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ON THE PROXIMINALITY OF STONE-WEIERSTRASS SUBSPACES

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Let S be a compact Hausdorff space, X a Banach space, $C(S, X)$ the Banach space of all continuous X-valued functions on S equipped with the supremum norm. In this paper a necessary and sufficient condition on X for every Stone-Weierstrass subspace of $C(S, X)$ to be proximinal is established. Furthermore, it is shown that every such subspace is proximinal if X is a dual locally uniformly convex space.

Introduction and notations. Let S be a compact Hausdorff space, X a Banach space, $C(S, X)$ the Banach space of all continuous functions on S with values in X, equipped with the supremum norm. The purpose of this paper is to study the proximinality of certain subspaces, the so-called Stone-Weierstrass subspaces (SW-subspaces) of $C(S, X)$. This problem has been studied by many authors: Mazur (unpublished, cf., e.g., [11]) proved that every SW-subspace of $C(S, X)$ is proximinal if X is the real line $\boldsymbol{R}$ (a subspace G of a normed linear space Y is called proximinal if every $y \in Y$ possesses an element of best approximation x_0 in G, i.e., if there is an $x_0 \in G$ such that $\|y - x_0\| \leqq \|y - x\|$ holds for every $x \in G$). Pelczynski [9] and Olech [8] asked for which Banach spaces X every SW-subspace of $C(S, X)$ is proximinal. Olech [8] and Blatter [2] showed that this is true if X is a uniformly convex Banach space and an L_1-predual space, respectively. It has been shown in [6] that there exists a Banach space X and a compact Hausdorff space S such that $C(S, X)$ has a non-proximinal SW-subspace. Thus, the above mentioned question of characterizing those Banach spaces X for which every SW-subspace is proximinal, arises naturally. Here we give such a characterization. Using a modification of a method due to Olech [8], we show further that if X is a locally uniformly convex space such that every compact subset of X has a Chebychev center (a point x_0 is called a Chebychev center of a bounded set F if x_0 is the center of a "smallest" ball containing F) then every SW-subspace of $C(S, X)$ is proximinal. Every dual space, e.g., has the latter property [3].

We use the following notations. $\boldsymbol{R}$ and N will denote the set of all real numbers and the set of all positive integers, respectively. Let X be a Banach space, $x \in X$, $r > 0$. $B(x, r)$ will denote the closed ball in X with center x and radius r. A set-valued function Φ from a topological space S into 2^X is said to be upper Hausdorff semicontinuous (u.H.s.c.) respectively lower Hausdorff semicontinuous (l.H.s.c.) if for every $s_0 \in S$ and every $\varepsilon > 0$ there is a neighborhood U of s_0 such that for every $s \in U$ we have

$$\sup_{x \in \Phi(s)} \operatorname{dist}(x, \Phi(s_0)) \leqq \varepsilon$$

respectively

$$\sup_{x \in \Phi(s_0)} \operatorname{dist}(x, \Phi(s)) \leqq \varepsilon$$

(cf. [10], [12]). The function Φ is Hausdorff continuous (H.c.) if Φ is both u.H.s.c. and l.H.s.c. Φ is l.s.c. respectively u.s.c. if Φ is lower semicontinuous respectively upper semicontinuous in the usual sense [7]. A Banach space X is said to be locally uniformly convex (l.u.c.) if for every $x \in X$ with $\|x\| = 1$ and every sequence $\{y_n\} \subset X$ with $\lim \|y_n\| \leqq 1$, $\lim \|x + y_n\| = 2$ implies $\lim \|x - y_n\| = 0$. For a Banach space X, $\mathscr{C}(X)$ will denote the class of all nonempty compact subsets of X. For a compact Hausdorff space S, $C(S, X)$ will denote the Banach space of all continuous functions f on S with values in X equipped with the norm $\|f\| = \sup_{s \in S} |f(s)|$, where $|\cdot|$ is the norm of X. A subspace V of $C(S, X)$ is said to be an SW-subspace if there is a compact Hausdorff space T and a continuous surjection $\varphi: S \to T$ such that V consists exactly of those elements f of $C(S, X)$ which have the form $f = g \circ \varphi$ for some $g \in C(T, X)$. Let Φ be a function from S into $\mathscr{C}(X)$. A function $f \in C(S, X)$ is said to be a best approximation of Φ in $C(S, X)$ if the number

