \left(\frac{\partial \varphi_{11}(1)}{\partial\theta}\right)^2 = 0$$

$$(7)\qquad \varphi_{20}(1) = \varphi_{11}(1) = \frac{\partial \varphi_{20}(1)}{\partial\theta} = \varphi_{02}(1)\cdot\frac{\partial^2 \varphi_{20}(1)}{\partial\theta^2} - 2\left(\frac{\partial \varphi_{11}(1)}{\partial\theta}\right)^2 = 0$$

$$(8)\qquad \varphi_{02}(1) = \varphi_{11}(1) = \frac{\partial \varphi_{02}(1)}{\partial\theta} = \varphi_{20}(1)\cdot\frac{\partial^2 \varphi_{02}(1)}{\partial\theta^2} - 2\left(\frac{\partial \varphi_{11}(1)}{\partial\theta}\right)^2 = 0$$

$$(9)\qquad \varphi_1(1) = \varphi_2(1) = \frac{\partial \varphi_1(1)}{\partial\theta} = \frac{\partial^2 \varphi_1(1)}{\partial\theta^2} = 0$$

$$(10)\qquad \varphi_{20}(1) = \varphi_{02}(1) = \frac{\partial \varphi_{20}(1)}{\partial\theta} = \frac{\partial^2 \varphi_{20}(1)}{\partial\theta^2} = 0$$

$$(11)\qquad \varphi_{20}(1) = \varphi_{02}(1) = \frac{\partial \varphi_{02}(1)}{\partial\theta} = \frac{\partial^2 \varphi_{02}(1)}{\partial\theta^2} = 0$$

$$(12)\qquad \varphi_{20}(1) = \varphi_{02}(1) = \varphi_{11}(1) = \frac{\partial \varphi_{11}(1)}{\partial\theta} = 0$$

$$(13)\qquad \frac{1}{\sigma_{33}}R_1(\sigma_{13}, \sigma_{23}, \sigma_{33}) = \frac{\partial}{\partial\alpha_3}\left(\frac{1}{\sigma_{33}}R_1(\sigma_{13}, \sigma_{23}, \sigma_{33})\right) = \frac{\partial^2}{\partial\alpha_3^2}\left(\frac{1}{\sigma_{33}}R_1(\sigma_{13}, \sigma_{23}, \sigma_{33})\right) = 0$$

$$(14)\qquad \varphi_{20}(1) = \varphi_{02}(1) = \frac{\partial \varphi_{20}(1)}{\partial\theta} = \frac{\partial}{\partial\alpha_3}R_1(\sigma_{13}, \sigma_{23}, \sigma_{33}) = 0$$

$$(15)\qquad \varphi_{20}(1) = \varphi_{02}(1) = \frac{\partial \varphi_{02}(1)}{\partial\theta} = \frac{\partial}{\partial\alpha_3}R_1(\sigma_{13}, \sigma_{23}, \sigma_{33}) = 0$$

$$(16)\qquad \frac{\partial^k \varphi_j(1)}{\partial\theta^k} = 0 \quad (k = 0, 1, \cdots, 6;\ j = j_1, j_2, j_3)\,,$$

where $\{j_1, j_2, j_3\}$ *is any three element subset of* $\{3, 4, 5, 6\}$.

(17) $$\frac{\partial^k\varphi_7(1)}{\partial\theta^k} = \frac{\partial^j\varphi_8(1)}{\partial\theta^j} = 0 \quad (k, j = 1, \cdots, 22)$$

(18) $$\psi_{20}(1) = \psi_{02}(1) = \frac{\partial\psi_{20}(1)}{\partial\theta} = 0$$

(19) $$\psi_{20}(1) = \psi_{02}(1) = \frac{\partial\psi_{02}(1)}{\partial\theta} = 0$$

(20) $$\psi_{20}(1) = \frac{\partial\psi_{20}(1)}{\partial\theta} = \psi_{11}(1) = 0$$

(21) $$\psi_{02}(1) = \frac{\partial\psi_{02}(1)}{\partial\theta} = \psi_{11}(1) = 0$$

(22) $$\psi_{02}(1) = \psi_{11}(1) = \psi_{20}(1) = 0$$

(23) $$\psi_1(1) = \frac{\partial\psi_1(1)}{\partial\theta} = \frac{\partial^2\psi_1(1)}{\partial\theta^2} = \frac{\partial^3\psi_1(1)}{\partial\theta^3} = 0$$

(24) $$\psi_2(1) = \frac{\partial\psi_2(1)}{\partial\theta} = \cdots = \frac{\partial^6\psi_2(1)}{\partial\theta^6} = 0$$

(25) $$R_1(3, 3, 1) \neq 0\ , \qquad \frac{\partial^6}{\partial\alpha_1^4\partial\alpha_2^2}\Big(\frac{1}{\sigma_{33}}R_1(\sigma_{13}, \sigma_{23}, \sigma_{33})\Big) \neq 0$$

(26) $$c_{05}c_{41} \neq c_{23}^2\ ; \quad c_{50}c_{23} \neq c_{32}c_{41}\ ; \quad c_{50}c_{14} \neq c_{32}^2\ ; \quad c_{14}c_{41} \neq c_{23}c_{32}\ ;$$
$$c_{05}c_{50} \neq c_{14}c_{41}\ ; \quad c_{05}c_{50} \neq c_{23}c_{32}\ ; \quad c_{32}c_{05} \neq c_{23}c_{14}\ .$$

Then

$$\sum_{x, y \in D} e(f(x, y)) \ll N^{1+\varepsilon}\sqrt[8]{\frac{N^{61/38}}{F} + \frac{F}{N^{85/38}}}\ .$$

Proof. I. Let $X \leqq N^{41/76}$. Applying Lemma 5 three times, we obtain:

$$S \ll \frac{1}{\sqrt{Q}} + \sqrt[8]{S_1}\ , \quad \text{where} \quad S_1 = \frac{1}{NQ^7}\sum_h \sum_{(x,y)\in D_1} e(f_1(x, y, h))$$
$$h = (h_1, \cdots, h_6) \neq 0\ , \quad |h_i| < q_i\ ;$$

$$\begin{aligned} D_1 &= D_1(h) \\ &= \{(x, y)\,|\,(x + h_1t_1 + h_3t_2 + h_5t_3,\ y + h_2t_1 + h_4t_2 + h_6t_3) \in D,\ 0 \leqq t_i \leqq 1\} \end{aligned}$$

$$f_1(x, y, h) = \int_0^1\int_0^1\int_0^1 \frac{\partial^3}{\partial t_1\,\partial t_2\,\partial t_3} f(x + h_1t_1 + h_3t_2 + h_5t_3,$$
$$y + h_2t_1 + h_4t_2 + h_6t_3)dt_1dt_2dt_3\ ;$$

$$q_1q_2 = Q = \min\{F^{-1/4}N^{61/152};\ F^{1/4}N^{-85/152}\}\ ; \qquad q_3q_4 = Q^2\ ; \qquad q_5q_6 = Q^4\ ;$$
$$q_1/q_2 = q_3/q_4 = q_5/q_6 = X/Y\ ; \quad XY = N\ ; \quad \delta \ll \sqrt{Q^4/N}\ .$$