$$\operatorname{dist}(f, \Phi) = \sup_{s \in S} \sup_{x \in \Phi(s)} \|x - f(s)\|$$

is equal to $\inf \operatorname{dist}(g, \Phi)$, where the infimum is taken over all $g \in C(S, X)$. Let F be a bounded subset of X. The number

$$r(F) = \inf_{x \in X} \sup_{y \in F} \|x - y\|$$

is called the Chebyshev radius of F. A point $x_0 \in X$ is said to be a Chebyshev center of F if $\|x_0 - y\| \leqq r(F)$ for all $y \in F$. The set of all Chebyshev centers of F will be denoted by $c(F)$. For a function $\Phi: S \to \mathscr{C}(X)$ we denote by r_Φ the number $\sup_{s \in S} r(\Phi(s))$. All Banach spaces in this paper are real.

SW-subspaces of $C(S, X)$. We first establish a simple lemma. Since its proof is straightforward, we omit it here.

LEMMA 1. *Let Φ be an* u.H.s.c. *function from a compact Hausdorff space T into $\mathscr{C}(X)$. Then the set $\bigcup_{t \in T} \Phi(t)$ is compact.*

We formulate now the main theorem of this paper.

THEOREM 2. *The following conditions on a Banach space X are equivalent*:

(i) *For every compact Hausdorff space T and for every* u.H.s.c *function $\Phi: T \to \mathscr{C}(X)$, the function*

$$\Psi_\Phi(t) = \bigcap_{x \in \Phi(t)} B(x, r_\Phi)\ , \quad t \in T\ ,$$

has a continuous selection.

(ii) *Every* u.H.s.c. *function Φ from an arbitrary compact Hausdorff space T into $\mathscr{C}(X)$ has in $C(T, X)$ a best approximation.*

(iii) *For any compact Hausdorff space S, every* SW-*subspace of $C(S, X)$ is proximinal.*

Proof. (i) $\Rightarrow$ (ii). If f is a continuous selection of Ψ_Φ, then $\operatorname{dist}(f, \Phi) = r_\Phi$. Further, we obviously have

$$\inf_{g \in C(T, X)} \operatorname{dist}(g, \Phi) \geqq r_\Phi\ . \tag{1}$$

It follows that f is a best approximation of Φ.

(ii) $\Rightarrow$ (i). It suffices to show that

$$\inf_{g \in C(T, X)} \operatorname{dist}(g, \Phi) = r_\Phi\ . \tag{2}$$

Let $r > r_\Phi$ be a fixed number. Let $\Psi_1: T \to 2^X$ be defined by

$$\Psi_1(t) = \{x \in X;\ \text{there is a neighborhood } U \text{ of } t \text{ such that } \Phi(t') \subset B(x, r) \text{ for all } t' \in U\}\ .$$

We show first that $\Psi_1(t) \neq \emptyset$ for every $t \in T$. Since $r(\Phi(t)) \leqq r_\Phi < r$, there is an $x_0 \in X$ for which

$$\Phi(t) \subset B(x_0, (r + r_\Phi)/2)$$

holds. Since Φ is u.H.s.c., there is a neighborhood U of t such that

$$\sup_{y \in \Phi(t')} \operatorname{dist}(y, \Phi(t)) < (r - r_\Phi)/2$$

for every $t' \in U$. It follows that $\Phi(t') \subset B(x_0, r)$ for all $t' \in U$. Hence $x_0 \in \Psi_1(t)$. For every $t \in T$ the set $\Psi_1(t)$ is obviously convex. It follows immediately from the definition of Ψ_1 that it is l.s.c. We put now $\Psi_2(t) = \operatorname{cl} \Psi_1(t)$, $t \in T$. The map Ψ_2 is still l.s.c. and therefore it has a continuous selection [7]. Denote this continuous selection by g. Let us show now that $\operatorname{dist}(g, \Phi) \leqq r$. To see this, let $\varepsilon > 0$ and $t \in T$ be given. There is an $x \in \Psi_1(t)$ with $\| g(t) - x \| < \varepsilon$. Consequently,

$$\Phi(t) \subset B(g(t), r + \varepsilon) .$$

Since ε and t has been arbitrary, we have $\operatorname{dist}(g, \Phi) \leqq r$. Since $r > r_\Phi$ has been arbitrary, it follows

$$\inf_{h \in C(T,X)} \operatorname{dist}(h, \Phi) \leqq r_\Phi .$$

Thus, by (1), we have (2).

(ii) $\Rightarrow$ (iii). This has been essentially proved in [8].

(iii) $\Rightarrow$ (ii). Let Φ be an u.H.s.c. function from T into $\mathscr{C}(X)$. We show that there is a compact Hausdorff space S, a continuous surjection $\varphi: S \to T$ and a function $f \in C(S, X)$ such that if, for some $g \in C(T, X)$, $g \circ \varphi$ is a best approximation of f in the corresponding SW-subspace V, then g is a best approximation of Φ.

By Lemma 1, there is a number $a > 0$ such that $\|x\| < a$ for all for all $x \in \Phi(t)$ and all $t \in T$. Choose an arbitrary $z \in X$ such that $\|z\| > a$ holds. Let R be the subset of X^T defined by

$$R = \{s \in X^T;\ \|s(t)\| < a \text{ for some } t \in T \text{ and } s(t') = z \\ \text{for all } t' \neq t\} .$$

Let $\psi: R \to T$ be a function which assigns to every $s \in R$ the only $t \in T$ with $\|s(t)\| < a$. We assume R to be equipped with the following topology τ: For every $s \in R$ the neighborhood base of s consists of all subsets $W_{\varepsilon,U}$ of R which have the form

$$W_{\varepsilon,U} = \{s' \in R;\ \psi(s') \in U \quad \text{and} \quad \|s'(\psi(s')) - s(\psi(s))\| < \varepsilon\} ,$$

where U is a neighborhood from a fixed neighborhood base of $\psi(s)$ and ε is a positive number. Let S be a subset of R consisting of all $s \in R$ for which $s(\psi(s)) \in \Phi(\psi(s))$ holds. We show that S equipped with the relative topology generated by τ is a compact Hausdorff space. To verify this, let $\{N_\alpha\}_{\alpha \in A}$ be a covering of S by open subsets of R. Let $t \in T$. For every $\alpha \in A$ with $\psi^{-1}(t) \cap N_\alpha \neq \varnothing$ let $O_\alpha = \{s(t);\ s \in \psi^{-1}(t) \cap N_\alpha\}$. Since $\{O_\alpha\}$ is a covering of $\Phi(t)$ by open subsets of X, there exists a finite subcovering $\{O_{\alpha_i(t)}\}$, $i = 1, \cdots, n(t)$. We will show now that there exists an $\varepsilon_t > 0$ and neighborhood U_0 of t such that we have

$$\{s;\ \psi(s) \in U_0\} \cap \{s;\ \operatorname{dist}(s(\psi(s)), \Phi(t)) < \varepsilon_t\} \subset \bigcup_{i=1}^{n(t)} N_{\alpha_i(t)} .$$

Suppose that this is not true. Then for every neighborhood U and every $n \in N$ there exists an $s_{U,n}$ with $\psi(s_{U,n}) \in U$ and $\operatorname{dist}(s_{U,n}(\psi(s_{U,n})), \Phi(t)) < 1/n$ which is not in the union of all $N_{\alpha_i(t)}$, $i = 1, \cdots, n(t)$. It follows from the compactness of $\Phi(t)$ that there is a cluster point $s_0 \in S$ of the net $\{s_{U,n}\}$ with $s_0(t) \in \Phi(t)$. The point s_0 cannot be in the union of all $N_{\alpha_i(t)}$, $i = 1, \cdots, n(t)$, which implies that $s_0(t)$ cannot be in the union of all $O_{\alpha_i(t)}$, $i = 1, \cdots, n(t)$. A contradiction.