Denoting $\theta = y/x$, we obtain:

$$\frac{\partial^{k+l}f_1(x, y, h)}{\partial x^k\partial y^l} = \frac{f(x, y)h_1h_3h_5}{x^ky^{l+3}}\varphi_{k,l}(\theta) + O\Big(\frac{F}{X^kY^l}\sqrt{\frac{Q^{11}}{N^4}}\Big)\ ,$$

$$\Delta_2(f_1(x, y)) = \frac{f^2(x, y)(h_1h_3h_5)^2}{x^2y^8}\varphi_1(\theta) + O\Big(\frac{F^2Q^7}{N^5}\sqrt{\frac{Q^4}{N}}\Big)$$

$$\Big(\frac{\partial^2 f_1(x, y)}{\partial x^2}\Big)^3 \times \frac{\partial}{\partial y}\Big(\frac{\Delta_2(f_1(x, y))}{\partial^2 f_1(x, y)/\partial y^2}\Big)$$
$$- \Big(\frac{\partial^2 f_1(x, y)}{\partial x^2}\Big)^2 \times \frac{\partial^2 f_1(x, y)}{\partial x \partial y} \times \frac{\partial}{\partial x}\Big(\frac{\Delta_2(f_1(x, y))}{(\partial^2 f_1(x, y))/\partial y^2}\Big)$$
$$= \frac{(f(x, y)h_1h_3h_5)^4}{x^8y^{13}}\varphi_2(\theta) + O\Big(\frac{F^4Q^{14}}{X^7N^7}\sqrt{\frac{Q^4}{N}}\Big).$$

Let $\varphi_{ij}(\theta_{ijk}) = 0$ $(k = 1, 2, 3)$, $\varphi_1(\theta_{1k}) = 0$ $(k = 1, \cdots, 6)$, $\varphi_2(\theta_{2k}) = 0$ $(k = 1, 2, \cdots, 12)$, $\varphi_i(\theta_{ik}) = 0$ $(i = 3, 4, 5,\ k = 1, 2, \cdots, 30)$. For all i, j, k such that $1/2 \leqq \theta_{ijk} \leqq 2$ and $1/2 \leqq \theta_{ik} \leqq 2$ we divide the domain D_1 into $O(N^\varepsilon)$ subdomains of the type $H_i \leqq h_i \leqq 2H_i$, $(y/x)\alpha_{ijk} \leqq |\theta - \theta_{ijk}| \leqq 2(y/x)\alpha_{ijk}$ and $(y/x)\alpha_{ijk} \leqq |\theta - \theta_{jk}| \leqq 2(y/x)\alpha_{ik}$, where $1/Q^4 \leqq \alpha_{i,j,k} \leqq \alpha_{i,j,k+1} \leqq \varepsilon_0$ and $1/Q^4 \leqq \alpha_{i,k} \leqq \alpha_{i,k+1} \leqq \varepsilon_0$ and a subdomain, which contains all the points of D_1 such that $|\theta - \theta_{i_0,k_0}| \leqq \delta_1/Q^4 \equiv q_1q_3q_5/h_1h_3h_5Q^4$. The number of integer points in the last subdomain D_2 for each h is $O(N\delta_1/Q_4)$ and $(1/NQ^7)\sum_h \sum_{(x,y)\in D_2} e(f_1(x, y, h)) \ll \log^3 N/Q^4$. We get the same estimate if $\delta_1 \gg Q^4$. We take one of the other subdomains, D_3 such that $\sum_{(x,y)\in\mathscr{D}_3} e(f(x, y))$ is the largest, and consider all possible cases:

(1) $\prod_{i=1}^3 \alpha_{20i} \ll \delta_1/Q^4$ and $\prod_{i=1}^3 \alpha_{02i} \ll \delta_1/Q^4$. If the domain is non-empty, then $\alpha_{202} \geqq \varepsilon_0$ or $\alpha_{022} \geqq \varepsilon_0$, otherwise $\varphi_{02}(\theta_0) = (\partial/\partial\theta)(\varphi_{20}(\theta_0 + \varepsilon_1(y/x))) = \varphi_{02}(\theta_0 + \varepsilon_2(y/x)) = (\partial/\partial\theta)(\varphi_{02}(\theta_0 + \varepsilon_0(y/x))) = 0$, which contradicts (1). So, $\alpha_{201} \ll \delta_1/Q^4$ or $\alpha_{021} \ll \delta_1/Q^4$ and

$$S_2 = \frac{1}{NQ^7}\sum_h \sum_{(x,y)\in D_3} e(f_1(x, y, h)) \ll \frac{\log^3 N}{Q^4}.$$

(2) (a) $\prod_{i=1}^3 \alpha_{20i} \ll \delta_1/Q^4$, $\alpha_{11} \leqq \varepsilon_0$ and $\alpha_{021} \leqq \varepsilon_0$

(b) $\prod_{i=1}^3 \alpha_{02i} \ll \delta_1/Q^4$, $\alpha_{11} \leqq \varepsilon_0$, $\alpha_{201} \leqq \varepsilon_0$.

In both cases it follows from (2) or (3) that $\alpha_{202} \geqq \varepsilon_0$ or $\alpha_{022} \geqq \varepsilon_0$ and $S_2 \ll \log^3 N/Q^4$.

(3) (a) $\prod_{i=1}^3 \alpha_{20} \ll \delta_1/Q^4$, $\alpha_{11} \leqq \varepsilon_0$, $\alpha_{021} \geqq \varepsilon_0$.

From (4) it follows that $\alpha_{13} \geqq \varepsilon_0$, $\alpha_{203} \geqq \varepsilon_0$ and $\alpha_{201} \times \alpha_{202} \leqq \delta_1/Q^4$; $\alpha_{201} \leqq \sqrt{\delta_1}/Q^2$. If $\alpha = \max\{\alpha_{202}, \alpha_{11}\}$ then $\theta\sigma_{13} = c_1 + O(\alpha)$, $\theta^2\sigma_{23} = c_2 + O(\alpha)$, $\theta^3\sigma_{33} = c_2 + O(\alpha)$, $Y/X \ll h_2/h_1, h_4/h_3, h_6/h_5 \ll Y/X$ and, from (27), $h_4 = c_4(h_3h_2/h_1) + O(\sqrt{\alpha}\,q_4)$, $h_6 = c_5(h_5h_2/h_1) + O(\sqrt{\alpha}\,q_6)$; $S_2 \ll (1/NQ^7)\sum_h \sum_{x,y} \alpha_{201} \ll \alpha_{201}\cdot(\sqrt{\alpha} + 1/Q^2)(\sqrt{\alpha} + 1/Q) \ll \log^3 N/Q^4$. If $\alpha_{11} \gg \alpha$, then from Lemma 4 it follows:

$$S_2 \ll \frac{1}{NQ^7}\sum_h\Big(\sqrt{\frac{F^2Q^7}{N^5}}\sum_{x,y} 1 + N\log N\sqrt{\frac{X^3}{Fh_1h_3h_5}}$$
$$+ \sqrt{\frac{F^2Q^7}{N^3}}\frac{\log N}{Y}\sqrt{\frac{N^5q_1^2q_3^2q_5^2}{F^2h_1^2h_3^2h_5^2Q^7\alpha_{11}}} + X\log N + \log N\frac{F^2Q^7}{N^3}\Big(\frac{X^3}{Fh_1h_3h_5}\Big)^{3/2}\Big)$$

$$\ll \frac{1}{Q^2}\sqrt{\frac{F^2Q^7}{N^5}} + \log N\sqrt[4]{\frac{N^3}{F^2Q^7}} + \frac{\log N}{Y} + \frac{\log N}{N}\sqrt[4]{\frac{F^2Q^7}{N^3}}$$
$$+ \frac{\log N}{Y\alpha_{11}}\Big(\sqrt{\alpha} + \frac{1}{Q^2}\Big)\Big(\sqrt{\alpha} + \frac{1}{Q}\Big)$$
$$\ll \sqrt{\frac{Q^3F^2}{N^5}} + N\sqrt[4]{\frac{N^3}{F^2Q^7}} + \frac{\log N}{Y} \ll Q^{-4}\log N\,.$$

(b) $\prod_{i=1}^3 \alpha_{02i} \ll \delta_1/Q^4$, $\alpha_{11} \leqq \varepsilon_0$, $\alpha_{201} \leqq \varepsilon_0$. Similarly to (a), it follows from (5) and (8) that

$$S_2 \ll Q^{-4}\log N\,.$$

(4) $\prod_{i=1}^3 \alpha_{20i} \ll \delta_1 Q^{-4}$, $\alpha_{021} \geqq \varepsilon_0$, $\alpha_{11} \geqq \varepsilon_0$ (or $\prod_{i=1}^3 \alpha_{02i} \ll \delta_1/Q^4$, $\alpha_{201} \geqq \varepsilon_0$, $\alpha_{11} \geqq \varepsilon_0$). Here $\alpha_{201} \ll Q^{-4/3}$ (or $\alpha_{021} \ll Q^{-4/3}$), and we can apply Lemma 4:

$$S_2 \ll \frac{1}{NQ^7}\sum_{h,x,y}\sqrt{\frac{F^2Q^7}{N^5}} + \log N\sqrt[4]{\frac{N^3}{F^2Q^7}} + \frac{\log^4 N}{NY}\sqrt{\frac{F^2Q^7}{N^3}}\cdot\sqrt{\frac{N^5}{F^3Q^7}}$$
$$+ \frac{\log N}{NQ^7}\sqrt[4]{\frac{F^2Q^7}{N^3}}\sum_h\Big(\frac{q_1q_3q_5}{h_1h_3h_5}\Big)^{3/2}$$
$$\ll \sqrt{\frac{F^2Q^7}{N^5}}\cdot Q^{-4/3} + \sqrt[4]{\frac{N^3}{F^2Q^7}}\log N + \frac{\log^4 N}{Y} + \frac{\log N}{N}\sqrt{\frac{FQ^7}{Y^3}}$$
$$\ll Q^{-4}\log^4 N\,.$$

(5) (a) $\prod_{i=1}^3 \alpha_{20i} \ll \delta_1/Q^4$, $\alpha_{11} \geqq \varepsilon_0$, $\alpha_{021} \leqq \varepsilon_0$, $\prod_{i=1}^3 \alpha_{02i} \gg \delta_1 Q^{-4}$. From (10) it follows that $\alpha_{203} \geqq \varepsilon_0$. If $\alpha_{202} \geqq \varepsilon_0$, then $S_2 \ll Q^{-4}\log^3 N$. If $\alpha_{202} \leqq \varepsilon_0$, then $\alpha_{201} \ll Q^{-2}\sqrt{\delta_1}$ and, from (1), $\alpha_{022} \geqq \varepsilon_0$; If $\alpha = \max\{\alpha_{202}, \alpha_{021}\}$, then, like in (3), $h_6 = c_1(h_5h_2/h_1) + O(\sqrt{\alpha}\,q_6)$, $h_4 = c_2h_3h_2/h_1 + O(\sqrt{\alpha}\,q_4)$ and, if $\alpha_{202} \leqq \alpha_{201}$, then from Lemma 4 it follows:

$$S_2 \ll \gamma_1 + \frac{\log N}{Q^7}\sqrt[4]{\frac{N^3}{F^2Q^7\alpha_{201}^2}}\sum_h 1$$
$$\ll \gamma_1 + \sqrt[4]{\frac{N^3}{F^2Q^7\alpha_{201}^2}} \times \Big(\sqrt{\alpha_{201}} + \frac{1}{Q^2}\Big) \times \Big(\sqrt{\alpha_{201}} + \frac{1}{Q}\Big)$$
$$\ll Q^{-4}\log N$$

if $\alpha_{202} > \alpha_{021}$, then

$$S_2 \ll \frac{1}{Q^7}\sum_h 1 \ll \alpha_{201}(\sqrt{\alpha_{202}} + Q^{-2})(\sqrt{\alpha_{202}} + Q^{-1}) \ll Q^{-4}\log^3 N\,.$$

(b) $\prod_{i=1}^3 \alpha_{02i} \ll \delta_1 Q^{-4}$, $\alpha_{11} \geqq \varepsilon_0$, $\alpha_{201} \leqq \varepsilon_0$, $\prod_{i=1}^3 \alpha_{20i} \gg Q^{-4}\delta_1$. Similarly to (a), from (11), (1) and (26) it follows that $S_2 \ll Q^{-4}\log N$.

(6) $\prod_{i=1}^6 \alpha_{1i} \ll \delta_1 Q^{-4}$, $\alpha_{201} \leqq \varepsilon_0$, $\alpha_{021} \leqq \varepsilon_0$. From (2), (3) and (6) it follows that $\alpha_{202} \geqq \varepsilon_0$, $\alpha_{022} \geqq \varepsilon_0$ and $\alpha_{13} \geqq \varepsilon_0$; $\alpha \ll \sqrt{\delta_1}/Q^2$. If to denote $\alpha = \max\{\alpha_{11}, \alpha_{201}, \alpha_{021}\}$, then, like in (3), $1/Q^7\sum_{h1} \ll \alpha_{11} + 1/Q^2$ and, if $\max\{\alpha_{201}, \alpha_{021}\} \ll \alpha$, or if $\alpha_{12} \gg \alpha$, or if $\alpha \ll Q^{-2}$, then $S_2 \ll Q^{-4}\log^3 N$;

if $\alpha_{201} \gg \alpha$ or $\alpha_{021} \gg \alpha$, then from Lemma 4 with the use of (9) we obtain:

$$S_2 \ll \sqrt{\frac{F^2Q^7}{N^5}} \cdot \frac{1}{Q^2} + \sum_h \sqrt[4]{\frac{N^3 \log N}{F^2Q^7\alpha^2}} \log N + \sqrt{\frac{F^2Q^7}{N^5}}$$
$$\times \min\left\{\frac{1}{Y}\sqrt{\frac{N^5}{F^2Q^7\prod_{i=1}^6 \alpha_{1i}}}; \sqrt[4]{\frac{N^3}{F^2Q^7}} \sqrt[3]{X^3\sqrt{\frac{N^3}{F^2Q^7}}}\right\}$$
$$\ll Q^{-4}\log^2 N + \frac{1}{Y}\sqrt[12]{\frac{F^2Q^7}{N^3}} \ll Q^{-4}\log^2 N .$$

(7) (a) $\prod_{i=1}^6 \alpha_{1i} \ll \delta_1/Q^4$, $\alpha_{201} \leqq \varepsilon_0$, $\alpha_{021} \geqq \varepsilon_0$. From (7) it follows that $\alpha_{13} \geqq \varepsilon_0$, $\alpha_{11} \ll \sqrt{\delta_1}/Q^2$ and like in (6) we can show that $S_2 \ll Q^{-4}\log N$.