Now, it follows from the assumption that Φ is u.H.s.c. that there is an open neighborhood U_t of t such that for all $t' \in U_t$ and all $y \in \Phi(t')$ we have $\operatorname{dist}(y, \Phi(t)) < \varepsilon_t$. Moreover, U_t can be chosen such that $U_t \subset U_0$. It follows that

$$\{s \in S;\ \psi(s) \in U_t\} \subset \bigcup_{i=1}^{n(t)} N_{\alpha_i(t)}\ .$$

Construct such a neighborhood U_t for every $t \in T$ and choose a finite subcovering $U_{t_1}, \cdots, U_{t_m}$, $m \in N$, of T. Then the sets $N_{\alpha_i(t_j)}$, $i = 1, \cdots, n(t_j)$, $j = 1, \cdots, m$, are obviously a finite subcovering of S.

The restriction φ of ψ to S is obviously a continuous surjection from S onto T. Let $f: S \to X$ be defined by $f(s) = s(\varphi(s))$. The function f is obviously continuous. Let $g \circ \varphi$ be a best approximation of f in the corresponding SW-subspace V. Then we have

$$\begin{aligned}\operatorname{dist}(g, \Phi) &= \|f - g\circ\varphi\| = \inf_{h \in C(T,X)} \|f - h\circ\varphi\| \\ &= \inf_{h \in C(T,X)} \operatorname{dist}(h, \Phi)\ .\end{aligned}$$

Hence g is a best approximation of Φ in $C(T, X)$. This completes the proof of the theorem.

Let Φ be an u.H.s.c. function from S into $\mathscr{C}(X)$. We establish now a sufficient condition for the existence of a continuous selection of Ψ_Φ.

DEFINITION. A Banach space X is said to have the property (QUCC) if $c(K) \neq \varnothing$ for every $K \in \mathscr{C}(X)$ and if the following is true: Given a set $K \subset \mathscr{C}(X)$, an element $x \in X$ and numbers $r > 0, \varepsilon > 0$, there is a $\delta > 0$ such that for every $y \in X$ there exists an element $z_y \in B(x, \varepsilon)$ satisfying

$$B(x, r + \delta) \cap B(y, r) \cap K \subset B(z_y, r) \cap K\ .$$

THEOREM 3. *Let X be a Banach space with the property* (QUCC), *S a compact Hausdorff space, $\Phi: S \to \mathscr{C}(X)$ an* u.H.s.c. *map. Then Ψ_Φ has a continuous selection.*

Proof. We show that Ψ_Φ is l.s.c. First, since for all $t \in T$ $c(\Phi(t)) \subset \Psi_\Phi(t)$, we have $\Psi_\Phi(t) \neq \varnothing$ for every $t \in T$. Let $t \in T$, $x \in \Psi_\Phi(t)$ and $\varepsilon > 0$ be given. For x, $K = \bigcup_{t \in T} \Phi(t)$ (which is a compact set by Lemma 1), $r = r_\Phi$ and ε find the corresponding δ. Since Φ is u.H.s.c., there is a neighborhood U of t with $\Phi(t') \subset B(x, r + \delta) \cap K$

for every $t' \in U$. For $t' \in U$ let $y \in \Psi_\Phi(t')$. Then $\Phi(t') \subset B(x, r + \delta) \cap B(y, r) \cap K \subset B(z_y, r) \cap K$. Hence $z_y \in B(x, \varepsilon) \cap \Psi_\Phi(t')$. The existence of a continuous selection of Ψ_Φ follows then from Michael's selection theorem [7].

The following proposition provides an example of a class of Banach spaces with the property (QUCC). To prove it, we need the following easy lemma which we state without proof.