(b) $\prod_{i=1}^6 \alpha_{1i} \ll \delta_1/Q^4$, $\alpha_{021} \leqq \varepsilon_0$, $\alpha_{201} \geqq \varepsilon_0$. Similarly to (a) with the use of (8) we get the estimate.

(8) $\prod_{i=1}^6 \alpha_{1i} \ll \delta_1/Q^4$, $\alpha_{021} \geqq \varepsilon_0$, $\alpha_{201} \geqq \varepsilon_0$. If $\alpha_{13} \leqq \varepsilon_0$, then from (9) it follows that $\alpha_{21} \geqq \varepsilon_0$, $\alpha_{11} \leqq Q^{-2/3}\delta_1^{1/6}$ and, like in (6), $S_2 \ll \gamma_1 + \sqrt{F^2Q^7/N^5Q^{-2/3}} \ll \gamma_1$. If $\alpha_{13} \geqq \varepsilon_0$, $\alpha_{11} \gg \sqrt{Q^8/N}$, then $\prod_{i=1}^6 \alpha_{1i} \gg Q^8/N$ and, like in (6),

$$S_2 \ll Q^{-4}\log N + \frac{1}{X}\sqrt{\frac{F^2Q^7}{N^5}}\sqrt{\frac{N^5}{F^2Q^7\prod_{i=1}^6 \alpha_{1i}}} \ll Q^{-4}\log N .$$

If $\alpha_{13} \geqq \varepsilon_0$, $\alpha_{11} \ll \sqrt{Q^8/N}$, then we apply Lemma 3:

$$S_2 \ll \frac{1}{NQ^7}\sum_h \sum_y \left(\frac{1}{X}\sqrt[4]{\frac{F^2Q^7}{N^3}}\sum_x 1 + X\sqrt[4]{\frac{N^3}{F^2Q^7}}\right)$$
$$\ll \sqrt{\frac{N^3}{F^2Q^7}} + \frac{1}{X}\sqrt[4]{\frac{F^2Q^7}{N^3}} \times \sqrt{\frac{Q^8}{N}} \ll Q^{-4}\log N .$$

(9) $\delta_1 Q^{-4} \ll \prod_{i=1}^3 \alpha_{20i} \ll \sqrt{N^3Q^9F^{-2}}$, $\delta_1 Q^{-4} \ll \prod_{i=1}^3 \alpha_{02i} \ll \sqrt{N^3Q^9F^{-2}}$, $\prod_{i=1}^6 \alpha_{1i} \gg \delta_1 Q^{-4}$. Like in (1), $\alpha_{202} \geqq \varepsilon_0$ or $\alpha_{022} \geqq \varepsilon_0$. Consider two cases:

(a) $\alpha_{202} \geqq \varepsilon_0$ and $\alpha_{022} \geqq \varepsilon_0$. For each point $(x, y) \in D_3$, $\varphi_{20}(\theta) = 0$ and $\varphi_{02}(\theta + \alpha\theta_0) = 0$, where $\alpha = \alpha_{201} + \alpha_{021}$, $|\theta_0| \leqq \theta$. $0 = R\varphi_{20}(\theta)$, $\varphi_{02}(\theta + \alpha\theta_0) = \alpha_1\alpha_2\alpha_3[\alpha_3^4(a\alpha_1^2 + b\alpha_1\alpha_2 + a\alpha_2^2) + \cdots] = \alpha_1\alpha_2(a\alpha_1^2 + b\alpha_1\alpha_2 + a\alpha_2^2)\alpha_3 \prod_{i=1}^4 (\alpha_3 - \alpha_3^i(\alpha_1, \alpha_2))$ where $a = a_1 + O(\alpha)$, $b = b_1 + O(\alpha)$, $b_1 = b_1(c_{ij})$, $a_1 = a_1(c_{ij})$, $a_1 \neq 0$ (see (26)). If $\beta \ll a\alpha_2^2 + b\alpha_1\alpha_2 + a\alpha_\alpha^2 \ll \beta$, then $\alpha_2 = c_1\alpha_1 + O(\sqrt{\beta})$ and from (13) it follows that $|\alpha_3 - \alpha_3^0(\alpha_1, \alpha_2)| \ll \sqrt{\alpha/\beta}$. From $\varphi_{20}(\theta) = \varphi_{02}(\theta + \alpha\theta_0) = 0$ it follows also that $\alpha_2 = c_2\theta + O(\sqrt{\beta}\theta)$, $\alpha_3 = c_3\theta + O(\sqrt{\beta}\theta)$ and $\alpha_3 = c_4\alpha_2 + O(\sqrt{\beta}\theta)$. So,

$$\frac{1}{Q^7}\sum_h \sqrt{\frac{q_1q_3q_5}{h_1h_3h_5}} \ll \left(\sqrt{\beta} + \frac{1}{Q}\right)\left(\min\{\sqrt{\beta}; \sqrt{\alpha/\beta}\} + \frac{1}{Q^2}\right) \ll \frac{1}{Q^2} + \sqrt{\alpha} .$$

Applying Lemma 4, we obtain:

$$S_2 \ll \frac{1}{NQ^7}\sum_h \Big(\sqrt{\frac{F^2Q^7}{N^5}}\sum_{x,y}1 + \sqrt[4]{\frac{N^3q_1^2q_3^2q_5^2}{F^2Q^7h_1^2h_3^2h_5^2\alpha^2}} + \frac{\log^4 N}{Y} + \frac{\log N}{N}\sqrt{\frac{FQ^7}{Y^3}}\Big)$$
$$\ll \frac{Q^4}{N} + \log N\sqrt[4]{\frac{N^3}{F^2Q^7}} + \frac{\log^4 N}{Y} + \frac{\log N}{N}\sqrt{\frac{FQ^7}{Y^3}} + \frac{1}{Q^4} \ll Q^{-4}\log N\,.$$

(b) $\alpha_{202}\leqq\varepsilon_0$ or $\alpha_{022}\leqq\varepsilon_0$. If $\varepsilon_1=\min\{\alpha_{202};\alpha_{022}\}$, $\varepsilon_2=\max\{\alpha_{201},\alpha_{021},\varepsilon_1\}$, $\varepsilon_3=\max\{\alpha_{201},\alpha_{021}\}$, then, like before,

$$\frac{1}{Q^7}\sum_h\sqrt{\frac{q_1q_3q_5}{h_1h_3h_5}} \ll \Big(\sqrt{\beta}+\frac{1}{Q}\Big)\Big(\min\{\sqrt{\beta};\sqrt{\varepsilon_3/\beta}\}+\frac{1}{Q^2}\Big) \ll \frac{1}{Q^2}+\varepsilon_2^{1/2}\,.$$

From (14) and (15), it follows also that $\alpha_1=c_1\theta+O(\varepsilon_2\theta)$, $\alpha_2=c_2\theta+O(\varepsilon_2\theta)$, $\alpha_3=c_3\theta+O(\varepsilon_2\theta)$ and

$$\frac{1}{Q}\sum_h\sqrt{\frac{q_1q_3q_5}{h_1h_3h_5}} \ll \Big(\varepsilon_2+\frac{1}{Q}\Big)\Big(\varepsilon_2+\frac{1}{Q^2}\Big) \ll \frac{1}{Q^2}+\varepsilon_2^2\,.$$

Applying Lemma 4, like in (a) we have:

$$S_2 \ll Q^{-4}\log N + \frac{1}{NQ^7}\sum_h\sqrt[4]{\frac{N^3q_1^2q_3^2q_5^2}{F^2Q^7h_1^2h_3^2h_5^2}}\cdot\frac{1}{\sqrt{\varepsilon_2\varepsilon_3}}$$
$$\ll Q^{-4}\log N + \sqrt[4]{\frac{N^3}{F^2Q^7}}\cdot\frac{1}{\sqrt{\varepsilon_2\varepsilon_3}}\cdot\min\{\varepsilon_2^2;\varepsilon_3^{1/2}\}$$
$$\ll Q^{-4}\log N + \sqrt[4]{N^3/(F^2Q^7)}\cdot Q^{2/5} \ll Q^{-4}\log N\,.$$

(10) $\prod_{i=1}^3\alpha_{20i}\gg\sqrt{N^3Q^9/F^2}$ (or $\prod_{i=1}^3\alpha_{02i}\gg\sqrt{N^3Q^9/F^2}$), $\prod_{j=1}^6\alpha_{1j}\gg 1/Q^4$ and for some i,j,k $\alpha_{ijk}\ll\sqrt{N^5/F^2Q^{15}}$ or $\alpha_{ij}\ll\sqrt{N^5/F^2Q^{15}}$, or $\prod_{j=1}^6\alpha_{1j}\ll N^5/F^2Q^{15}$, or $|h_1h_2\cdots h_6|\ll Q^7\cdot\sqrt{N^5/F^2Q^{15}}$.

Applying Lemma 4, we obtain:

$$S_2 \ll Q^{-4}\log N + (NQ^7)^{-1}\sum_h F\sqrt{Q^7/N^3}\cdot\Big(\prod_{j=1}^6\alpha_{1j}\Big)^{-1/2}\sum_{x,y}1 \ll Q^{-4}\log N\,,$$

because $\sum_{x,y}1\ll N$ and $\sum_{x,y}1\ll\sqrt{N^5/F^2Q^{15}}$ if, say, $\alpha_{ijk}\ll\sqrt{N^5/F^2Q^{15}}$.

(11) $\prod_{i=1}^3\alpha_{20i}\gg\sqrt{N^3Q^9/F^2}$ (or $\prod_{i=1}^3\alpha_{02i}\gg\sqrt{N^3Q^9/F^2}$), $\prod_{j=1}^6\alpha_{1j}\gg N^5/F^2Q^{15}=\delta_0$, $\alpha_{ij}\gg\sqrt{\delta_0}$, $\alpha_{i,j,k}\gg\sqrt{\delta_0}$, $|h_1\cdots h_6|\gg Q^7\sqrt{\delta_0}$.

Applying Lemma 4, we obtain:

$$S_2 \ll Q^{-4}\log N + (NQ^7)^{-1}\sum_h q_1q_3q_5\Big(h_1h_3h_5\cdot F\cdot\sqrt{Q^7N^{-3}}\cdot\sqrt{\prod_{i=1}^6\alpha_{1j}}\Big)^{-1}$$
$$\times\Big|\sum_{y_1,y_2\in\mathscr{D}_1}e(g(y_1,y_2))\Big|\,,$$

where $g(y_1,y_2)=f(\varphi_1,\varphi_2)-y_1\varphi_1-y_2\varphi_2$, $\mathscr{D}_1$, φ_1, φ_2 are defined as in Lemma 4,

$$\sum_{(y_1,y_2)\in\mathscr{D}_1}1 \ll (Fh_1h_3h_5)^2(q_1q_5)^{-2}Q^7N^{-3}\prod_{i=1}^6\alpha_{1i}\sum_{(x,y)\in\mathscr{D}}1\,.$$

One can verify, that for $k+l\geqq 2$

$$\frac{\partial^{k+l} g(y_1, y_2)}{\partial y_1^k \partial_2^l} = \Phi_{k,l}(f(\varphi_1, \varphi_2)) \cdot (\Delta(f(\varphi_1, \varphi_2)))^{3-2k-2l} \tag{27}$$

where $\Phi_{k,l}$ and Δ are defined as in the notation for the theorem. From (18) we get that either $\alpha_{7,23} \geqq \varepsilon_0$ or $\alpha_{8,23} \geqq \varepsilon_0$. We consider three cases.

(a) $|\alpha_{7,23}| \leqq \varepsilon_0$. Then $\alpha_{8,23} \geqq \varepsilon_0$ and $\prod_{j=1}^{60} \alpha_{8,j} \gg \delta_0^{11}$. Also, from (17) we obtain that either $\alpha_{6,6} \geqq \varepsilon_0$ (and $\prod_{i=1}^{30} \alpha_{6,j} \gg \delta_0^3$) or $\alpha_{4,6} \geqq \varepsilon_0$ (and $\prod_{i=1}^{30} \alpha_{4,i} \gg \delta_0^3$). Applying three times Lemma 5 with $n = 1$ to the sum over y_2, we obtain:

$$\begin{aligned} |S_3|^8 &= |\sum_{(y_1,y_2)\in\mathscr{D}} e(g(y_1, y_2))|^8 \\ &\ll Q_1^{-4} \cdot (\sum_{(y_1,y_2)} 1)^8 + (\sum_{(y_1,y_2)} 1)^7 \cdot Q_1^{-7} \sum_{l_1=1}^{Q_1} \sum_{l_2=1}^{Q_1^2} \sum_{l_3=1}^{Q_1^4} |\sum_{y_1,y_2} e(g_1(y_1, y_2))| \end{aligned} \tag{28}$$