LEMMA 4. *Let $\{s_n\}$, $\{t_n\}$ be two sequences in a Banach space X. Let for some $r > 0$ $\lim \|s_n\| \leqq r$, $\lim \|t_n\| \leqq r$. Let*

$$u_n = \lambda_n s_n + (1 - \lambda_n) t_n$$

be such that we have $\beta_0 \leqq \lambda_n \leqq \eta_0$ for some $0 < \beta_0 < 1$, $0 < \eta_0 < 1$ and every $n \in N$, and such that $\lim \|u_n\| \geqq r$. Then $\lim \|(s_n + t_n)/2\| \geqq r$ for suitable subsequences.

PROPOSITION 5. *Let X be a* l. u. c. *space such that $c(K) \neq \emptyset$ for every $K \in \mathscr{C}(X)$. Then X has the property* (QUCC).

Proof. Assume the contrary. Then there exist positive numbers ε and r, an element $x \in X$ and a compact set $K \subset X$, such that for every $n \in N$ there is a $y_n \in X$ and a $w_n \in K$ with $\|x - w_n\| \leqq r + 1/n$, $\|y_n - w_n\| \leqq r$, and $\|z_n - w_n\| > r$, where

$$z_n = (1 - \varepsilon/2a_n)x + (\varepsilon/2a_n)y_n$$

and $a_n = \|x - y_n\|$. One can obviously assume $a_n > \varepsilon$ for every $n \in N$. Without loss of generality we can further assume that w_n converges to some $w_0 \in K$. It follows that $\|x - w_0\| \leqq r$, $\|y_n - w_0\| \leqq r + \eta_n$, $\|z_n - w_0\| > r - \eta_n$ for every $n \in N$ holds, where $\eta_n = \|w_n - w_0\|$. For every $n \in N$ denote $t_0 = x - w_0$, $s_n = y_n - w_0$, $u_n = z_n - w_0$. Without loss of generality one can now assume that $\lim \|s_n\| \leqq r$ and $\lim \|u_n\| \geqq r$. Thus, by Lemma 4, we have $\lim \|(t_0 + s_n)/2\| \geqq r$ which, together with $\|t_0 - s_n\| = a_n > \varepsilon$, $n \in N$, contradicts the local uniform convexity of X.

The following corollary is an immediate consequence of Theorems 2 and 3 and Proposition 5.

COROLLARY 6. *Let X be a dual* l.u.c. *space. Let S be a compact Hausdorff space. Then every* SW-*subspace of $C(S, X)$ is proximinal.*

Proof. By a result of Garkavi [3], $c(F)$ is nonempty even for every bounded subset of X.

It is an easy consequence of Lindenstrauss' well-known theorem concerning intersection properties of balls in L_1-predual spaces with centers in a compact set that these spaces also have the property (QUCC). So we have the following result of Blatter [2].

COROLLARY 7. *Let X be an L_1-predual space, S a compact Hausdorff space. Then every* SW-*subspace of $C(S, X)$ is proximinal.*

Ward [13] proved that $c(F) \neq \emptyset$ for every bounded subset of $C(S, X)$ if X is a Hilbert space and S is an arbitrary topological space. Amir [1] and Lau [4], independently, improved this result by showing that this is true for every X uniformly convex. We show now that, for compact subsets of $C(S, X)$ with S compact Hausdorff, this still remains true, if X has the property (QUCC).

THEOREM 8. *Let S be a compact Hausdorff space, X a Banach space with the property* (QUCC). *Then $c(K) \neq \emptyset$ for every compact subset K of $C(S, X)$.*

Proof. Let

$$\Phi(s) = \{x \in X;\ x = f(s) \quad \text{for some} \quad f \in K\}\ , \quad s \in S\ .$$

Then Φ obviously is a H.c. map from S into $\mathscr{C}(X)$. Furthermore, it is easy to show that $r(K) \geqq r_\Phi$. Hence every continuous selection of Ψ_Φ is in $c(K)$. The assertion of the theorem follows then from Theorem 3.

COROLLARY 9. *Let X be a dual* l. u. c. *space, S a compact Hausdorff space. Then $c(K) \neq \emptyset$ for every compact subset K of $C(S, X)$.*

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