where $Q_1 = [\delta_0^{-1}]$, $g_1(y_1, y_2) = \int_0^1\int_0^1\int_0^1 g(y_1, y_2 + l_1 t_1 + l_2 t_2 + l_3 t_3) dt_1 dt_2 dt_3$, and the last sum is over y_1, y_2 such that $(y_1, y_2) \in \mathscr{D}_1$, $(y_1, y_2 + h_1 + h_2 + h_3) \in \mathscr{D}_1$. Using (27), we get that either

$$\left|\frac{\partial^2 g_1(y_1, y_2)}{\partial y_1^2}\right| \gg \delta_0^3 l_1 l_2 l_3 \cdot \left(\prod_{i=1}^{6} \alpha_{1j}\right)^{-7} \cdot (q_1 q_3 q_5/h_1 h_3 h_5)^4 \cdot N^8 y F^{-4} Q^{-14}$$

or

$$\left|\frac{\partial^2 g_1(y_1, y_2)}{\partial y_2^2}\right| \gg \delta_0^3 l_1 l_2 l_3 \cdot \left(\prod_{i=1}^{6} \alpha_{1j}\right)^{-7} \cdot (q_1 q_3 q_5/h_1 h_3 h_5)^4 \cdot y^5 N^6 F^{-4} Q^{-14} .$$

Also, from $\prod_{j=1}^{60} \alpha_{8,j} \gg \delta_0^{11}$ we obtain

$$|\Delta(g_1(y_1, y_2))| \gg \delta_0^{11} l_1^2 l_2^2 l_3^2 \cdot \left(\prod_{i=1}^{6} \alpha_{1j}\right)^{-14} \cdot (q_1 q_3 q_5/h_1 h_3 h_5)^8 \cdot N^{14} y^6 F^{-8} Q^{-28} ;$$

trivially,

$$|\Delta(g_1(y_1, y_2))| \ll l_1^2 l_2^2 l_3^2 \cdot \left(\prod_{i=1}^{6} \alpha_{1i}\right)^{-14} \cdot N^{14} y^6 F^{-8} Q^{-28} .$$

Applying Lemma 4, we get

$$\begin{aligned} |S_3|^8 &\ll (\sum_{y_1,y_2} 1)^8 \cdot \left[Q_1^{-4} + Q_1^{-7} \sum_{l_1,l_2,l_3} l_1 l_2 l_3 \cdot \left(\prod_{i=1}^{6} \alpha_{1i}\right)^{-7} N^7 y^3 F^{-4} Q^{-14}\right] \\ &\quad + (\sum_{y_1,y_2} 1)^7 Q_1^{-7} \sum_{l_1,l_2,l_3} \Big\{F \sqrt{Q^7/N^3} \cdot X^{-1} \\ &\quad \times \left[\delta_0^3 l_1 l_2 l_3 \cdot \left(\prod_{i=1}^{6} \alpha_{1j}\right)^{-7} \cdot (q_1 q_3 q_5/h_1 h_3 h_5)^4 \cdot y N^8 F^{-4} Q^{-14}\right]^{-1/2} \log^2 N \\ &\quad + \left[l_1 l_2 l_3 \cdot \left(\prod_{i=1}^{6} \alpha_{1i}\right)^{-5} \cdot (q_1 q_3 q_5/h_1 h_3 h_5)^3 \cdot F^{-3} N y^2 \cdot (Q^7/N^3)^{-3/2} + 1\right] \\ &\quad \times \left[\delta_0^{11} l_1^2 l_2^2 l_3^2 \cdot \left(\prod_{i=1}^{6} \alpha_{1j}\right)^{-14} \cdot (q_1 q_3 q_5/h_1 h_3 h_5)^8 \cdot N^{14} y^6 F^{-8} Q^{-28}\right]^{-1/2} \log^2 N\Big\} \\ &\ll Q_1^{-4} (\sum_{y_1,y_2} 1)^8 \log^2 N \end{aligned}$$

and

$$S_2 \ll Q^{-4}\log^2 N .$$

(b) $\alpha_{7,23} \leqq \varepsilon_0$. Then $\alpha_{7,23} \geqq \varepsilon_0$ and $\prod_{j=1}^{60}\alpha_{7,j} \gg \delta_0^{11}$. Also, from (17) we get that either $\alpha_{3,6} \geqq \varepsilon_0$ (and $\prod_{i=1}^{30}\alpha_{3,i} \gg \delta_0^3$) or $\alpha_{5,6} \geqq \varepsilon_0$ (and $\prod_{i=1}^{30}\alpha_{5,i} \gg \delta_0^3$). Applying three times Lemma 5 with $n=1$ (twice to the sum over y_2 and once to the sum over y_1) we obtain the inequality similar to (28) with $g_1(y_1, y_2) = \int_0^1\int_0^1\int_0^1 g(y_1 + l_1t_1, y_2 + l_2t_2 + l_3t_3)dt_1dt_2dt_3$. Arguing similarly to (a) we can show that $S_2 \ll Q^{-4}\log^2 N$.

(c) $\alpha_{7,23} \geqq \varepsilon_0$, $\alpha_{8,23} \geqq \varepsilon_0$. Using (17), we get that for some $j \in [3, 6]$ $\alpha_{j,6} \geqq \varepsilon_0$, and similar reasonings as in (a) (if $j=4$ or $j=6$) or in (b) (if $j=3$ or $j=5$) will show that

$$S_2 \ll Q^{-4}\log^2 N .$$

II. If $N^{15/33} \leqq Y \leqq N^{35/76}$, then we apply Lemma 5 one time with $n=1$ and two times with $n=2$

$$S \ll \frac{1}{\sqrt{Q}} + \sqrt[8]{\frac{1}{NQ^7}\Big(\sum_{h_0=1}^{Q-1}\sum_{|h_i|=1}^{q_1-1}\sum_{x,y\in D_1}^{q_4-1} e(f_1(x, y, h)) + \textstyle\sum'\Big)} .$$

Where $Q^2 = q_1q_2 = \sqrt{q_3q_4}$, $q_1/X = q_2/Y$, $q_3/X = q_4/Y$; $Q \ll N^{-1/12}$; $\delta \ll \sqrt{Q^4/N}$, Q is as in I,

$$f_1(x, y, h) = \int_0^1\int_0^1\int_0^1 \frac{\partial^3}{\partial t_1\partial t_2\partial t_3} f(x + h_0t_1 + h_1t_2 + h_3t_3,\ y + h_2t_2 + h_4t_3)dt_1dt_2dt_3$$

$$D_1 \equiv D_1(h) = \{x, y; (x + h_0t_1 + h_1t_2 + h_3t_3,\ y + h_2t_2 + h_4t_3) \in D,\ 0 \leqq t_i \leqq 1\} .$$

$\sum'$ is the sum over h_i such that $\sum_{i=0}^4 |h_i| > 0$, $h_j = 0$. It will be seen later that $\sum'$ can be estimated like the sum

$$S_1 = (NQ^7)^{-1}\sum_h \sum_{(x,y)\in\mathscr{D}_1} e(f_1(x, y, h))$$

and the estimate is smaller.

$$\begin{aligned}\frac{\partial^{k+l} f_1(x, y, h)}{\partial x^k\partial y^l} &= h_0[h_1h_3 f_{x^{k+3}y^l}(x, y) + (h_1h_4 + h_2h_3)f_{x^{k+2}y^{l+1}}(x, y) \\ &\quad + h_2h_4 f_{x^{k+1}y^{l+2}}(x, y)] \\ &= \frac{f(x, y)h_1h_3h_0}{x^{k+1}y^{l+2}}\psi_{k,l}(\theta) + O\Big(\frac{FQ^4}{X^{k+1}Y^lN}\sqrt{\frac{Q^4}{N}}\Big) .\end{aligned}$$

Let $\psi_{k,l}(\theta_{k,l,i}) = 0$, $\psi_1(\theta_{1i}) = 0$, $\varphi_2^1(\theta_{2i}) = 0$. For $1/3 \leqq \theta_{kli} \leqq 3$, $1/3 \leqq \theta_{ij} \leqq 3$ and each h we divide the domain D_1 on $O(N^\varepsilon)$ subdomains of the type $\alpha_{kli} \leqq \theta - \theta_{kli} \leqq 2\alpha_{kli}$ ($k, l = 0, 1, 2$; $i = 1, 2$), $\alpha_{ki} \leqq \theta - \theta_{ki} \leqq 2\alpha_{ki}$ ($k = 1, 2$; $i = 1, 2, \cdots, 8$) and subdomain where $|\theta - \theta_{ki}| \ll \delta_2/Q^4 \equiv (q_1q_3/h_1h_3)Q^{-4}$ or $|\theta - \theta_{kli}| \ll \delta_2/Q^4$. The sum over h and (x, y) from the last subdomain is $\ll Q^{-4}\ln^3 N$. If $\delta_2 \gg Q^4$, then we get the same estimate.

We take one of the remaining subdomains, D_2, such that $|\sum_{(x,y)\in\mathscr{D}_2} e(f_1(x, y))|$ is the largest, and like in I, we consider all possible cases:

(1) $\alpha_{201}\alpha_{202} \ll \delta_2 Q^{-4}$ and $\alpha_{021} \leqq \varepsilon_0$ (or $\alpha_{201} \leqq \varepsilon_0$ and $\alpha_{021}\alpha_{022} \ll \delta_2 Q^{-4}$). From (18)-(19) it follows that $\alpha_{202} \geqq \varepsilon_0$ (or $\alpha_{022} \geqq \varepsilon_0$) and $S_1 = (1/NQ^7)\sum_h \sum_{x,y\in D_2} e(f_1(x, y)) \ll \log^3 N/Q^4$.

(2) $\alpha_{201}\alpha_{202} \ll \delta^2 Q^{-4}$ (or $\alpha_{021}\alpha_{022} \ll \delta_2 Q^{-4}$) and $\alpha_{11} \leqq \varepsilon_0$. With the use of (20)-(21) we get $S_1 \ll Q^{-4}\log^3 N$.

(3) $\alpha_{201}\alpha_{202} \ll \delta_2 Q^{-4}$ (or $\alpha_{021}\alpha_{022} \ll \delta_2 Q^{-4}$), $\alpha_{11} \geqq \varepsilon_0$ and $\alpha_{021} \geqq \varepsilon_0$ (or $\alpha_{201} \geqq \varepsilon_0$). From Lemma 4 we obtain:

$$S_1 \ll \left(\frac{FQ^2}{N^2X} + \sqrt{\frac{NX}{FQ^4}} + \frac{1}{Y} + \frac{1}{Q^4} + \sqrt{\frac{FQ^4}{N^4X}}\right)\log^7 N \ll Q^{-4}\log^7 N .$$

(4) $\prod_{i=1}^4 \alpha_{1i} \ll \delta_2 Q^{-4}$, $\alpha_{201} \leqq \varepsilon_0$ (or $\alpha_{021} \leqq \varepsilon_0$). From (20)-(21) it follows that $\alpha_{12} \geqq \varepsilon_0$ and $S_1 \ll Q^{-4}\log^3 N$.

(5) $\delta_2/Q^4 \ll \alpha_{201}\alpha_{202} \ll NXQ^4/F$ and $\delta_2 Q^{-4} \ll \alpha_{021}\alpha_{022} \ll NXQ^4F^{-1}$. If $\alpha = \max\{\alpha_{201}, \alpha_{021}\}$, then $\alpha_{11} \geqq \varepsilon_0$, $\sigma_{12} = c_1\theta + O(\alpha\theta)$, $\sigma_{22} = c_2\theta + O(\alpha\theta)$ (because of (29) and (15)), $\alpha_2 = c_3\alpha_1 + O(\sqrt{\alpha}\,\theta)$,

$$\frac{1}{Q^7}\sum_h (h_1h_3)^{-1/2} \ll (\sqrt{\alpha} + Q^{-2})(q_1q_3)^{-1/2} \ll \sqrt{\alpha}\,(q_1q_3)^{-1/2}$$

and from Lemma 4 it follows that

$$S_1 \ll \left(\alpha^{3/2}\frac{FQ^4}{N^2X} + \sqrt{\frac{NX}{FQ^4}} + \frac{1}{Y} + \frac{1}{Q^4}\right)\log^7 N \ll Q^{-4}\log^7 N .$$

(6) $\delta_2 Q^{-4} \ll \prod_{i=1}^4 \alpha_{1i} \ll Q^{-1/2}$, $\alpha_{201} \geqq \varepsilon_0$, $\alpha_{021} \geqq \varepsilon_0$. From (23) it follows that $\alpha_{14} \geqq \varepsilon_0$, $\alpha_{11} \ll Q^{-1/6}$, and, applying Lemma 4, we obtain:

$$S_1 \ll \left(\frac{FQ^{23/6}}{N^2X} + \sqrt{\frac{NX}{FQ^4}} + \frac{Q^2}{X} + Q^{-4}\right)\log^7 N \ll Q^{-4}\log^7 N .$$

(7) $\prod_{i=1}^4 \alpha_{1i} \gg Q^{-1/2}$, $\alpha_{201}\alpha_{202} \gg NXQ^4F^{-1}$, $\alpha_{021}\alpha_{022} \gg NNQ^4F^{-1}$. Using Lemma 4, we obtain

$$S_1 \ll Q^{-4}\log^2 N + (NQ^7)^{-1}\sum_h\left[\left(\prod_{i=1}^4 \alpha_{1i}\right)\cdot(h_0h_1h_3)^2\cdot(q_1q_3)^{-2} \times F^2Q^6X^{-2}N^{-4}\right]^{-1/2}\left|\sum_{y_1,y_2} e(g(y_1, y_2))\right| , \tag{29}$$

where $g(y_1, y_2) = f(\varphi_1, \varphi_2) - y_1\varphi_1 - y_2\varphi_2$, $\varphi_1 = \varphi_1(y_1, y_2)$, $\varphi_2 = \varphi_2(y_1, y_2)$ are defined as in Lemma 4, $\sum_{y_1,y_2} 1 \ll (\prod_{i=1}^4 \alpha_{1j})\cdot(h_0h_1h_3)^2(q_1q_3)^{-2} \times F^2Q^6X^{-2}N^{-4}\sum_{(x,y)\in\mathscr{D}_2} 1$ and $\sum_{y_1} 1 \ll FQ^3X^{-2}N^{-1}$. If $\prod_{i=1}^{14} \alpha_{2i} \ll \delta_2^5 Q^{-5/6}$, then, using (24), we get $\alpha_{26} \geqq \varepsilon_0$, $\alpha_{21} \ll \delta_2 Q^{-1/6}$, $\sum_{(x,y)\in\mathscr{D}_2} 1 \ll N\delta_2 Q^{-1/6}$ and, estimating $S_2 = |\sum_{y_1,y_2} e(g(y_1, y_2))|$ trivially, we obtain: $S_1 \ll Q^{-4}\log^2 N$.

We obtain a similar result if $\delta_2 \ll Q^{-1/6}$. If $\prod_{i=1}^{14}\alpha_{2i} \gg \delta_2^6 Q^{-5/6}$ and $\delta_2 \gg Q^{-1/6}$, then we apply van der Corput's theorem (see, for example, Lemma 5.13, page 93, [2]): if $M_4^{-1} \leqq f^{(4)}(x) \leqq AM_4^{-1}$, then

$$\Big|\sum_{a\leqq x\leqq b} e(f(x))\Big| \ll (b-a)M_4^{-1/14}A^{1/14} + (b-a)^{3/4}M_4^{1/14}$$

(which can be obtained by applying Lemma 5 twice with the appropriate choice of q, and after that Lemma 2). We obtain:

$$\begin{aligned}\Big|\sum_{y_1,y_2} e(g(y_1,y_2))\Big| &\leqq \sum_{y_1}\Big|\sum_{y_2} e(g(y_1,y_2))\Big| \\ &\ll A^{1/14}M_4^{-1/14}\sum_{y_1,y_2}1 + M_4^{1/14}\sum_{y_1}\Big(\sum_{y_2}1\Big)^{3/4} \\ &\ll A^{1/14}M_4^{-1/14}\sum_{y_1,y_2}1 + M_4^{1/14}\Big(\sum_{y_1}1\Big)^{1/4}\Big(\sum_{y_1,y_2}1\Big)^{3/4},\end{aligned}$$

where

$$AM_4^{-1} = \sup_{y_1,y_2}\left|\frac{\partial^4 g(y_1,y_2)}{\partial y_2^4}\right| \ll \left(\prod_{i=1}^{4}\alpha_{1i}\right)^{-5}\cdot(Qq_1q_3/h_0h_1h_3)^{-5}\cdot y(FQ^4)^{-3}N^6$$

and

$$M_4^{-1} = \inf_{y_1,y_2}\left|\frac{\partial^4 g(y_1,y_2)}{\partial y_2^4}\right| \gg \left(\prod_{i=1}^{4}\alpha_{1i}\right)^{-5}\left(\prod_{i=1}^{14}\alpha_{2i}\right)\cdot(Qq_1q_3/h_0h_1h_3)^{-2}yN^6Q^{-12}\,.$$

Putting the estimates into (29), we obtain $S_1 \ll Q^{-4}\log^2 N$.

III. If $Y \ll N^{15/38}$, we apply Lemma 5 three times with $n=1$:

$$S \ll \frac{1}{\sqrt{Q}} + \sqrt[8]{\frac{1}{NQ^7}\left|\sum_{h_1=1}^{Q-1}\sum_{h_2=1}^{Q^2-1}\sum_{h_3=1}^{Q^4-1}\sum_{x,y} e(f(x,y))\right|}\,,$$

where

$$f_1(x,y) = \int_0^1\int_0^1\int_0^1 \frac{\partial^3}{\partial t_1\partial t_2\partial t_3} f(x+h_1t_1+h_2t_2+h_3t_3,\, y)\,dt_1dt_2dt_3\,.$$

Applying Lemma 4, we obtain:

$$S_1 \ll \frac{1}{Q^4} + \frac{FQ^7}{X^3N} + \sqrt{\frac{X^3}{FQ^7}}\,;$$

we can choose Q such that

$$S_1^8 \ll \left[\left(\frac{F}{N^{107/38}}\right)^{4/11} + N^{-1/3} + \sqrt[8]{\frac{N^5}{F^4}}\right]\ln^7 N\,.$$

This completes the proof of the theorem:

$$S_1 \ll \left(\frac{N^{61/38}}{F} + \frac{F}{N^{85/38}}\right)\ln^7 N\,.$$

3. Consequences of the theorem.

Corollary 1. $\Delta(R) \ll R^{(35/108)+\varepsilon}$.

Proof. Like in [1], from the theorem it follows:

$$\Delta(R) \ll \frac{R}{V} + \frac{R^{1+\varepsilon}}{V^{3/2}}\frac{V^2}{R}\left(\frac{V^{42/19}}{R^{61/38}} + \frac{R^{85/38}}{V^{66/19}}\right)^{1/8} \ll R^{(35/108)+\varepsilon}$$

if $V = R^{73/108}$.

COROLLARY 2. $\zeta(1/2 + it) \ll t^{(35/216)+\varepsilon}$.

Proof. Like in [1];

$$\zeta\left(\frac{1}{2} + it\right) \ll \frac{t^\varepsilon}{\sqrt{X}}\Big|\sum_{X \leqq x \leqq X_1} x^{-it}\Big| + 1 ,$$

where $X_1 \leqq 2X \leqq \sqrt{t}$. If $X \ll t^{137/324}$, then

$$\zeta\left(\frac{1}{2} + it\right) \ll \sqrt{X}\cdot\sqrt[14]{\frac{t}{X^4}} + X^{1/4}\sqrt[14]{\frac{t}{X^4}}t^\varepsilon \ll t^{(35/216)+\varepsilon} ;$$

if $X \gg t^{137/324}$, then with the use of theorem we get that

$$\zeta\left(\frac{1}{2} + it\right) \ll \frac{t^\varepsilon}{\sqrt{X}}\left(\frac{X}{\sqrt{q}} + \sqrt[4]{\frac{X^5}{qt}} + \sqrt{X}\right.$$
$$\left. + \sqrt[4]{\frac{X^5}{tq^3}}\sqrt{\frac{tq^2}{X^2}\sqrt[8]{t^{-61/38}\frac{tq^{42/19}}{X} + t^{85/38}\frac{tq^{-66/19}}{X}}}\right)$$

and if we choose $q = t^{19/108}$,

$$\zeta\left(\frac{1}{2} + it\right) \ll t^{(35/216)+\varepsilon} .$$

REFERENCES

1. G. A. Kolesnik, *On the estimation of some exponential sums*, Acta Arithmetica, **XXV** (1973), 7-30.

2. E. K. Titchmarsh, *The theory of the Riemann Zeta-Function*, Oxford, 1951.

3. ———, *On Epstein's zeta-function*, Proc. London Math. Soc., 2, **36** (1934), 485-500.

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